On the Hong-Krahn-Szego inequality

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References

Some of the results here presented are contained in

L. B., A. Pratelli, *Sharp stability of some spectral inequalities*, submitted

The inequality we are going to discuss has been discovered (at least) 3 times...

- **E. Krahn**, *Acta Comm. Univ. Dorpat.* **A9** (1926)
- **I. Hong**, *Kōdai Math. Sem. Rep.* **6** (1954)
- **G. Pólya**, *Math. Zeitschr.* **63** (1955)

Outline

Eigenvalues of the Laplacian

$$\Omega \subset \mathbb{R}^N \text{ open set with } |\Omega| < +\infty$$

The boundary value problem

$$\begin{cases} -\Delta u = \lambda\, u, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases}$$

have solutions $\neq 0$ only for a **discrete set** of **real and positive**

$$0 < \lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \dots$$

- $\lambda_j(\Omega)$ are called **eigenvalues of the Dirichlet-Laplacian**
- corresponding solutions u_j are the **eigenfunctions** and, renormalized, give an orthonormal basis of $L^2(\Omega)$
- if for $\lambda > 0$ this problem has m linearly independent solutions $\Longrightarrow \lambda = \lambda_{j+1}(\Omega) = \cdots = \lambda_{j+m}(\Omega)$ has **multiplicity** m

The first two eigenspaces: properties

$$\mathcal{R}_\Omega(u) = \frac{\|\nabla u\|^2_{L^2(\Omega)}}{\|u\|^2_{L^2(\Omega)}}$$ **Rayleigh quotient**

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2. Ω connected: $\lambda_1(\Omega)$ is **simple** and u_1 has **constant sign**
3. $\Omega = \Omega_1 \cup \cdots \cup \Omega_i \ldots$ disconnected: $\lambda_1(\Omega) = \min_i \lambda_1(\Omega_i)$

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The second

1. $\lambda_2(\Omega) = \min\{\mathcal{R}_\Omega(u) \ : \ u \in H^1_0(\Omega) \quad \text{and} \quad \langle u, u_1 \rangle = 0\}$
2. Ω connected: u_2 has to **change sign** and

   $\Omega_+ = \{u_2 > 0\} \qquad \Omega_- = \{u_2 < 0\}$ **nodal domains**
3. Ω disconnected: **gather** and **order** the λ_1 and λ_2 of connected components, then choose the 2nd

Goal of the talk: the second eigenvalue λ_2

We consider the **spectral optimization problem**

- $\min\{\lambda_2(\Omega) : |\Omega| = c\}$ *equivalently* $\min |\Omega|^{2/N}\lambda_2(\Omega)$

We will see that there exists a unique class of optimal sets, we aim to prove **stability**, i.e.

"*almost optimal sets are near to the space of optimizers*"

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Quantify? Let $\mathcal{O}$ the "manifold" of optimizers, λ_2^* the minimum

$$|\Omega|^{2/N}\lambda_2(\Omega) - \lambda_2^* \geq \varphi(d(\Omega, \mathcal{O}))$$

with φ positive increasing function, $\varphi(0) = 0$ and d a "distance"

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Sharp? for some Ω_ε slight pertubations of an optimizer

$$|\Omega_\varepsilon|^{2/N}\lambda_2(\Omega_\varepsilon) - \lambda_2^* \simeq \varphi(d(\Omega_\varepsilon, \mathcal{O})) \quad \text{as} \quad \varepsilon \to 0$$

One step back: the first eigenvalue λ_1

"*Among sets of given volume, the* **ball** *is the only set minimizing* λ_1"

Theorem [Faber-Krahn]

$$|\Omega|^{2/N}\lambda_1(\Omega) \geq |B|^{2/N}\lambda_1(B) =: \lambda_1^*$$

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(Sharp?) Quantitative version [Fusco-Maggi-Pratelli]

$$|\Omega|^{2/N}\lambda_1(\Omega) - \lambda_1^* \geq c_N\,\mathcal{A}(\Omega)^4$$

where $\mathcal{A}$ is the L^1 distance from optimizers, i.e.

$$\mathcal{A}(\Omega) = \min\left\{\frac{\|1_\Omega - 1_B\|_{L^1}}{|\Omega|}\ :\ B \text{ ball}, |B| = |\Omega|\right\}\ (\textit{asymmetry})$$

The **conjectured sharp exponent** for $\mathcal{A}$ is 2

1. Introduction and goal of the talk
   - Warm-up
   - Goal

2. Second eigenvalue of the Laplacian under volume constraint
   - The inequality...
   - ...in quantitative form
   - Sharpness?

Hong-Krahn-Szego[1] inequality

"*Among sets of given volume, the* **disjoint union of equal balls** *is the only set minimizing* λ_2"

Theorem [Hong-Krahn-Szego]

$$|\Omega|^{2/N}\lambda_2(\Omega) \geq 2^{2/N}|B|^{2/N}\lambda_1(B) =: \lambda_2^*$$

where B is a ball and with equality **if and only if** Ω is a disjoint union of equal balls

Remark

For $\Theta_2 = B_1 \cup B_2$ with $|B_1| = |B_2|$ and $B_1 \cap B_2 = \emptyset$, we have

$$|\Theta_2|^{2/N}\lambda_2(\Theta_2) = 2^{2/N}|B_i|^{2/N}\lambda_1(B_i)$$

[1] This is Peter Szego, son of Gabor Szegő

Proof

1. given Ω, we can find $\Omega_+, \Omega_- \subset \Omega$ **disjoint** such that

$$\lambda_2(\Omega) = \max\{\lambda_1(\Omega_+), \lambda_1(\Omega_-)\}$$

Who are these sets Ω_+ and Ω_-?

nodal domains of u_2 or connected components of Ω

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2. **use Faber-Krahn inequality** to say

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3. **hence** optimizer of λ_2 is a disjoint union of balls $B_1 \cup B_2$
4. use the homogeneity of λ_2 to conclude that $|B_1| = |B_2|$

HKS inequality in quantitative form

We introduce the **deficit**

$$HKS(\Omega) := |\Omega|^{2/N}\lambda_2(\Omega) - 2^{2/N}|B|^{2/N}\lambda_1(B)$$

Quantitative HKS inequality [B.-Pratelli]

$$HKS(\Omega) \geq c_N\, \mathcal{A}_2(\Omega)^{2(N+1)}$$

where $\mathcal{A}_2$ is the L^1 distance from optimizers, i.e.

$$\mathcal{A}_2(\Omega) := \inf\left\{ \frac{\|1_\Omega - 1_{B_1\cup B_2}\|_{L^1}}{|\Omega|} \ :\ |B_1 \cap B_2| = 0 \text{ with } |B_i| = \frac{|\Omega|}{2} \right\}$$

Steps of the proof

Reminder: we know that $\lambda_2(\Omega) = \max\{\lambda_1(\Omega_+), \lambda_1(\Omega_-)\}$

1. first goal use the **quantitative Faber-Krahn** so to obtain

$$HKS(\Omega) \gtrsim \mathcal{A}(\Omega_+)^4 + \left|\frac{1}{2} - \frac{|\Omega_+|}{|\Omega|}\right| + \mathcal{A}(\Omega_-)^4 + \left|\frac{1}{2} - \frac{|\Omega_-|}{|\Omega|}\right|$$

which means

"In terms of the deficit, I can control how Ω_+ and Ω_- are far from being two balls having measure $|\Omega|/2$"

2. second goal pass from this quantity to $\mathcal{A}_2$

$$\mathcal{A}_2(\Omega)^{(N+1)/2} \lesssim \mathcal{A}(\Omega_+) + \left|\frac{1}{2} - \frac{|\Omega_+|}{|\Omega|}\right| + \mathcal{A}(\Omega_-) + \left|\frac{1}{2} - \frac{|\Omega_-|}{|\Omega|}\right|$$

Optimality of the exponent for $\mathcal{A}_2$?

Alert! The **exponent** obtained for $\mathcal{A}_2$ is **dimension-dependent**!

Reasonable suspect: Maybe the exponent is not sharp...

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Discussion

The **two steps** of the proof **are sharp**, in the following sense:

- the first step does not require to know the sharp exponent for the Faber-Krahn inequality, indeed the same proof provide

$$HKS(\Omega) \geq c_N\, \mathcal{A}_2(\Omega)^{\kappa_1 \cdot (N+1)/2}$$

with $\kappa_1 =$ sharp exponent (2?) for the Faber-Krahn

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- for the second step, take $\Omega_\varepsilon = \Omega_\varepsilon^+ \cup \Omega_\varepsilon^-$ union of **two equal balls slightly overlapping**, then

$$\mathcal{A}(\Omega_\varepsilon^+) + \mathcal{A}(\Omega_\varepsilon^-) \simeq \mathcal{A}_2(\Omega_\varepsilon)^{(N+1)/2}$$

Conclusions and open questions

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3. We believe **this conjecture** to be **false**: indeed
   - to be sharp in the quantitative Faber-Krahn you have to be smooth (Barchiesi-B.-Fusco-Pratelli, work in progress)...
   - ...while Ω_+ and Ω_- are nodal sets of u_2 and in general they develop singularities (ex. corners) where $\{u_2 = 0\}$ touches the boundary!

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4. to (at least!) conjecture the sharp exponent, you need **precise asymptotics of λ_2** for the two slightly overlapping balls

Further readings

On the quantitative Faber-Krahn

- **A. Melas**, *J. Differential Geom.* **36** (1992)
- **W. Hansen, N. Nadirashvili**, *Potential Anal.* **3** (1994)
- **T. Bhattacharya**, *Electron. J. Diff. Eq.* **35** (2001)
- **N. Fusco, F. Maggi, A. Pratelli**, *Ann. Sc. Norm. Sup.* **8** (2009)