Generalized Bernoulli problems

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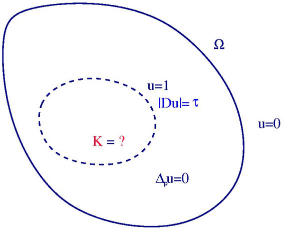
The Classical Bernoulli problem

Let Ω be a bounded domain of $\mathbb{R}^N$ and $\tau > 0$.
We look for a function u_τ and a domain $\mathrm{K} \subseteq \Omega$, satisfying the problem:

$$[Pb] \begin{cases} \Delta_{\mathrm{p}} u(x) = 0 & \text{in } \Omega \setminus \mathrm{K}, \\ u = 0 & \text{on } \partial\Omega, \\ u = 1, \ |Du| = \tau & \text{on } \partial \mathrm{K}. \end{cases}$$



$$\Delta_{\mathrm{p}} u = \operatorname{div}(|Du|^{\mathrm{p}-2} Du), \qquad 1 < \mathrm{p} < \infty.$$

Classical problem: existence of a solution

Let Ω be an open bounded convex domain of $\mathbb{R}^N$.
Then $\exists\ \Lambda(\Omega)$ = **Bernoulli constant** of Ω such that:

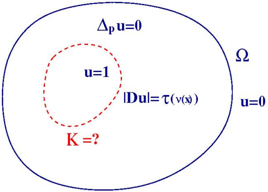
1. There is a solution to $[Pb]$ if and only if $\tau \geq \Lambda(\Omega)$;
2. If a solution (u_τ, K) exists, then K is convex (and u_τ is quasi-concave).
3. For every $p > 1$ there exists exactly one solution for $\tau = \Lambda(\Omega)$.

[A. Henrot, H. Shahgholian '00]
[P. Cardaliaguet, R. Tahraoui '02]
[CB, P. Salani '09]

Generalized Bernoulli problem

Let Ω be a bounded domain of $\mathbb{R}^N$ and let $g : S^{N-1} \to \mathbb{R}^+$ be a continuous function s.t. $0 < c_0 \leq g(\cdot) \leq c_1$.
We look for a function u and a domain $K \subseteq \Omega$, satisfying the problem:

$$[\text{PBg}] \begin{cases} \Delta_p u(x) = 0 & \text{in } \Omega \setminus \overline{K}, \\ u = 0 & \text{su } \partial\Omega, \\ u = 1, \ |Du(x)| = g(\nu(x)) & \text{su } \partial K \,, \end{cases}$$



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What can we say about existence and convexity of solutions?.

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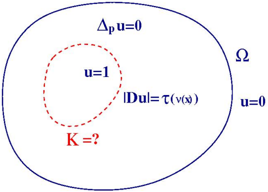

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What can we say about existence and convexity of solutions?.

Generalized problem: subsolutions

Def. Let $\Omega \subseteq \mathbb{R}^N$ be a convex domain. $\mathscr{F}^-(\Omega, g)$ is the family of **subsolutions** corresponding to Ω and $g : S^{N-1} \to \mathbb{R}^+$,
that is the class of $(u, K) \in \mathrm{Lip}(\overline{\Omega}) \times \mathscr{K}^N$ such that

$$\begin{cases} \Delta_p u(x) \geq 0 & \text{in } \Omega \setminus \overline{K}, \\ u = 0 & \text{on } \partial\Omega, \\ u = 1, \ |Du(x)| \leq g(\nu(x)) & \text{on } \partial K\,. \end{cases}$$

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Generalized problem: main results

Theorem. Let $\Omega \subseteq \mathbb{R}^N$ be a bounded convex C^1 domain, and $g : S^{N-1} \to \mathbb{R}$ be a continuous function s.t. $0 < c \leq g \leq C$.
If $\mathscr{F}^-(\Omega, g(\nu))$ is non empty, then there exists a C^1 convex domain K with $\overline{K} \subseteq \Omega$ such that the p-capacitary potential u of $\Omega \setminus \overline{K}$ is a classical solution to the Bernoulli problem [PBg].

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Preliminary result

Lemma. Let $(u_0, K_0), (u_1, K_1) \in \mathscr{F}^-(\Omega, g)$.
Then $(v, K^*) \in \mathscr{F}^-(\Omega, g)$,
where $K^* = \text{conv}(K_0 \cup K_1)$, and v is the p-potential of $\Omega \setminus \overline{K}^*$.

Proof. We only need to prove $|Dv| \leq g(\nu)$ on ∂K^*.
$u := \max\{u_0, u_1\}$ and u^* its quasi-concave envelope.

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by the viscosity comparison principle: $|Dv| \leq |Du^*|$ on ∂K^*.
$\rightsquigarrow$ compare $|Du^*(y)|$ with $g(\nu_{K^*}(y))$: $|Du^*(y)| \leq g(\nu_{K^*}(y))$
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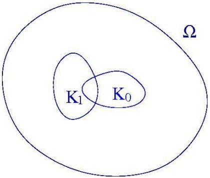

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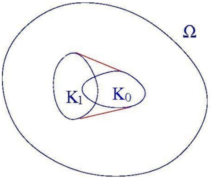

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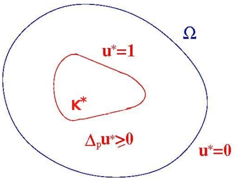

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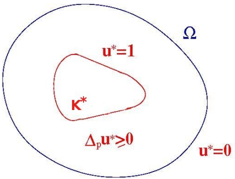

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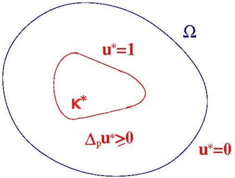

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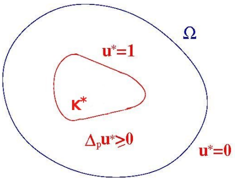

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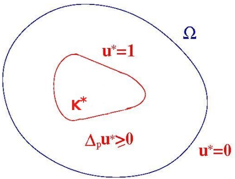

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Proof of the main result

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$g : S^{N-1} \to \mathbb{R}$ a continuous function s.t. $0 < c \leq g \leq C$.

$\mathscr{F}^-(\Omega, g(\nu)) \neq \emptyset \implies \exists K$ convex, $\overline{K} \subset \Omega$ s.t.
the p-potential u of $\Omega \setminus \overline{K}$ is a classical solution to [PBg] in $\Omega \setminus \overline{K}$.

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Consider $u = \sup\{v \in \mathscr{F}^-(\Omega, g)\}$ $\rightsquigarrow$ **AIM:** u is a solution.

let u_n be a max. sequence, $u_n \in \mathscr{F}^-(\Omega, g)$
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with $|Du_n(x)| \leq g(\nu_{K_n}(x))$ on ∂K_n $\forall n$.

Let $K = \lim K_n \rightsquigarrow K$ convex and u is the p-potential of $\Omega \setminus \overline{K}$,
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We need to show that in fact $|Du(x)| = g(\nu_K(x))$!.

Assume $\exists y \in \partial K$ s.t. $|Du(y)| < g(\nu(y))$
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Let $K = \lim K_n$ $\rightsquigarrow$ K convex and u is the p-potential of $\Omega \setminus \overline{K}$, with $|Du(x)| \leq g(\nu_K(x))$ on ∂K.

We need to show that in fact $|Du(x)| = g(\nu_K(x))$!.

Assume $\exists y \in \partial K$ s.t. $|Du(y)| < g(\nu(y))$
$\rightsquigarrow$ we construct a perturbation w of u s.t. $w \in \mathscr{F}^-(\Omega, g)$ and $w = u$ in $\Omega \setminus B(y, \varepsilon)$, $w > u$ at some points.

$\rightsquigarrow$ contradiction!

Proof of the main result

Consider $u = \sup\{v \in \mathscr{F}^-(\Omega, g)\}$ $\rightsquigarrow$ **AIM:** u is a solution.

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What can we say about the existence of a subsolution?

- no characterizations of functions g s.t. $\mathscr{F}^-(\Omega, g) \neq \emptyset$ are known;

however...

- if $\min_{\nu\in S^{N-1}} g(\nu) \geq \Lambda(\Omega)$, then a solution exists,
  indeed: $\emptyset \neq \mathscr{F}^-(\Omega, \Lambda(\Omega)) \subseteq \mathscr{F}^-(\Omega, g)$;
- if $M = \max_{\nu\in S^{N-1}} g(\nu) < \Lambda(\Omega)$, then there is no solution,
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