

Qualitative behavior of global solutions to some nonlinear fourth order differential equations

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The equation

Consider the equation

$$w''''(s) + kw''(s) + f(w(s)) = 0 \quad s \in \mathbb{R}$$

where $k \in \mathbb{R}$ and f is a locally Lipschitz function

- $\Rightarrow k > 0$ Swift-Hohenberg equation
- $\Rightarrow k \leq 0$ extended Fisher-Kolmogorov equation

- L.A. Peletier, W.C. Troy, Birkhäuser Boston Inc. (2001)

The suspension bridge model

Beam equation

$$u_{tt} + u_{xxxx} + u^+ - 1 = 0 \quad x \in \mathbb{R}$$

- O.H. Ammann, T. Von Karman, G.B. Woodruff, (1941)
- P.J. McKenna, W. Walter, SIAM J. Appl. Math. (1990)

The suspension bridge model

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- O.H. Ammann, T. Von Karman, G.B. Woodruff, (1941)
- P.J. McKenna, W. Walter, SIAM J. Appl. Math. (1990)

Travelling waves of the form $\implies u(x, t) = 1 + w(x - ct)$:

$$w''''(s) + c^2 w''(s) + f(w(s)) = 0 \quad s \in \mathbb{R}$$

Piece-wise nonlinearity

$$f(w) = (w + 1)^+ - 1$$

Smoothed nonlinearity

$$f(w) = e^w - 1$$

- Y. Chen, P.J. McKenna, J. Diff. Eq. (1997);
J. Horak, P.J. McKenna, Progr. Nonlin. Diff. Eq. (2003)

Other models

$$w''''(s) + kw''(s) + f(w(s)) = 0 \quad (s, k \in \mathbb{R})$$

⇒ Configuration of a nonlinearly supported elastic strut

$$f(w) = w - w^2; \quad f(w) = w - w^3 + w^5$$

- G.W. Hunt, H.M. Bolt, J.M.T. Thompson, Proc. R.S.L. (1989)
- C.J. Amick, J.F. Toland, Eur. J. Appl. Math. (1992)
- L.A. Peletier, SIAM J. Math. Anal. (2001)

⇒ Pattern formation in physical, chemical or biological systems

$$f(w) = w^3 - w$$

- L.A. Peletier, W.C. Troy, Birkhäuser Boston Inc.(2001)
- D. Bonheure, L. Sanchez, Handbook Diff. Eq. III, Elsevier (2006)

A connected biharmonic Gelfand-type equation

Let $u = u(r)$, $r := |x|$ be a radial solution to

$$\Delta^2 u + e^u = \frac{1}{|x|^4} \quad \text{in } \mathbb{R}^4 \setminus \{0\}$$

with the change of variables

$$s = \log r \quad w(s) := u(e^s) + 4s \quad s \in \mathbb{R}$$

w solves

$$w''''(s) - 4w''(s) + e^{w(s)} - 1 = 0 \quad s \in \mathbb{R}$$

- C.S. Lin, (1998); S.Y.A. Chang, W. Chen (2001);
- G.Arioli, F.Gazzola, H.-Ch.Grunau, (2006); G.Arioli, F.Gazzola, H.-Ch.Grunau, E.Mitidieri, (2005); E.Berchio, D.Cassani, F.Gazzola, (2010); E.Berchio, F.Gazzola, (2005); J.Dávila, L.Dupaigne, I.Guerra, M.Montenegro, (2007); J.Dávila, I.Flores, I.Guerra, (2009)

Main assumption on f

$$w''''(s) + kw''(s) + f(w(s)) = 0 \quad s \in \mathbb{R}$$

we assume

$$(S) \quad f \in \text{Lip}_{\text{loc}}(\mathbb{R}), \quad f(t) t > 0 \quad \text{for every } t \in \mathbb{R} \setminus \{0\}$$

- E.B., A. Ferrero, F. Gazzola, P. Karageorgis, to appear J. Diff. Eq.

Global continuation

Theorem

Let $k \in \mathbb{R}$ and assume that f satisfies **(S)**.

(i) If a local solution to

$$w''''(s) + kw''(s) + f(w(s)) = 0 \quad s \in \mathbb{R}$$

blows up at some finite $R \in \mathbb{R}$, then

$$\liminf_{s \rightarrow R} w(s) = -\infty \quad \text{and} \quad \limsup_{s \rightarrow R} w(s) = +\infty.$$

(ii) If f also satisfies

$$\text{(L)} \quad \limsup_{t \rightarrow +\infty} \frac{f(t)}{t} < +\infty \quad \text{or} \quad \limsup_{t \rightarrow -\infty} \frac{f(t)}{t} < +\infty.$$

Then, **any local solution exists for all** $s \in \mathbb{R}$.

- For every $k \geq 0$, $f(w) = -24w^5 - 2kw^3$ satisfies **(L)** but not **(S)** and $w(s) = \frac{1}{s}$ is a local solution to

$$w''''(s) + kw''(s) - 24w^5 - 2kw^3 = 0 \quad s \in \mathbb{R}$$

which **blows up** in finite time.

⇒ If **(S)** doesn't hold, global continuation may fail.

- For every $k \geq 0$, $f(w) = -24w^5 - 2kw^3$ satisfies **(L)** but not **(S)** and $w(s) = \frac{1}{s}$ is a local solution to

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⇒ If **(S)** doesn't hold, global continuation may fail.

- Fix $n \geq 5$, $f(w) = \left(\frac{n(n-4)}{4}\right)^2 t + |t|^{8/(n-4)}t$ satisfies **(S)** but not **(L)** and the corresponding equation (with $k := -\frac{n^2-4n+8}{2}$) admits a solution which **blows up** in finite time with wide oscillations.
- F. Gazzola, R. Pavani, preprint 2011
- L. D'Ambrosio, E. Mitidieri, preprint 2011

⇒ If **(L)** doesn't hold, global continuation may fail.

A connected biharmonic Gelfand-type equation

Corollary

Let B_R be the ball in \mathbb{R}^4 with radius $0 < R < +\infty$ and center the origin. Then, any radial solution to

$$\Delta^2 u + e^u = \frac{1}{|x|^4} \quad \text{in } B_R \setminus \{0\}$$

admits a radial extension to $\mathbb{R}^4 \setminus \{0\}$.

In particular, the equation subject to the boundary condition

$$\lim_{|x| \rightarrow R} u(x) = \infty,$$

admits **no** radial solution.

Explicit solution:

$$u(x) = -4 \log |x|$$

Qualitative behavior when $k \geq 0$

$$w''''(s) + kw''(s) + f(w(s)) = 0 \quad s \in \mathbb{R}$$

Theorem

Let $k \geq 0$ and f satisfy **(S)**. If w is a global solution, then

$$\liminf_{s \rightarrow +\infty} w(s) \leq 0 \leq \limsup_{s \rightarrow +\infty} w(s),$$

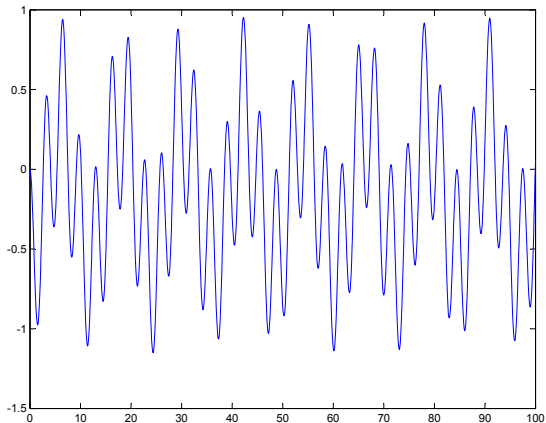
so that if $\lim_{s \rightarrow +\infty} w(s)$ exists then

$$\lim_{s \rightarrow +\infty} w(s) = 0.$$

Furthermore, if $w \not\equiv 0$ then $w(s)$ **changes sign infinitely many times** as $s \rightarrow +\infty$. Similar statements hold for $s \rightarrow -\infty$.

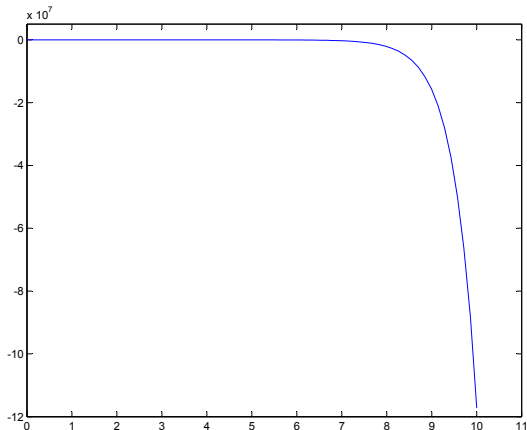
$$w''''(s) + 4w''(s) + e^{w(s)} - 1 = 0 \quad s \in \mathbb{R}$$

$$w(0) = 0 \quad w'(0) = 0 \quad w''(0) = -2 \quad w'''(0) = 0$$



$$w''''(s) - 4w''(s) + e^{w(s)} - 1 = 0 \quad s \in \mathbb{R}$$

$$w(0) = 0 \quad w'(0) = 0 \quad w''(0) = -2 \quad w'''(0) = 0$$



Qualitative behavior when $k < 0$

$$w''''(s) + kw''(s) + f(w(s)) = 0 \quad s \in \mathbb{R}$$

Theorem

Let $k < 0$ and assume that f satisfies **(S)** and

$$\inf_{t \in \mathbb{R}} f(t) = -M > -\infty, \quad (1)$$

then there exists a global solution w which is **eventually negative**, decreasing, and concave as $s \rightarrow +\infty$ and

$$\lim_{s \rightarrow +\infty} w(s) = -\infty.$$

Similar statement if

$$\sup_{t \in \mathbb{R}} f(t) = M < +\infty.$$

Sketch of the proof

Let $k < 0$ and $\inf_{t \in \mathbb{R}} f(t) = -M > -\infty$. Consider a solution to

$$w''''(s) + kw''(s) + f(w(s)) = 0 \quad s \in \mathbb{R}$$

satisfying

$$w(0) = 0, \quad w'(0) = 0, \quad w''(0) < -\frac{M}{|k|} < 0, \quad w'''(0) = 0.$$

$$\implies w''''(0) = -kw''(0) < -M.$$

Set

$$\bar{s} := \sup\{s > 0 : w''''(\sigma) < 0 \text{ for all } \sigma \in (0, s)\} \in (0, +\infty].$$

Claim: $\bar{s} = +\infty$.

Sketch of the proof

$$\bar{s} := \sup\{s > 0 : w''''(\sigma) < 0 \text{ for all } \sigma \in (0, s)\} \in (0, +\infty].$$

$$\text{Were } \bar{s} < +\infty \implies w''''(\bar{s}) = 0.$$

$$\implies w''''(s) \searrow \text{ in } (0, \bar{s}], w''''(0) = 0 \implies w''''(s) < 0 \text{ in } (0, \bar{s}]$$

$$\implies w'''(s) \searrow \text{ in } (0, \bar{s}], w'''(0) < 0 \implies w'''(s) < 0 \text{ in } (0, \bar{s}]$$

$$\implies w''(s) \searrow \text{ in } (0, \bar{s}], w''(0) = 0 \implies w''(s) < 0 \text{ in } (0, \bar{s}]$$

$$\implies w'(s) \searrow \text{ in } (0, \bar{s}], w'(0) = 0 \implies w'(s) < 0 \text{ in } (0, \bar{s}]$$

$$w''''(\bar{s}) = |k|w''(\bar{s}) - f(w(\bar{s})) \leq |k|w''(0) - M < 0,$$

a contradiction $\implies \bar{s} = +\infty$.

Qualitative behavior when $k < 0$

$$w''''(s) + kw''(s) + f(w(s)) = 0 \quad s \in \mathbb{R}$$

Theorem

Let $k < 0$ and f satisfy **(S)**, $\sup_{t \in \mathbb{R}} f(t) = M < +\infty$ and

$$\lim_{t \rightarrow -\infty} \frac{f(t)}{t} = +\infty.$$

Then any solution w is global and

$$\sup_{s \in \mathbb{R}} w(s) = +\infty \quad \text{and} \quad \inf_{s \in \mathbb{R}} w(s) > -\infty.$$

Energy functions

$$F(w) := \int_0^w f(\tau) d\tau,$$

$$E(s) := \frac{1}{2} w''(s)^2 - \frac{k}{2} w'(s)^2 - F(w(s)) \quad s \in \mathbb{R}.$$

If w is a solution and $w'(s_1) = w'(s_2) = 0 \implies E(s_1) = E(s_2)$.

$$\mathcal{E}(s) := E(s) - w'(s)w'''(s) \quad s \in \mathbb{R}.$$

If w is a solution $\implies \mathcal{E}'(s) = 0 \implies \mathcal{E}(s) = C$, for some $C \in \mathbb{R}$.

Energy functions

$$H(s) := w'(s)w''(s) - w(s)w'''(s) - kw(s)w'(s)$$

and its antiderivative

$$G(s) := w'(s)^2 - w(s)w''(s) - \frac{k}{2}w(s)^2 \quad s \in \mathbb{R}.$$

w solution, $k \leq 0 \implies H'(s) = w''(s)^2 - kw'(s)^2 + w(s)f(w(s)) \geq 0$

$\implies H$ is nondecreasing and G is convex.

An application

Proposition

Let $k \leq 0$ and f satisfy **(S)**. Then, the equation

$$w''''(s) + kw''(s) + f(w(s)) = 0 \quad s \in \mathbb{R}$$

has no nontrivial bounded solutions. In particular, the equation has **no nontrivial homoclinic solutions**.

- L.A. Peletier, W.C. Troy, Birkhäuser Boston Inc. (2001)

A simpler proof

w homoclinic $\Leftrightarrow \lim_{s \rightarrow \pm\infty} w(s) = 0 \Rightarrow \lim_{s \rightarrow \pm\infty} w^i(s) = 0$ for $i = 1, \dots, 4$

Recall the energy function

$$H(s) := w'(s)w''(s) - w(s)w''''(s) - kw(s)w'(s)$$

$H'(s) \geq 0$ for $k \leq 0 \Rightarrow \lim_{s \rightarrow \pm\infty} H(s) = 0 \Rightarrow H = H' \equiv 0 \Rightarrow w \equiv 0$

Dynamical system

Put

$$Y(s) = (y_1(s), y_2(s), y_3(s), y_4(s)) = (w(s), w'(s), w''(s), w'''(s))$$

the equation

$$w''''(s) + kw''(s) + f(w(s)) = 0 \quad s \in \mathbb{R}$$

may be rewritten as

$$\begin{cases} y_1' = y_2 \\ y_2' = y_3 \\ y_3' = y_4 \\ y_4' = -ky_3 - f(y_1). \end{cases}$$

(S) \Rightarrow unique stationary point $O = (0, 0, 0, 0)$ corresponding to $w \equiv 0$

Proposition

Assume **(S)** and f differentiable near the origin: $f'(0) = 1$.

- (i) If $k < -2$, O has a 2-dimensional **stable** manifold and a 2-dimensional **unstable** manifold, both not oscillating near O ;
- (ii) if $k = -2$, O has a 2-dimensional **stable** manifold and a 2-dimensional **unstable** manifold;
- (iii) if $-2 < k < 2$, O has a 2-dimensional **stable** manifold and a 2-dimensional **unstable** manifold, both having locally the form of a spiral near O ;
- (iv) if $k = 2$, the linearized problem at O has 2 (opposite) double **purely imaginary** eigenvalues;
- (v) if $k > 2$, the linearized problem at O has 4 **purely imaginary** eigenvalues.

Theorem

Assume that f satisfies **(S)**.

- (i) If $k \leq -2$ and f satisfies one of the following

$$f(t) \geq t \quad \text{near } t = 0 \quad \text{or} \quad f(t) \leq t \quad \text{near } t = 0,$$

any global solution : $\lim_{s \rightarrow +\infty} w(s) = 0$ is of **one sign** as $s \rightarrow +\infty$.

Similar statement with $+\infty$ replaced by $-\infty$.

- (ii) If $-2 < k < 0$, $f'(0) = 1$ and

$$\liminf_{|t| \rightarrow +\infty} \frac{f(t)}{t} > k^2,$$

any global solution **changes sign infinitely many times** both as $s \rightarrow \pm\infty$.

Sketch of the proof of (i)

$$\lim_{s \rightarrow +\infty} w(s) = 0 \quad \Rightarrow \quad \lim_{s \rightarrow +\infty} (w(s), w'(s), w''(s), w'''(s)) = (0, 0, 0, 0)$$

$k \leq -2 \Rightarrow 4$ real eigenvalues $\pm\lambda, \pm\mu: \lambda \geq \mu > 0$.

$$\Rightarrow (\partial_s - \lambda)(\partial_s - \mu)(\partial_s + \lambda)(\partial_s + \mu)w(s) = w(s) - f(w(s))$$

$$f(t) \geq t \text{ near } t = 0 \Rightarrow (e^{-\lambda s}(\partial_s - \mu)(\partial_s + \lambda)(\partial_s + \mu)w(s))' \leq 0 \text{ s large}$$

$$e^{-\lambda s}(\partial_s - \mu)(\partial_s + \lambda)(\partial_s + \mu)w(s) \rightarrow 0 \text{ as } s \rightarrow +\infty$$

$$\Rightarrow e^{-\mu s}((\partial_s + \lambda)(\partial_s + \mu)w(s)) \nearrow 0$$

$$(\partial_s + \lambda)(\partial_s + \mu)w(s) \leq 0 \text{ for s large}$$

$$\Rightarrow s \mapsto e^{\lambda s}(w'(s) + \mu w(s)) \text{ decreasing.}$$

Sketch of the proof of (i)

$\Rightarrow s \mapsto e^{\lambda s}(w'(s) + \mu w(s))$ decreasing.

Case 1 $\exists s_0: w'(s_0) + \mu w(s_0) < 0$

$\Rightarrow w'(s) + \mu w(s) < 0$ for all $s \geq s_0 \Rightarrow e^{\mu s} w(s) \searrow \ell \in \mathbb{R}$ as $s \rightarrow +\infty$

If $\ell > 0$ or $\ell < 0 \Rightarrow w$ is eventually of one sign.

If $\ell = 0 \Rightarrow e^{\mu s} w(s) \searrow 0 \Rightarrow w$ is eventually positive.

Case 2 $w' + \mu w \geq 0$ for every $s \in R$

$\Rightarrow e^{\mu s} w(s) \nearrow \ell$ as $s \rightarrow +\infty \dots$

Homoclinics

$$w \text{ homoclinic} \iff \lim_{s \rightarrow \pm\infty} w(s) = 0$$

Proposition

Let $k \leq 0$ and f satisfy **(S)**. Then, the equation

$$w''''(s) + kw''(s) + f(w(s)) = 0 \quad s \in \mathbb{R}$$

has no nontrivial bounded solutions. In particular, the equation has **no nontrivial homoclinic** solutions.

- L.A. Peletier, W.C. Troy, Birkhäuser Boston Inc. (2001)

Homoclinics

Theorem

Let $k > 0$.

(i) $k \leq 2$ and $\frac{f(t)}{t} \geq 1$ for all $t \neq 0$

\implies **no homoclinics**

(ii) f satisfies **(S)** and $f'(0) = 1$, $\{s_m\}_{m \geq 1}$ increasing sequence of zeroes of an homoclinic (as $s \rightarrow +\infty$)

$$\implies \liminf_{m \rightarrow +\infty} (s_{m+1} - s_m) \geq \frac{\pi \sqrt{k + \sqrt{k^2 + 12}}}{\sqrt{6}}.$$

Similar statement as $s \rightarrow -\infty$.

(iii) $k < 2$, f satisfies **(S)** and $f'(0) = 1$, w is a homoclinic

$\implies w \in H^2(\mathbb{R})$

Sketch of the proof of (i)

w homoclinic, $k > 0 \Rightarrow w$ changes sign infinitely many times.
Let $s_1 < s_2$ any two of its roots

$$\begin{aligned} 2 \int_{s_1}^{s_2} w'(s)^2 ds &= -2 \int_{s_1}^{s_2} w(s)w''(s) ds \\ &= \int_{s_1}^{s_2} [w(s)^2 + w''(s)^2] ds - \int_{s_1}^{s_2} [w(s) + w''(s)]^2 ds \\ &\leq \int_{s_1}^{s_2} [w(s)^2 + w''(s)^2] ds. \end{aligned}$$

\Rightarrow there exist two roots $\bar{s}_1 < \bar{s}_2$ such that $<$ holds (otherwise $w''(s) + w(s) = 0$ for all $s \in \mathbb{R}$, contradicting w homoclinic).

If $s_1 \leq \bar{s}_1 < \bar{s}_2 \leq s_2 \Rightarrow \int_{s_1}^{s_2} [w(s)^2 + w''(s)^2] ds > 2 \int_{s_1}^{s_2} w'(s)^2 ds$.

Sketch of the proof of (i)

$$H(s) := w'(s)w''(s) - w(s)w'''(s) - kw(s)w'(s)$$

Since $\frac{f(t)}{t} \geq 1$

$$\begin{aligned} H(s_2) - H(s_1) &= \int_{s_1}^{s_2} [w''(s)^2 - kw'(s)^2 + w(s)f(w(s))] ds \\ &\geq \int_{s_1}^{s_2} [w''(s)^2 - kw'(s)^2 + w(s)^2] ds > (2-k) \int_{s_1}^{s_2} w'(s)^2 ds \geq 0, \end{aligned}$$

whenever $s_1 \leq \bar{s}_1 < \bar{s}_2 \leq s_2$

w homoclinic $\implies H(s) \rightarrow 0$ as $s \rightarrow \pm\infty$, a contradiction.