Sharp Hardy Inequalities in the Half Space with Trace Remainder Term

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joint work with
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Geometric Properties for Parabolic and Elliptic PDE's
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The Framework

HARDY'S INEQUALITY Let $n \geq 3$ $\mathbb{R}^n_+ = \{(x,t) \in \mathbb{R}^n : x \in \mathbb{R}^{n-1}, t > 0\}$ and $u \in W^{1,2}(\mathbb{R}^n_+)$

$$\frac{(n-2)^2}{4} \int_{\mathbb{R}^n} \frac{u(x,t)^2}{|x|^2 + t^2} dxdt \leq \int_{\mathbb{R}^n} |\nabla u|^2(x,t) dxdt$$

KATO'S INEQUALITY. Let $n \geq 3$, $\mathbb{R}^n_+ = \{(x,t) \in \mathbb{R}^n : x \in \mathbb{R}^{n-1}, t > 0\}$ and $u \in W^{1,2}(\mathbb{R}^n_+)$

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$\Gamma(s) = \int_0^\infty t^{s-1} \mathrm{e}^{-t} dt$

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No existence of minimizers ⇒ REMAINDER TERMS

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No existence of minimizers $\Rightarrow$ Remainder Terms

Maz'ya (1972), Brezis, Vazquez (1997)

Brezis, Marcus (1998) Vazquez, Zuazua (2000)

Barbatis, Filippas, Tertikas (2003, ...) Bartsch, Weth, Willem (2003)

Detalla, Horiuci, Ando (2004) Gazzola, Gruneau, Mitidieri (2004)

Tidblom (2005) Filippas, Maz'ya, Tertikas (2006,....)

Maz'ya, Shaposhnikova (2008,....)

Gossoub, Moradifam (2008,...) Cianchi, F. (2008, ...)

Alvino, Volpicelli, Volzone (2010) ....

Remainder terms in Hardy's Inequality

HARDY'S INEQUALITY WITH REMAINDER TERM Let $n \geq 3$ and $u \in W_0^{1,2}(\Omega)$,

$$\text{REMAINDER} + \frac{(n-2)^2}{4} \int_\Omega \frac{u(x,t)^2}{|x|^2 + t^2} dxdt \leq \int_\Omega |\nabla u|^2(x,t) dxdt$$

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What About $u_{|\partial \mathbb{R}^n_+}$

The Kato's Inequality

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Trace Sobolev Inequality. Let $n \geq 3$ and $u \in W^{1,2}(\mathbb{R}^n_+)$

$$\left(\frac{n-2}{2}\right)^{1/2}\omega_{n-1}^{\frac{1}{2(n-1)}}\parallel u\parallel_{L^{\frac{2(n-1)}{n-2}}(\partial\mathbb{R}^n_+)} \leq \parallel \nabla u\parallel_{L^2(\mathbb{R}^n_+)}$$

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Existence of minimizers — Lions 1985

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Existence of minimizers		Lions	1985
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Optimal constant	Herbst	1997
	Davila, Dupaigne, Montenegro	2008
Application to PDEs	Ishige, Ishiwata	2010

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Davila, Dupaigne and Montenegro Result (2008)

Kato's Inequality with Remainder Term Let $n \geq 3$, $1 \leq q < 2$ and $u \in C_0^\infty(\overline{\mathbb{R}^n_+} \cap B)$,

$$C\|u\|^2_{W^{1,q}(\mathbb{R}^n_+\cap B)} + 2\frac{\Gamma^2\left(\frac{n}{4}\right)}{\Gamma^2\left(\frac{n-2}{4}\right)}\int_{\partial\mathbb{R}^n_+\cap B}\frac{u^2}{|x|}d\mathcal{H}^{n-1} \leq \int_{\mathbb{R}^n_+\cap B}|\nabla u|^2(x,t)\,dxdt$$

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Brezis, Vazquez (1997), Vazquez, Zuazua (2000),
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If $u \in W^{1,2}(\mathbb{R}^n_+) \Rightarrow$ No Remainder Terms

Aim

Can we consider a Hardy's inequality without loosing informations on $u_{|\partial\mathbb{R}^n_+}$?

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Sharp Hardy's Inequality

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$$\frac{(\beta-2)^2}{4}\int_{\mathbb{R}^n_+}\frac{u(x,t)^2}{|x|^2+t^2}dxdt < \int_{\mathbb{R}^n_+}|\nabla u|^2(x,t)dxdt$$

$\beta = n$ Sharp Hardy's Inequality

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$\beta = n$ Sharp Hardy's Inequality

$2 < \beta < n$

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Can we consider a Hardy's inequality without loosing informations on $u_{|\partial \mathbb{R}^n_+}$?

$$H(n,\beta) \quad \int_{\partial \mathbb{R}^n_+} \frac{u^2}{|x|} d\mathcal{H}^{n-1} + \frac{(\beta-2)^2}{4} \int_{\mathbb{R}^n_+} \frac{u(x,t)^2}{|x|^2+t^2} dxdt < \int_{\mathbb{R}^n_+} |\nabla u|^2(x,t)\, dxdt$$

$\beta = n$ Sharp Hardy's Inequality

$2 < \beta < n$ Extra *Kato's* Term

Aim

CAN WE CONSIDER A HARDY'S INEQUALITY WITHOUT LOOSING INFORMATIONS ON $u_{|\partial \mathbb{R}^n_+}$?

$$H(n,\beta) \quad \int_{\partial \mathbb{R}^n_+} \frac{u^2}{|x|} d\mathcal{H}^{n-1} + \frac{(\beta-2)^2}{4} \int_{\mathbb{R}^n_+} \frac{u(x,t)^2}{|x|^2+t^2} dxdt < \int_{\mathbb{R}^n_+} |\nabla u|^2(x,t) dxdt$$

$\beta = n$ SHARP HARDY'S INEQUALITY

$2 < \beta < n$ EXTRA *Kato's* TERM

$\beta = 2$

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$$2\frac{\Gamma^2\left(\frac{n}{4}\right)}{\Gamma^2\left(\frac{n-2}{4}\right)}\int_{\partial\mathbb{R}^n_+}\frac{u^2}{|x|}d\mathcal{H}^{n-1}$$

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$\beta = n$ — Sharp Hardy's Inequality

$2 < \beta < n$ — Extra *Kato's* Term

$\beta = 2$ — Sharp Kato's Inequality

Main Theorem [Alvino,F.,Volpicelli]

Let $n \geq 3$ and let $u \in W^{1,2}(\mathbb{R}^n_+)$. Then for any $2 \leq \beta < n$, there exists a positive constant $H(n,\beta)$ such that

$$H(n,\beta)\int_{\partial\mathbb{R}^n_+}\frac{u^2}{|x|}d\mathcal{H}^{n-1}+\frac{\beta-2}{4}\int_{\mathbb{R}^n_+}\frac{u^2(x,t)}{|x|^2+t^2}dxdt \leq \int_{\mathbb{R}^n_+}|\nabla u|^2(x,t)dxdt$$

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The optimal constant $H(n,\beta)$ is given by

$$H(n,\beta)=2\frac{\Gamma\left(\frac{n+\beta}{4}-\frac{1}{2}\right)\Gamma\left(\frac{n-\beta}{4}+\frac{1}{2}\right)}{\Gamma\left(\frac{n+\beta}{4}-1\right)\Gamma\left(\frac{n-\beta}{4}\right)}$$

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$\beta = 2$ Alternative proof of the SHARP KATO'S INEQUALITY

Sketch of the Proof

- $J(u) = \dfrac{\int_{\mathbb{R}^n_+} \left(|\nabla u|^2(x,t) - \frac{(\beta-2)^2}{4} \frac{u^2(x,t)}{|x|^2+t^2} \right) dxdt}{\int_{\partial \mathbb{R}^n_+} \frac{u^2}{|x|} d\mathcal{H}^{n-1}}$

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-
$$\begin{cases} \Delta\varphi + \frac{(\beta-2)^2}{4} \frac{\varphi}{|x|^2+t^2} = 0 & (x,t) \in \mathbb{R}^n_+ \\ \varphi(x,0) = |x|^{-\frac{n}{2}+1} \end{cases} \tag{1}$$

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- $$\begin{cases} \Delta\varphi + \frac{(\beta-2)^2}{4} \frac{\varphi}{|x|^2+t^2} = 0 & (x,t) \in \mathbb{R}^n_+ \\ \varphi(x,0) = |x|^{-\frac{n}{2}+1} \end{cases} \tag{1}$$

$$\varphi(x,t) = \rho^{-\frac{n}{2}+1}\Phi(\sin\theta^2) \qquad \begin{cases} |x| = \rho\cos\theta \\ t = \rho\sin\theta \end{cases}$$

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Φ solves the Hypergeometric Equation

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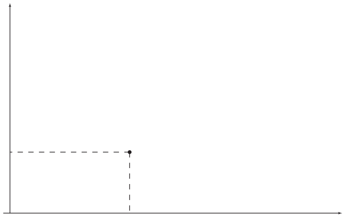

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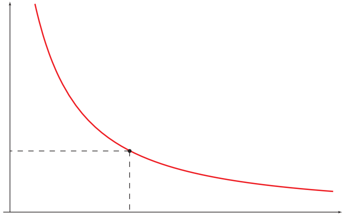

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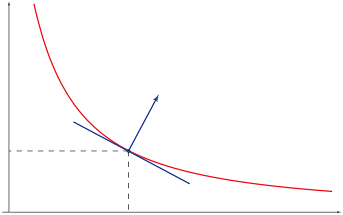
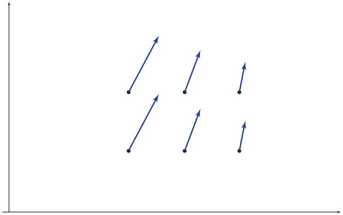


The Weierstrass's Method

The Mayer's field

$$(1, \mathbf{p}(x, y)) \equiv (1, \nabla \varphi_{k(x,y)}(x)) \in \mathbb{R} \times \mathbb{R}^n$$



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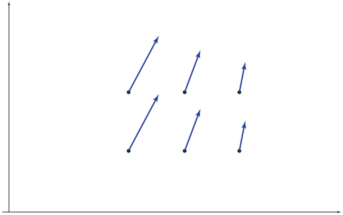

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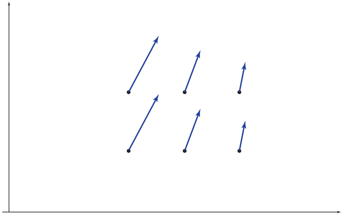

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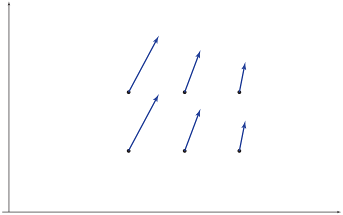

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$$f = f(x,y,\mathbf{p}(x,y)) \quad \Rightarrow \quad \mathrm{d}iv\mathbf{F} = 0$$

Sketch of the Proof $\beta = 2$

- Let $u \in C_c^\infty(\overline{\mathbb{R}_+^n} \cap B(\mathbf{0}, R))$

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$$+ \int_{\mathbb{R}^n_+ \setminus B_r(0)} \left(2\langle \mathbf{p}, \nabla u\rangle - |\mathbf{p}|^2\right) dxdt$$

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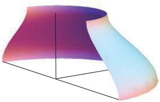

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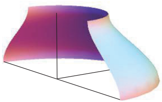

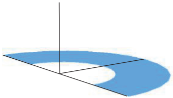

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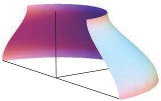

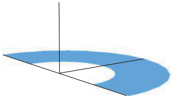

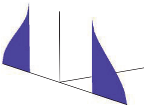

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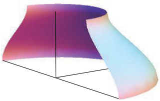

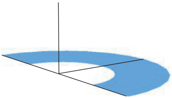

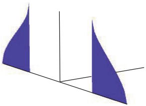

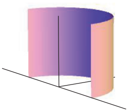

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$\mathrm{div}\,\mathbf{F} = 0$

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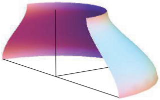
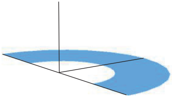
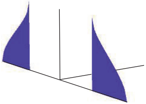
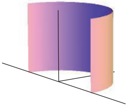
$$= \int_{\{(y,0):|y|>r\}} dx \int_0^{u(x,0)} \frac{-\varphi_t(x,0)}{\varphi(x,0)} 2v dv + o(r^{n-1})$$









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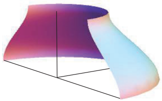

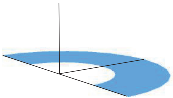

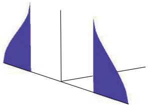

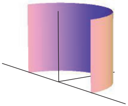

Sketch of the Proof $\beta = 2$

$\mathrm{div}\, \mathbf{F} = 0$

$$\int_{\mathbb{R}^n_+ \setminus B_r(0)} |\nabla u|^2 dxdt \geq \int_{\mathrm{Graph}\ u} \langle \mathbf{F}, \nu \rangle d\mathcal{H}^n$$

$$= \int_{\Sigma_1} \langle \mathbf{F}, \nu_1 \rangle d\mathcal{H}^n + \int_{\Sigma_2} \langle \mathbf{F}, \nu_2 \rangle d\mathcal{H}^n + o(r^{n-1})$$

$$= \int_{\{(y,0):|y|>r\}} \frac{H(n,2)}{|x|} u^2(x,0)dx + o(r^{n-1})$$