Stability and instability of asymptotic profiles of solutions for fast diffusion equations

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1. Introduction

Aim of this talk

We deal with the Cauchy-Dirichlet for the fast diffusion equation,

$$\partial_t \left(|u|^{m-2} u \right) = \Delta u \quad \text{in } \Omega \times (0, \infty), \tag{1}$$

$$u = 0 \quad \text{on } \partial\Omega \times (0, \infty), \tag{2}$$

$$u(\cdot, 0) = u_0 \quad \text{in } \Omega, \tag{3}$$

where $m > 2$ and Ω is a bounded domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$.

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where $m > 2$ and Ω is a bounded domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$.

Put $w = |u|^{m-2}u$ to reformulate (1) as

$$\partial_t w = \Delta\left(|w|^{m'-2}w\right), \quad 1 < m' = m/(m-1) < 2.$$

Background: singular diffusion of plasma ($m = 3$ by Okuda-Dawson '73).

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Aim of this talk

Our aim of this talk is to discuss asymptotic profiles of solutions as well as the stability and instability of profiles.

Assumptions

Throughout this talk, we assume that

$$m < 2^* := \frac{2N}{(N-2)_+} \quad \text{and} \quad u_0 \in H_0^1(\Omega).$$

Definition of solutions

Definition (Weak solutions)

A function $u : \Omega \times (0, \infty) \to \mathbb{R}$ is said to be a (weak) solution of (1)–(3), if the following conditions are all satisfied:

- $u \in C([0,\infty); H_0^1(\Omega))$ and $|u|^{m-2}u \in C^1([0,\infty); H^{-1}(\Omega))$,
- For all $t \in (0, \infty)$ and $\psi \in C_0^\infty(\Omega)$,

$$\left\langle \frac{d}{dt}\left(|u|^{m-2}u\right)(t), \psi \right\rangle_{H_0^1} + \int_\Omega \nabla u(x,t) \cdot \nabla \psi(x) dx = 0,$$

- $u(\cdot, t) \to u_0$ strongly in $H_0^1(\Omega)$ as $t \to +0$.

For any $u_0 \in H_0^1(\Omega)$, the problem (1)–(3) admits a unique solution.

Extinction of solutions in finite time

Berryman-Holland ('80) proved

$$\forall u_0 \in H^1_0(\Omega), \quad \exists t_* = t_*(u_0) > 0 \quad \text{s.t.}$$
$$\|u(\cdot,t)\|_{1,2} \propto (t_* - t)_+^{1/(m-2)}.$$

Namely, every solution $u = u(x,t)$ vanishes at $t_* = t_*(u_0)$ at the rate $(t_* - t)^{1/(m-2)}$.

Here, $t_* = t_*(u_0)$ is called the <u>extinction time</u> (of solutions) for u_0.

$$\begin{array}{rccc} t_* : & H^1_0(\Omega) & \longrightarrow & [0,\infty) \\ & u_0 & \longmapsto & t_*(u_0) \end{array}$$

Asymptotic profiles of vanishing solutions

One can define the asymptotic profile $\phi = \phi(x)$ of $u = u(x,t)$ by

$$\phi(x) := \lim_{t \nearrow t_*} (t_* - t)^{-1/(m-2)} u(x,t) \quad \text{in } H_0^1(\Omega).$$

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In order to characterize ϕ, we apply the following transformation:

(4) $v(x,s) := (t_* - t)^{-1/(m-2)} u(x,t)$ and $s := \log(t_*/(t_* - t))$.

t	0	$\nearrow$	t_*
s	0	$\nearrow$	∞

Then the asymptotic profile $\phi = \phi(x)$ of $u = u(x,t)$ is reformulated as

$$\phi(x) := \lim_{s \nearrow \infty} v(x,s) \quad \text{in } H_0^1(\Omega).$$

Asymptotic profiles of vanishing solutions (contd.)

The Cauchy-Dirichlet problem (1)–(3) for $u = u(x,t)$ is rewritten by

(5) $$\partial_s \left(|v|^{m-2}v\right) = \Delta v + \lambda_m |v|^{m-2}v \quad \text{in } \Omega \times (0,\infty),$$

(6) $$v = 0 \quad \text{on } \partial\Omega \times (0,\infty),$$

(7) $$v(\cdot,0) = v_0 \quad \text{in } \Omega,$$

where $v_0 = t_*(u_0)^{-1/(m-2)} u_0$ and $\lambda_m = \frac{m-1}{m-2} > 0$.

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where $v_0 = t_*(u_0)^{-1/(m-2)} u_0$ and $\lambda_m = \frac{m-1}{m-2} > 0.$

Theorem 1 (Asymptotic profiles)

For any sequence $s_n \to \infty$, there exist a subsequence (n') of (n) and $\phi \in H^1_0(\Omega) \setminus \{0\}$ such that $v(s_{n'}) \to \phi$ strongly in $H^1_0(\Omega)$. Moreover, ϕ is a nontrivial stationary solution of (5)–(7), that is,

$$-\Delta\phi = \lambda_m |\phi|^{m-2}\phi \ \text{ in } \Omega, \quad \phi = 0 \ \text{ on } \partial\Omega. \tag{8}$$

See also [Berryman-Holland '80], [Kwong '88], [Savaré-Vespri '94].

Asymptotic profiles of vanishing solutions (contd.)

Moreover,

- $U(x,t) = (1-t)_+^{1/(m-2)}\phi(x)$ solves (1)–(3) with $U(0) = \phi(x)$.
- $t_*(\phi) = 1$ and the profile of $U(x,t)$ is $\phi(x)$.

Then we notice that

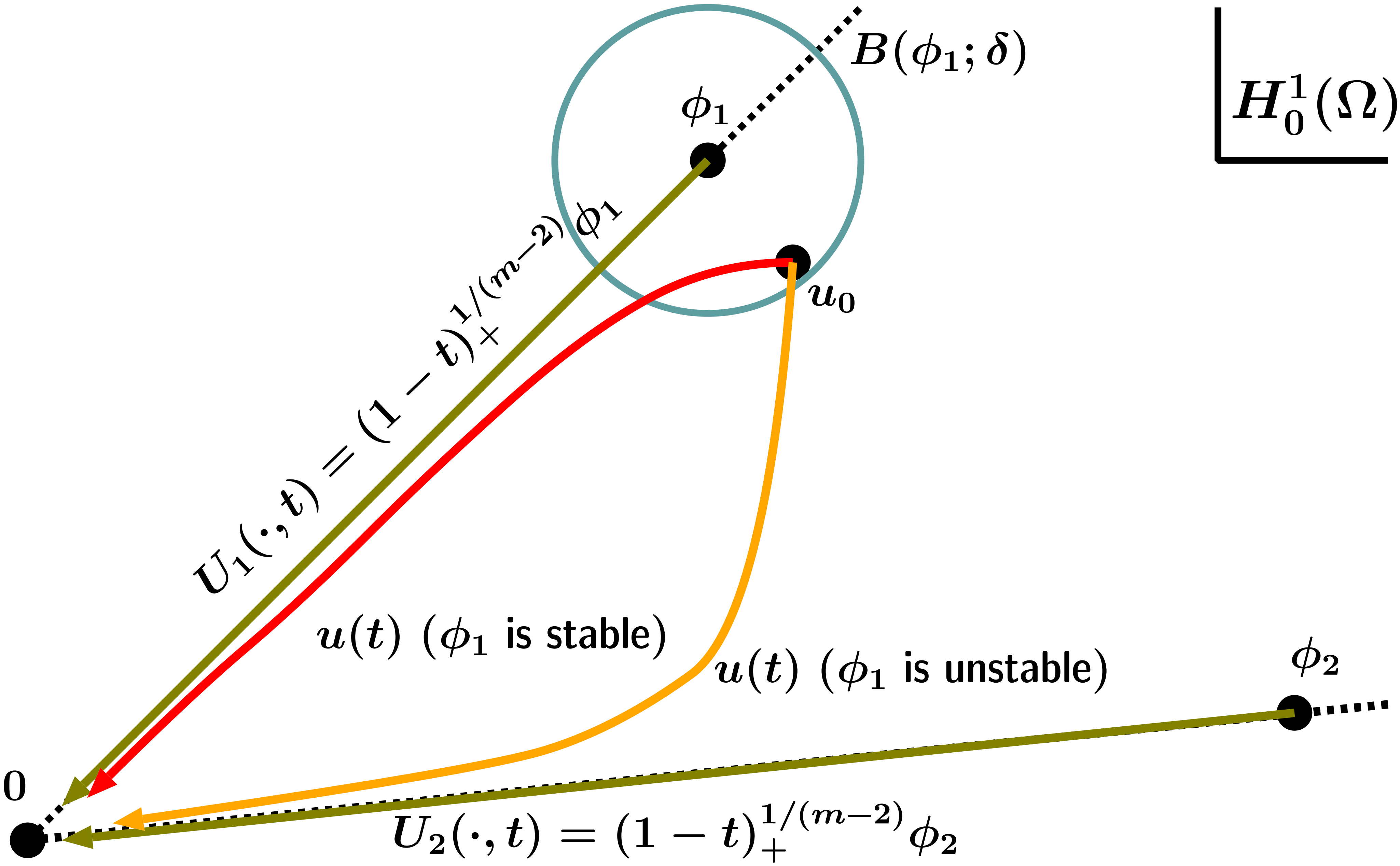
$$\{\text{Asymptotic profiles of } u(x,t)\} = \{\text{Nontrivial solutions } \phi(x)\} =: \mathcal{S}$$

Stability/instability of asymptotic profiles

Problem Let ϕ be an asymptotic profile and set

$$u_0 = \phi + p \quad \text{with a perturbation } \; p \in H_0^1(\Omega).$$

If $u_0 \in H_0^1(\Omega)$ is sufficiently close to ϕ (i.e., p is small), does the asymptotic profile of $u = u(x,t)$ also coincide with ϕ ? or not ?



Stability/instability of asymptotic profiles

Transformation and the set of initial data

Let us recall the transformation,

$$v(x,s) = (t_* - t)^{-1/(m-2)} u(x,t) \quad \text{and} \quad s = \log(t_*/(t_* - t)) \geq 0.$$

In particular, $v_0 = t_*(u_0)^{-1/(m-2)} u_0$.

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$$u_0 \in H_0^1(\Omega) \setminus \{0\} \quad \Leftrightarrow \quad v_0 \in \mathcal{X},$$

where

$$\mathcal{X} := \left\{ t_*(u_0)^{-1/(m-2)} u_0;\ u_0 \in H_0^1(\Omega) \setminus \{0\} \right\}.$$

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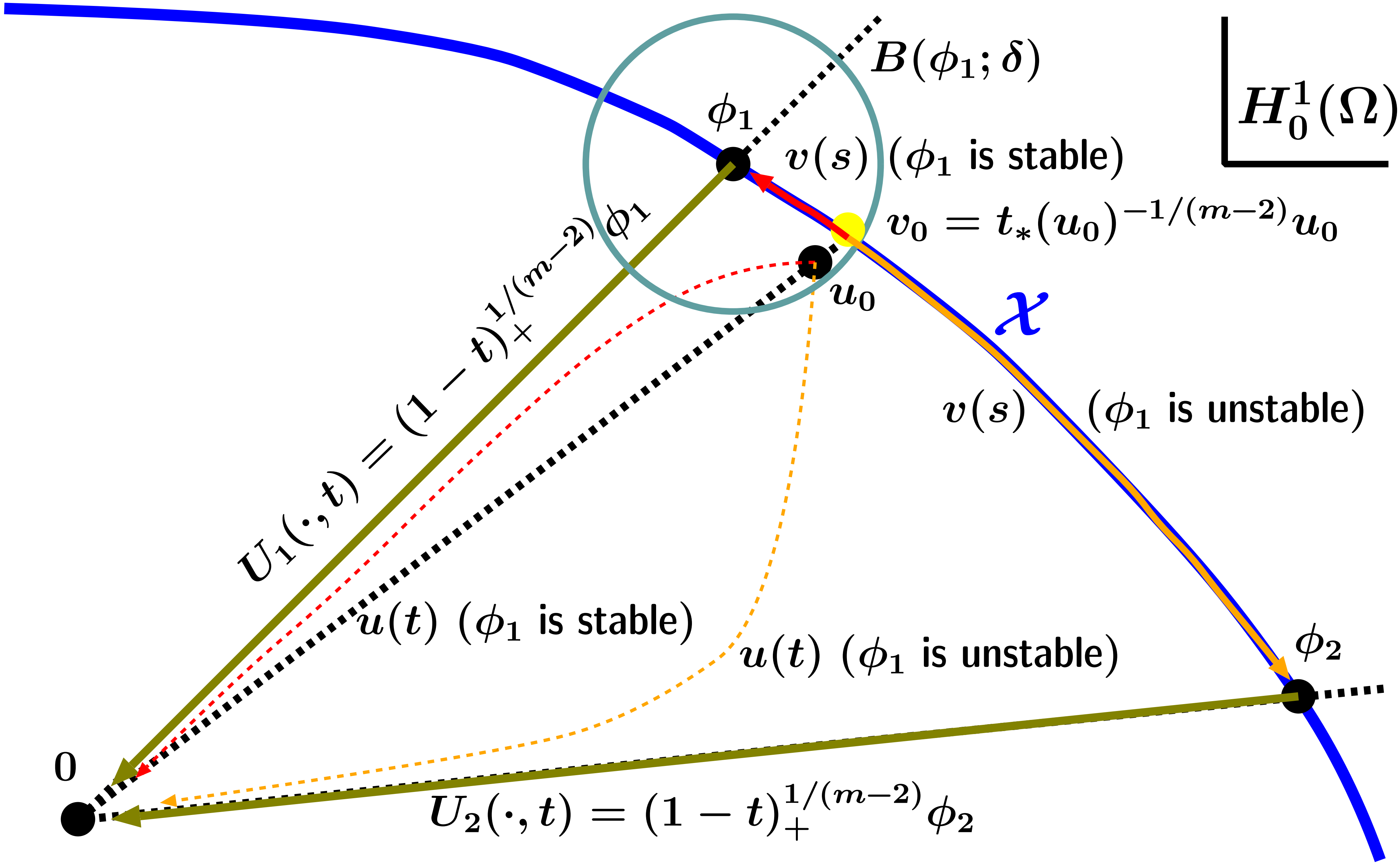
Then we note that

(i) $v_0 \in \mathcal{X} \ \Rightarrow \ v(s) \in \mathcal{X} \ \ \forall s \geq 0$.

(ii) $\mathcal{X} = \left\{ v_0 \in H_0^1(\Omega);\ t_*(v_0) = 1 \right\}$, which is homeomorphic to a unit sphere in $H_0^1(\Omega)$.

(iii) $\mathcal{S} \subset \mathcal{X}$ by $t_*(\phi) = 1$ for $\phi \in \mathcal{S}$.



Stability/instability of asymptotic profiles

Definition of the stability/instability of profiles

Definition 2 (Stability and instability of profiles)

Let $\phi \in H_0^1(\Omega)$ be an asymptotic profile of vanishing solutions.

(i) ϕ is said to be stable, if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that any solution v of (5)–(7) satisfies

$$v(0) \in \mathcal{X} \cap B(\phi; \delta) \quad \Rightarrow \quad \sup_{s \in [0,\infty)} \|v(s) - \phi\|_{1,2} < \varepsilon.$$

(ii) ϕ is said to be unstable, if ϕ is not stable.

(iii) ϕ is said to be asymptotically stable, if ϕ is stable, and moreover, there exists $\delta_0 > 0$ such that any solution v of (5)–(7) satisfies

$$v(0) \in \mathcal{X} \cap B(\phi; \delta_0) \quad \Rightarrow \quad \lim_{s \nearrow \infty} \|v(s) - \phi\|_{1,2} = 0.$$

2. Stability Analysis

Gradient system on the surface $\mathcal{X}$

Problem (5)–(7) can be written as a (generalized) gradient system,

$$\frac{d}{ds}|v|^{m-2}v(s) = -\nabla J(v(s)), \quad s > 0, \qquad v(0) = v_0 \in \mathcal{X},$$

where ∇J stands for the gradient of the functional

$$J(w) = \frac{1}{2}\|w\|_{1,2}^2 - \frac{\lambda_m}{m}\|w\|_m^m.$$

Hence $s \mapsto J(v(s))$ is non-increasing. Moreover,

$$\phi \text{ is an asymptotic profile} \quad \Leftrightarrow \quad \nabla J(\phi) = 0, \ \phi \neq 0.$$

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Therefore one can reveal the stability/instability of profiles by investigating the geometry of the functional J over $\mathcal{X} = \{w \in H_0^1(\Omega);\ t_*(w) = 1\}$.

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Therefore one can reveal the stability/instability of profiles by investigating the geometry of the functional J over $\mathcal{X} = \{w \in H_0^1(\Omega);\ t_*(w) = 1\}$.

Cf. Since $m > 2$, J forms a mountain pass structure in $H_0^1(\Omega)$. Hence 0 is stable and nontrivial critical points are unstable in $H_0^1(\Omega)$.

Main result 1 (stability)

Let d_1 be the least energy of J over nontrivial solutions, i.e.,

$$d_1 := \inf_{v \in \mathcal{S}} J(v), \quad \mathcal{S} := \{ \text{ nontrivial solutions of (8)} \}.$$

A least energy solution ϕ_1 of (8) means $\phi_1 \in \mathcal{S}$ satisfying $J(\phi_1) = d_1$.
Every least energy solution of (8) is sign-definite.

Theorem 3 (Stability of profiles)

Let ϕ be a least energy solution of (8). Then

(i) ϕ is a stable profile, if ϕ is isolated in $H_0^1(\Omega)$ from the other least energy solutions.

(ii) ϕ is an asymptotically stable profile, if ϕ is isolated in $H_0^1(\Omega)$ from the other sign-definite solutions.

Main result 2 (instability)

Theorem 4 (Instability of profiles)

Let ϕ be a sign-changing solution of (8). Then

(i) ϕ is not an asymptotically stable profile.

(ii) ϕ is an unstable profile, if ϕ is isolated in $H_0^1(\Omega)$ from the set $\{\psi \in \mathcal{S};\ J(\psi) < J(\phi)\}$.

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Roughly speaking,

- least energy solutions of (8) are asymptotically stable profiles;
- sign-changing solutions of (8) are unstable profiles

under appropriate isolations of profiles.

Let us see several situations that such isolations of profiles hold...

Corollary of the main result 1 (stability)

We first note that sign-definite solutions are isolated in $H_0^1(\Omega)$ from sign-changing solutions. Moreover, least energy solutions are also isolated from sign-definite ones in the following cases:

Corollary 5 (Examples of asymptotically stable profiles)

Least energy solutions are asymptotically stable profiles in the following cases:

- Ω is a ball and $2 < m < 2^*$ (Gidas-Ni-Nirenberg '79).
- $\Omega \subset \mathbb{R}^2$ is bounded and convex and $2 < m < 2^*$ (Lin '94).
- $\Omega \subset \mathbb{R}^N$ is bounded and $2 < m < 2 + \delta$ (Dancer '03).
- $\Omega \subset \mathbb{R}^N$ is symmetric w.r.t. hyperplanes $[x_i = 0]$ and convex in x_i for $i = 1, 2, \ldots, N$ and $2^* - \delta < m < 2^*$ (Grossi '00).

Corollary of the main result 2 (instability)

Corollary 6 (Instability of sign-changing least energy profiles)

'Sign-changing least energy solutions' are unstable.

- (8) always admits a 'sign-changing least energy solution' ϕ_2, provided that $m < 2^*$.

  $$\phi_2 \in \mathcal{SC} \text{ satisfying } J(\phi_2) = d_2, \text{ where}$$
  $$d_2 := \inf \{J(\psi);\ \psi \in \mathcal{SC}\},\ \mathcal{SC} = \{\text{sign-changing sol. of (8)}\}.$$

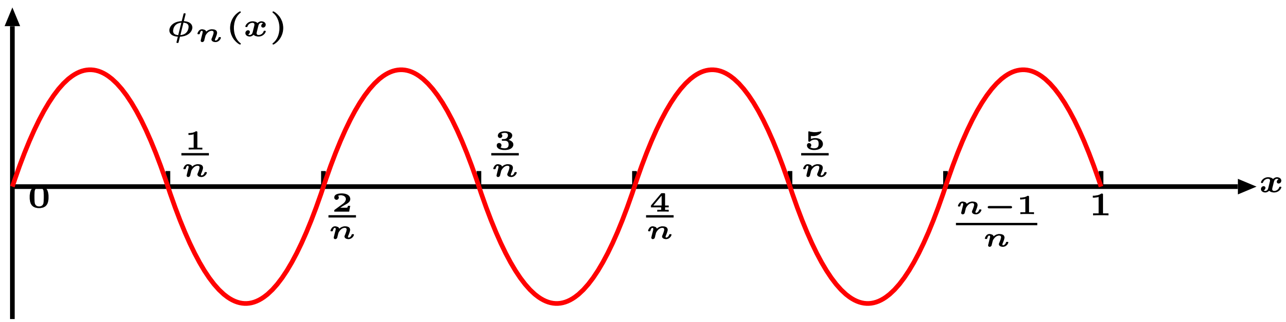
- Each sign-changing least energy solution ϕ_2 is isolated in $H_0^1(\Omega)$ from $\{\psi \in \mathcal{S};\ J(\psi) < d_2\}$.

One-dimensional case

In case $N = 1$ and $\Omega = (0, 1)$, the Dirichlet problem (8) is written by

(9) $$-\phi'' = \lambda_m |\phi|^{m-2}\phi \text{ in } (0,1), \quad \phi(0) = \phi(1) = 0.$$

Then one can obtain all nontrivial solutions $\{\pm\phi_n\}_{n\in\mathbb{N}}$.



$$J(\pm\phi_1) < J(\pm\phi_2) < \cdots < J(\pm\phi_n) \to \infty \quad \Rightarrow \quad \phi_n \text{ is isolated !}$$

Corollary 7 (Stability and instability of profiles in $N = 1$)

- Sign-definite profiles $\pm\phi_1$ are asymptotically stable.
- All the other profiles $\pm\phi_n$ $(n \neq 1)$ are unstable.

3. Proof of Theorem 3

Theorem 3 (Stability of profiles)

Let ϕ be a least energy solution of (8). Then

(i) ϕ is a stable profile, if ϕ is isolated in $H_0^1(\Omega)$ from the other least energy solutions.

(ii) ϕ is a asymptotically stable profile, if ϕ is isolated in $H_0^1(\Omega)$ from the other sign-definite solutions.

Proof of Theorem 3

From the continuous dependence of solutions on initial data, we have

Proposition 8 (Continuity of $t_*(\cdot)$)

Assume $m < 2^*$.

$$u_{0,n} \to u_0 \quad \text{weakly in } H_0^1(\Omega) \quad \Rightarrow \quad t_*(u_{0,n}) \to t_*(u_0).$$

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$$u_{0,n} \to u_0 \quad \text{weakly in } H_0^1(\Omega) \quad \Rightarrow \quad t_*(u_{0,n}) \to t_*(u_0).$$

Let us recall that $\mathcal{X} = \{w \in H_0^1(\Omega);\ t_*(w) = 1\}$.

Lemma 9 (Closedness of $\mathcal{X}$)

$$u_n \in \mathcal{X} \quad \text{and} \quad u_n \to u \ \text{ weakly in } H_0^1(\Omega) \quad \Rightarrow \quad u \in \mathcal{X}.$$

Proof of Theorem 3

Lemma 10 (Variational feature of $\mathcal{X}$)

Let $d_1 = \inf_{\mathcal{S}} J$. Then

$$\mathcal{X} \subset [d_1 \leq J] := \left\{ v_0 \in H_0^1(\Omega);\ d_1 \leq J(v_0) \right\}.$$

Moreover, if $v_0 \in \mathcal{X}$ and $J(v_0) = d_1$, then $\nabla J(v_0) = 0$.

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Moreover, if $v_0 \in \mathcal{X}$ and $J(v_0) = d_1$, then $\nabla J(v_0) = 0$.

(Proof) Let $v_0 \in \mathcal{X}$. Then

$$v(s_n) \to \phi \quad \text{strongly in } H_0^1(\Omega) \quad \text{and} \quad \phi \in \mathcal{S}.$$

Since $s \mapsto J(v(s))$ is non-increasing, $J(v_0) \geq J(v(s)) \geq J(\phi) \geq d_1$. Hence $d_1 \leq J(v_0)$.

If $v_0 \in \mathcal{X}$ and $J(v_0) = d_1$, then $J(v_0) = \min_{\mathcal{X}} J$. Hence $v(s) \equiv v_0$.

Proof of Theorem 3

Let $\mathcal{LES} := \{$least energy solutions of (8)$\}$. By assumption,

$$\exists r > 0 \quad \text{s.t.} \quad B(\phi; r) \cap \mathcal{LES} = \{\phi\}.$$

Claim 1: For any $\varepsilon \in (0, r)$

$$c := \inf\{J(v);\ v \in \mathcal{X},\ \|v - \phi\|_{1,2} = \varepsilon\} > d_1.$$

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Assume that $c = d_1$, i.e.,

$$\exists v_n \in \mathcal{X};\ \|v_n - \phi\|_{1,2} = \varepsilon \ \text{ and } \ J(v_n) \to d_1.$$

Since $m < 2^*$, up to a subsequence,

$$v_n \to v_\infty \quad \text{weakly in } H_0^1(\Omega) \ \text{ and strongly in } L^m(\Omega).$$

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By two lemmas,

$$v_\infty \in \mathcal{X}, \quad \text{and hence,} \quad d_1 \le J(v_\infty).$$

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Hence

$$\begin{aligned}\frac{1}{2}\|v_n\|_{1,2}^2 &= J(v_n) + \frac{\lambda_m}{m}\|v_n\|_m^m \\ &\to d_1 + \frac{\lambda_m}{m}\|v_\infty\|_m^m \leq J(v_\infty) + \frac{\lambda_m}{m}\|v_\infty\|_m^m = \frac{1}{2}\|v_\infty\|_{1,2}^2.\end{aligned}$$

Thus $v_n \to v_\infty$ strongly in $H_0^1(\Omega)$. Hence $\|v_\infty - \phi\|_{1,2} = \varepsilon$ and $J(v_\infty) = d_1$.

Proof of Theorem 3

We have known that

$$v_\infty \in \mathcal{X}, \quad J(v_\infty) = d_1 \quad \text{and} \quad \|v_\infty - \phi\|_{1,2} = \varepsilon.$$

Hence $v_\infty \in \mathcal{LES}$. However, the fact that $\|v_\infty - \phi\|_{1,2} = \varepsilon < r$ contradicts the isolation of ϕ. □

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Let $\varepsilon \in (0, r)$ be arbitrarily given. Choose $\delta \in (0, \varepsilon)$ so small that

$$J(v) < c \quad \forall v \in B(\phi; \delta)$$

(it is possible, because $c > d_1 = J(\phi)$ by Claim 1, and J is continuous in $H_0^1(\Omega)$).

For any $v_0 \in \mathcal{X} \cap B(\phi; \delta)$, let $v(s)$ be a solution of (5)–(7). Then $v(s) \in \mathcal{X}$.

Proof of Theorem 3

Claim 2: For any $s \geq 0$, $v(s) \in B(\phi; \varepsilon)$ (hence ϕ is stable).

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Assume $v(s_0) \in \partial B(\phi; \varepsilon)$ at some $s_0 \geq 0$. By the definition of c,

$$c \leq J(v(s_0)).$$

However, it contradicts the facts that $J(v(s_0)) \leq J(v_0) < c$. $\square$

4. Proof of Theorem 4

Theorem 4 (Instability of profiles)

Let ϕ be a sign-changing solution of (8). Then

(i) ϕ is not an asymptotically stable profile.

(ii) ϕ is an unstable profile, if ϕ is isolated in $H_0^1(\Omega)$ from the set $\{\psi \in \mathcal{S};\ J(\psi) < J(\phi)\}$.

Proof of Theorem 4

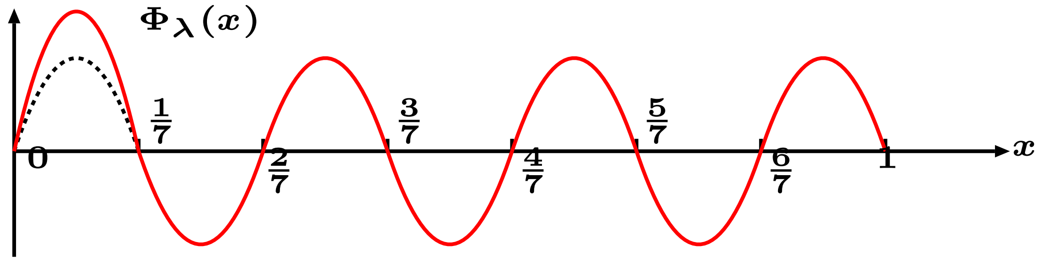
Let ϕ be a sign-changing solution of (8) (hence ϕ admits more than two nodal domains).

Claim 1: ϕ is not an asymptotically stable profile.

Let D be a nodal domain of ϕ and define

$$\phi_\mu(x) := \begin{cases} \mu\phi(x) & \text{if } x \in D, \\ \phi(x) & \text{if } x \in \Omega \setminus D \end{cases} \qquad \text{for } \mu \geq 0$$

(Note: ϕ_μ might not belong to $\mathcal{X}$).
Then one can observe that



- $\phi_\mu \longrightarrow \phi$ strongly in $H_0^1(\Omega)$ as $\mu \longrightarrow 1$.
- if $\mu \neq 1$, then $J(c\phi_\mu) < J(\phi)$ for any $c \geq 0$.

Proof of Theorem 4

Set

$$c_\mu := t_*(\phi_\mu)^{-1/(m-2)}, \quad v_{0,\mu} := c_\mu \phi_\mu \in \mathcal{X}.$$

It follows that

- $t_*(\phi_\mu) \to t_*(\phi) = 1$ and $v_{0,\mu} \to \phi$ strongly in $H^1_0(\Omega)$ as $\mu \to 1$.
- if $\mu \neq 1$, then $J(v_{0,\mu}) = J(c_\mu \phi_\mu) < J(\phi)$.

Hence solutions $v_\mu(s)$ of (5)–(7) with $v_\mu(0) = v_{0,\mu}$ never converges to ϕ as $s \to \infty$, since $J(v_\mu(s)) \leq J(v_{0,\mu}) < J(\phi)$.

Therefore ϕ is not an asymptotically stable profile.

Proof of Theorem 4

We further assume that

$$\exists R > 0 \quad \text{s.t.} \quad \overline{B(\phi; R)} \cap \{\psi \in \mathcal{S};\ J(\psi) < J(\phi)\} = \emptyset.$$

Claim 2: If $\mu \neq 1$, then $v_\mu(s) \notin \overline{B(\phi; R)}$ for any $s \gg 1$.

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Claim 2: If $\mu \neq 1$, then $v_\mu(s) \notin \overline{B(\phi; R)}$ for any $s \gg 1$.

Assume that $v_\mu(s_n) \in \overline{B(\phi; R)}$ with some sequence $s_n \to \infty$.

Then

$$v_\mu(s_n) \longrightarrow {}^\exists\psi \in \overline{B(\phi; R)} \cap \mathcal{S} \quad \text{strongly in } H_0^1(\Omega).$$

Moreover,

$$J(\psi) \leq J(v_{0,\mu}) < J(\phi).$$

It contradicts the isolation of ϕ. Thus ϕ is an unstable profile. □

5. Characterization of $\mathcal{X}$

Characterization of $\mathcal{X}$

$\mathcal{X}$ is a separatrix for (5)–(7) !

Proposition 11 (Characterization of $\mathcal{X}$)

Let $v(s)$ be a solution of (5)–(7) with $v(0) = v_0$.

(i) If $v_0 \in \mathcal{X} = \{v_0 \in H_0^1(\Omega);\ t_*(v_0) = 1\}$,then

$$v(s_n) \longrightarrow \phi \in \mathcal{S} \quad \text{strongly in } H_0^1(\Omega) \text{ as } s_n \longrightarrow \infty.$$

(ii) If $v_0 \in \mathcal{X}^+ := \{v_0 \in H_0^1(\Omega);\ t_*(v_0) > 1\}$, then $v(s)$ diverges as $s \longrightarrow \infty$. Hence $\mathcal{X}^+$ is an unstable set.

(iii) If $v_0 \in \mathcal{X}^- := \{v_0 \in H_0^1(\Omega);\ t_*(v_0) < 1\}$, then $v(s)$ vanishes in finite time. Hence $\mathcal{X}^-$ is a stable set.

Separatrix $\mathcal{X}$ and Nehari manifold $\mathcal{N}$

Proposition 11 classifies the whole of energy space $H_0^1(\Omega)$ in terms of large-time behaviors of solutions for (5)–(7):

$$\begin{aligned} \partial_s \left(|v|^{m-2}v\right) &= \Delta v + \lambda_m |v|^{m-2}v && \text{in } \Omega \times (0,\infty), \\ v &= 0 && \text{on } \partial\Omega \times (0,\infty), \\ v(\cdot,0) &= v_0 \in H_0^1(\Omega) && \text{in } \Omega. \end{aligned}$$

Moreover, we emphasize that the separatrix $\mathcal{X}$ between the stable and unstable sets does not coincides with the Nehari manifold of J,

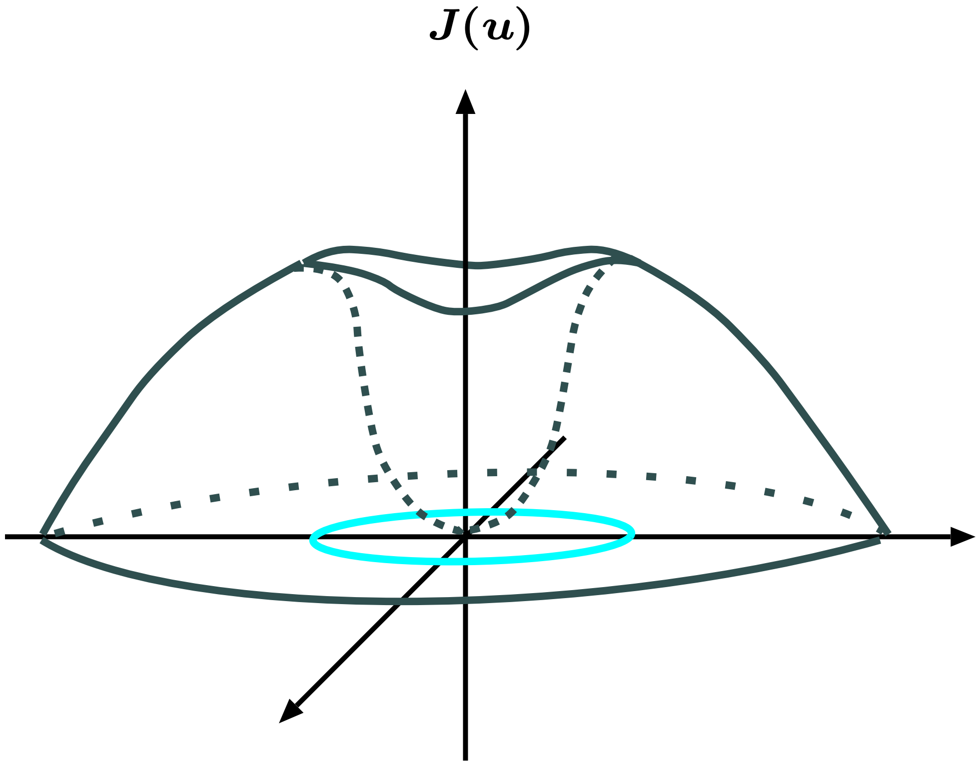
$$\mathcal{N} := \left\{ w \in H_0^1(\Omega);\ \langle \nabla J(w), w \rangle = 0 \right\}.$$

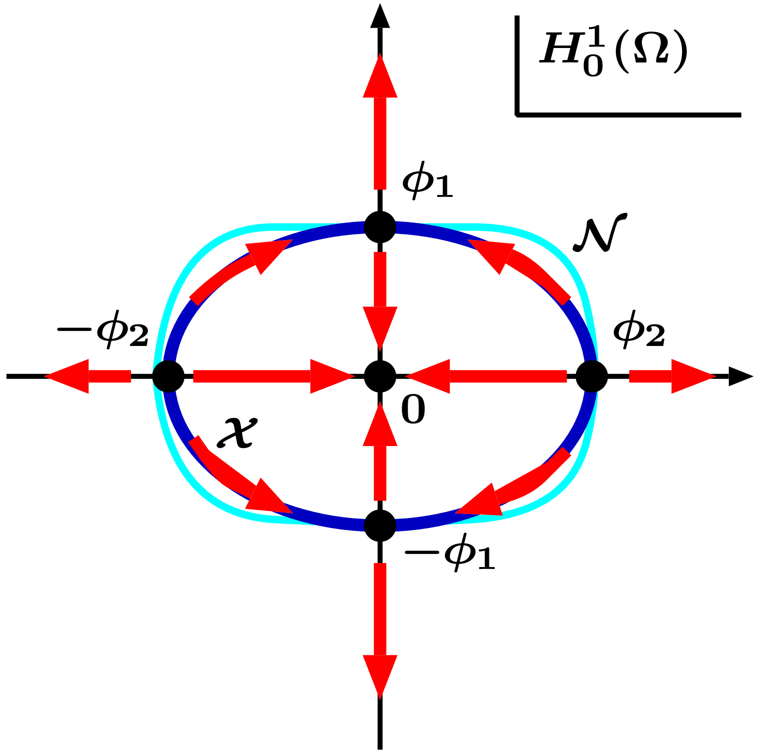
We further observe that

$\mathcal{X}$ is surrounded by $\mathcal{N}$ and $\mathcal{N} \cap \mathcal{X} = \mathcal{S}$.

The geometry of the functional J

$$J(w) = \frac{1}{2}\|w\|_{1,2}^2 - \frac{\lambda_m}{m}\|w\|_m^m, \quad w \in H_0^1(\Omega), \quad m > 2.$$





$\pm\phi_1$: (asymptotic) stable, $\pm\phi_n$ ($n \neq 1$): unstable

Thank you for your attention.

6. Stability and symmetry

Asymptotic profiles in a ball

Let us particularly consider the ball domain,

$$\Omega := \{x \in \mathbb{R}^2;\ |x| < 1\}.$$

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admits the unique positive radial solution ϕ, and no other positive solution.

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admits the unique positive radial solution ϕ, and no other positive solution.

Hence ϕ is the unique asymptotic profiles of positive solutions for (1)–(3).

By Theorem 3, the positive radial profile ϕ is asymptotically stable.

Asymptotic profiles in an annulus

Let us next treat the annular domain,

$$\Omega := \{x \in \mathbb{R}^2;\ a < |x| < b\}, \quad 0 < a < b.$$

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$$\Omega := \{x \in \mathbb{R}^2;\ a < |x| < b\}, \quad 0 < a < b.$$

Let $\phi > 0$ be a positive radial solution of

(10) $$-\Delta\phi = \lambda_m \phi^{m-1} \quad \text{in } \Omega,$$

(11) $$\phi = 0 \quad \text{on } \partial\Omega.$$

Then ϕ becomes an asymptotic profiles of solutions $u = u(x,t)$ for (1)–(3).

Remark and question

Remark.

- (10), (11) admits the unique radial solution ϕ and infinitely many non-radial solutions. Moreover, $J(\phi) > d_1 := \inf_{\mathcal{N}} J$.
  Hence, ϕ is sign-definite but does not take least energy.

- Our preceding results cannot judge the stability/instability of ϕ.

Question.

Is the radial profile $\phi > 0$ (asymptotic) stable or unstable ?

Answer to the question

Our result reads,

Theorem 12 (Instability of positive radial profiles)

Let Ω be the annular domain.

Let ϕ be the unique positive radial solution of (10), (11).

Then ϕ is not an asymptotically stable profile of solutions for (1)–(3).

Remark. Due to Theorem 4, we have already known that all the sign-changing profiles are unstable.

Perturbations to radial solutions/profiles

Define $u_{0,\varepsilon} : \Omega \to \mathbb{R}$ with a parameter $\varepsilon > 0$ by

$$u_{0,\varepsilon}(x) = \sigma_\varepsilon(\theta)\phi(r) \quad \text{for } x = (r\cos\theta, r\sin\theta) \in \Omega$$

with the function

$$\sigma_\varepsilon(\theta) = 1 + \varepsilon \sin\theta \quad \text{for } \theta \in [0, 2\pi] \text{ and } \varepsilon > 0.$$

Then we have:

Proposition 13 (Perturbations to radial solutions/profiles)

Assume that

(12) $$0 < (b-a)/a < \sqrt{\pi(m-2)}.$$

Then there exist $c_0 \in (0,1)$ and $\varepsilon_0 > 0$ such that $J(cu_{0,\varepsilon}) < J(\phi)$ for any $\varepsilon \in (0, \varepsilon_0)$ and $c > c_0$.

Sketch of proof (1/2)

Let $\varepsilon > 0$. Then we remark that

$$u_{0,\varepsilon} \to \phi \quad \text{strongly in } H_0^1(\Omega),$$
$$t_*(u_{0,\varepsilon}) \to t_*(\phi) = 1 \quad \text{as} \ \ \varepsilon \to 0.$$

Put $v_{0,\varepsilon} := t_*(u_{0,\varepsilon})^{-1/(m-2)} u_{0,\varepsilon}$ and denote by $v_\varepsilon = v_\varepsilon(x,s)$ the unique solution of (5)–(7) with the initial data $v_{0,\varepsilon}$.

Choose $\varepsilon_1 > 0$ such that $c_\varepsilon > c_0$, where c_0 is given by Proposition 6.5, for all $\varepsilon \in (0, \varepsilon_1)$. Then by Proposition 6.5, one can assure that

$$J(v_{0,\varepsilon}) = J(c_\varepsilon u_{0,\varepsilon}) < J(\phi) \quad \text{for } \varepsilon \text{ sufficiently close to } 0.$$

Sketch of proof (2/2)

Moreover, it holds that

$$v_{0,\varepsilon} \longrightarrow \phi \quad \text{strongly in } H_0^1(\Omega).$$

Hence noting that

$$J(v_\varepsilon(s)) \leq J(v_{0,\varepsilon}) < J(\phi) \quad \text{for all } s \geq 0,$$

we deduce that $v_\varepsilon(s)$ never converges to ϕ strongly in $H_0^1(\Omega)$ as $s \longrightarrow \infty$.

Thus the positive radial profile ϕ is not asymptotically stable. $\square$

Asymmetry of least energy solutions

As a corollary of our method of proof, we have:

Corollary 14 (Asymmetry of least energy solutions)

Let Ω be the annulus and assume that (12) holds. Then the Dirichlet problem (10), (11) admits a non-radial positive solution with a lower energy than that of the unique radial positive solution.

Hence least energy solutions of (10), (11) are not radially symmetric.

Asymmetry of least energy solutions

Proof. The unique radial positive solution ϕ has the minimum energy among all the non-trivial radial solutions. In the previous arguments,

$$J(v_\varepsilon(s)) \leq J(v_{0,\varepsilon}) < J(\phi) \quad \text{for all } s > 0.$$

Moreover, we have also verified

$$v_\varepsilon(s_n) \longrightarrow \exists \phi_\varepsilon \in \mathcal{S} \quad \text{strongly in } H_0^1(\Omega) \text{ as } s_n \to \infty,$$

which implies $J(\phi_\varepsilon) < J(\phi) = \inf\{J(\psi);\ \psi \text{ is a radial solution}\}$.

Hence ϕ_ε is never radially symmetric. □

Remarks

- We can extend these results to the following cases:
  - N-dimensional cases,
  - cylindrical domains,
  - toroidal domains.
- The asymmetry of least energy solutions for (10), (11) in annular domains has been proved by Coffman ($N = 2$), Li ($N \geq 4$) and Byeon ($N = 3$), provided that $(b - a)/a$ is sufficiently small. However, their result does not provide any estimates for the smallness.
- Our proof of the asymmetry of least energy solutions for the elliptic problem relies on fast diffusion flow.

Nonlinear diffusion

Let us consider a solution $u = u(x,t)$ of the nonlinear parabolic equation:

$$\partial_t \left(|u|^{m-2} u \right) = \Delta u, \qquad x \in \Omega \subset \mathbb{R}^N, \quad t > 0,$$

where $\partial_t = \partial/\partial t$ and $1 < m < \infty$.

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By setting $w = |u|^{m-2}u$, one can transform it into a usual form,

$$\partial_t w = \Delta\left(|w|^{m'-2}w\right) = \nabla \cdot \Big(\underbrace{(m'-1)|w|^{m'-2}}_{\text{Diffusion coefficient } D} \nabla w\Big)$$

with $m' = \dfrac{m}{m-1}$.

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with $m' = \dfrac{m}{m-1}$.

In this talk, we address ourselves to the case that

$$m > 2 \qquad \text{(equivalently, } \; m' < 2\text{)}.$$

Then the diffusion coefficient D will be singular when $w(x,t) = 0$.

Nonlinear diffusion

Equation	m	m'	D	Properties of diffusion
Heat/Diffusion	2	2	1	Infinite-speed propagation Decaying as $t \longrightarrow \infty$
Porous medium (PME)	$(1, 2)$	$(2, \infty)$	Degenerate	Finite-speed propagation Decaying as $t \longrightarrow \infty$
Fast diffusion (FDE)	$(2, \infty)$	$(1, 2)$	Singular	Infinite-speed propagation Extinction in finite time

Extinction of solutions in finite time

Let us first consider a separable solution, $u(x,t) = \rho(t)\psi(x)$, where $\rho(t) \geq 0$.

$$\frac{d}{dt}\rho(t)^{m-1} = -\lambda\rho(t) \;\text{ for }\; t > 0, \quad \rho(0) = 1,$$
$$-\Delta\psi(x) = \lambda|\psi|^{m-2}\psi(x) \;\text{ for }\; x \in \Omega, \quad \psi|_{\partial\Omega} = 0$$

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with a constant $\lambda > 0$. By solving the ODE of ρ,

$$\rho(t) = C(t_* - t)_+^{1/(m-2)} \ \text{ for } \ t > 0 \quad \text{with } \ t_* := \frac{1}{\lambda} \cdot \frac{m-1}{m-2},$$

and hence, $\rho(t)$ vanishes at a finite time t_*.
As for ψ, due to $m < 2^*$, the elliptic equation admits (infinitely-many) non-trivial solutions.

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Hence these nontrivial separable solutions vanish in finite time at the rate $(t_* - t)^{1/(m-2)}$. This fact also holds for general solutions (Sabinina '62).