Shape optimization problems for variational functionals under geometric constraints

Ilaria Fragalà

2^{nd} Italian-Japanese Workshop
Cortona, June 20-24, 2011

The variational functionals

- The first Dirichlet eigenvalue of the Laplacian

$$\lambda_1(\Omega) := \inf \left\{ \frac{\int_\Omega |\nabla u|^2 \, dx}{\int_\Omega u^2 \, dx} \ : \ u \in H^1_0(\Omega) \, , \int_\Omega u^2 \, dx > 0 \right\}$$

- The torsional rigidity

$$\frac{1}{\tau(\Omega)} := \inf \left\{ \frac{\int_\Omega |\nabla u|^2 \, dx}{\left(\int_\Omega |u| \, dx \right)^2} \ : \ u \in H^1_0(\Omega) \, , \int_\Omega |u| \, dx > 0 \right\}$$

- The Newtonian capacity (for $n \geq 3$)

$$\mathrm{Cap}(\Omega) := \inf \left\{ \int_{\mathbb{R}^n \setminus \Omega} |\nabla u|^2 \, dx \ : \ u \in C^\infty_0(\mathbb{R}^n) \, , \ u \geq \chi_\Omega \right\}$$

For any of these functionals F it holds:

– F is a Dirichlet energy, $F(\Omega) = \int |\nabla u_\Omega|^2 \, dx$

$$\begin{cases} -\Delta u = \lambda_1(\Omega) u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n \setminus \overline{\Omega} \\ u = 1 & \text{on } \partial\Omega \\ u(x) \to 0 & \text{as } |x| \to +\infty \end{cases}$$

– F is monotone by inclusions

– F is homogeneous under dilations (of degree $\alpha = -2, n+2, n-2$)

– F is continuous with respect to Hausdorff convergence

– F is shape differentiable: if $\Omega_t = (I + tV)(\Omega)$,

$$\frac{d}{dt} F(\Omega_t)_{|t=0} = \pm \int_{\partial\Omega} (V \cdot \nu) |\nabla u_\Omega|^2 \, d\mathcal{H}^{n-1}$$

The geometric constraints

- The volume $|\Omega|$
- The perimeter $|\partial\Omega|$ (for sets with finite perimeter, $\chi_\Omega \in BV$)
- The mean width (for convex sets, $\Omega = \mathrm{int}(K)$)

$w_K(\xi) := h_K(\xi) + h_K(-\xi)$, $\quad h_K(\xi) := \sup_{x \in K}(x \cdot \xi)$ $\quad$ for $\xi \in S^{n-1}$
$\uparrow$
the distance between the two support planes of K normal to ξ

$$M(K) := \frac{2}{\mathcal{H}^{n-1}(S^{n-1})} \int_{S^{n-1}} h_K(\xi)\, d\mathcal{H}^{n-1}(\xi) \,.$$

The problems under study

Find *extremal* domains for

$$F(\Omega) = \lambda_1(\Omega)\,,\ \tau(\Omega)\,,\ \mathrm{Cap}(\Omega)$$

under one of the constraints

$$|\Omega|\,,\ |\partial\Omega|\,,\ M(\Omega) = const.$$

- The meaningful problems are:

$$\min_{|\Omega|=c} \lambda_1(\Omega) \qquad \max_{|\Omega|=c} \tau(\Omega) \qquad \min_{|\Omega|=c} \mathrm{Cap}(\Omega)$$

$$\min_{|\partial\Omega|=c} \lambda_1(\Omega) \qquad \max_{|\partial\Omega|=c} \tau(\Omega) \qquad \min_{|\partial\Omega|=c} \mathrm{Cap}(\Omega)$$

$$\min_{M(\Omega)=c} \lambda_1(\Omega) \qquad \max_{M(\Omega)=c} \tau(\Omega) \qquad \max_{M(\Omega)=c} \mathrm{Cap}(\Omega)$$

- We are interested as well in finding *stationary* domains for these problems.

Outline of the talk

1. Volume constrained problems

2. Perimeter constrained problems

3. Mean width constrained problems

4. Some results about a conjecture by Pólya-Szegö

1. Volume constrained problems

Assume $|\Omega| = |B|$. Then:

- $\lambda_1(\Omega) \geq \lambda_1(B)$ [Faber-Krahn]
- $\tau(\Omega) \leq \tau(B)$ [Pólya]
- $\mathrm{Cap}(\Omega) \geq \mathrm{Cap}(B)$ [Szegö]

Proof. By Schwarz symmetrization. □

Under the assumption $\partial\Omega \in \mathcal{C}^2$, if there exists a solution $\mathcal{C}^2$ up to the boundary to any of the following *overdetermined b.v.p.*, necessarily $\Omega = B$:

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ \left|\frac{\partial u}{\partial \nu}\right| = c & \text{on } \partial\Omega \end{cases} \qquad \begin{cases} -\Delta u = \lambda_1(\Omega) u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ \left|\frac{\partial u}{\partial \nu}\right| = c & \text{on } \partial\Omega \end{cases}$$

[SERRIN '71]

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n \setminus \overline{\Omega} \\ u = 1 & \text{on } \partial\Omega \\ u \to 0 & \text{as } |x| \to +\infty \\ \left|\frac{\partial u}{\partial \nu}\right| = c & \text{on } \partial\Omega \end{cases}$$

[REICHEL '97]

Proof. By moving planes or by many different methods! □

2. Perimeter constrained problems

Assume $|\partial\Omega| = |\partial B|$. Then the extremality of balls under volume constraint, combined with the isoperimetric inequality

$$\frac{|\Omega|^{1/n}}{|\partial\Omega|^{1/(n-1)}} \leq \frac{|B|^{1/n}}{|\partial B|^{1/(n-1)}} \; ,$$

yields:

- $\lambda_1(\Omega) \geq \lambda_1(B)$
- $\tau(\Omega) \leq \tau(B)$

Under the assumption $\partial\Omega \in \mathcal{C}^2$, if there exists a solution $\mathcal{C}^2$ up to the boundary to any of the following *overdetermined b.v.p.*, necessarily $\Omega = B$:

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ \left|\frac{\partial u}{\partial \nu}\right|^2 = c\, H_\Omega & \text{on } \partial\Omega \end{cases} \qquad \begin{cases} -\Delta u = \lambda_1(\Omega) u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ \left|\frac{\partial u}{\partial \nu}\right|^2 = c\, H_\Omega & \text{on } \partial\Omega \end{cases}$$

[SERRIN '71]

- **Conjecture:** [PÓLYA-SZEGÖ '51]

  Among *convex bodies* $K \subset \mathbb{R}^3$, with given *surface measure* $S(K)$, the *planar disk D* minimizes the Newtonian capacity.

– The convexity constraint is irremissible!

– $S(K)$ it is meant as $\mathcal{H}^2(\partial K)$ if $\mathrm{int}(K) \neq \emptyset$ and $2\mathcal{H}^2(K)$ otherwise.

– The solution cannot be the ball!!

- **Conjecture:** [Crasta-F.-Gazzola '05]

  Among open smooth and strictly convex sets,
  balls are the unique *stationary domains* for the PS problem.

$$
\begin{cases}
\Delta u = 0 & \text{in } \mathbb{R}^n \setminus \overline{\Omega} \\
u = 1 & \text{on } \partial\Omega \\
u \to 0 & \text{as } |x| \to +\infty \\
\left|\frac{\partial u}{\partial \nu}\right|^2 = c\, H_\Omega & \text{on } \partial\Omega
\end{cases}
$$

3. Mean width constrained problems

Remark: The mean width is a *Minkowski linear* functional.

Recall that $K + L$ is defined by the equality $h_{K+L} = h_K + h_L$.

Concavity inequalities in the Minkowski structure:

$F(\Omega) = \lambda_1(\Omega), \tau(\Omega), \mathrm{Cap}(\Omega)$ satisfy a *Brunn-Minkowski type inequality*:

$$F^{1/\alpha}(K+L) \geq F^{1/\alpha}(K) + F^{1/\alpha}(L) \qquad \forall K, L \in \mathcal{K}^n\,,$$

with strict inequality for non-homothetic sets.

[Brascamp-Lieb '73, Borell '83, '85,
Caffarelli-Jerison-Lieb '96, Colesanti '96]

Theorem (shapeopt under mean width constraint).
[Bucur-F.-Lamboley '11]
Assume that $F : \mathcal{K}^n \to \mathbb{R}^+$ satisfies a BM-type inequality, is invariant under rigid motions, and continuous in the Hausdorff distance.

Consider the quotient $\mathcal{E}(K) := \dfrac{F^{1/\alpha}(K)}{M(K)}$. Then:

(i) the maximum of $\mathcal{E}$ over $\mathcal{K}^n$ is attained only on balls;

(ii) if $n = 2$, the minimum of $\mathcal{E}$ over $\mathcal{K}^2$ can be attained only on triangles or on segments.

In particular, if $M(\Omega) = M(B)$:

- $\lambda_1(\Omega) \geq \lambda_1(B)$
- $\tau(\Omega) \leq \tau(B)$
- $\mathrm{Cap}(\Omega) \leq \mathrm{Cap}(\mathrm{B})$

Proof

(i) *Hadwiger's Theorem*: For every $K \in \mathcal{K}^n$ (with $\dim K > 0$) there exists a sequence K_h of rotation means of K which converges in Hausdorff distance to a ball.

Then:

$$\frac{F^{1/\alpha}(K)}{M(K)} \leq \frac{F^{1/\alpha}(K_h)}{M(K_h)} \rightarrow \frac{F^{1/\alpha}(B)}{M(B)}$$

(ii) In dimension $n = 2$, the unique Minkowski indecomposable bodies are triangles and segments. (No longer true in higher dimensions!)

Then: if K is not a triangle or a segment, $K = K_1 + K_2 \Longrightarrow$

$$\frac{F^{1/\alpha}(K)}{M(K)} > \frac{F^{1/\alpha}(K_1) + F^{1/\alpha}(K_2)}{M(K_1) + M(K_2)} \geq \min\left\{\frac{F^{1/\alpha}(K_1)}{M(K_1)}, \frac{F^{1/\alpha}(K_2)}{M(K_2)}\right\}.$$

□

Theorem (Gaussian curvature overdetermined b.v.p.) [F. '11]

Under the assumption $\Omega = \operatorname{int} K$ for some $K \in \mathcal{K}^n_0$, with $\partial\Omega$ of class $\mathcal{C}^2$, if there exists a solution $\mathcal{C}^2$ up to the boundary to any of the following *overdetermined b.v.p.*, necessarily $\Omega = B$:

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ \left|\frac{\partial u}{\partial \nu}\right|^2 = c\, G_\Omega & \text{on } \partial\Omega \end{cases} \qquad \begin{cases} -\Delta u = \lambda_1(\Omega) u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ \left|\frac{\partial u}{\partial \nu}\right|^2 = c\, G_\Omega & \text{on } \partial\Omega \end{cases}$$

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n \setminus \overline{\Omega} \\ u = 1 & \text{on } \partial\Omega \\ u \to 0 & \text{as } |x| \to +\infty \\ \left|\frac{\partial u}{\partial \nu}\right|^2 = c\, G_\Omega & \text{on } \partial\Omega \end{cases}$$

Proof. By concavity, a stationary domain for the quotient functional $\mathcal{E} = \frac{F^{1/\alpha}}{M}$ is necessarily a maximizer. □

Concavity inequalities in the Blaschke structure:

We say that $F : \mathcal{K}_0^n \to \mathbb{R}^+$ satisfies a *Kneser-Süss type inequality* if

$$F^{(n-1)/\alpha}(K \sharp L) \geq F^{(n-1)/\alpha}(K) + F^{(n-1)/\alpha}(L) \qquad \forall K, L \in \mathcal{K}_0^n \, ,$$

with equality if and only if K and L are homothetic.

- $K \sharp L$ is defined by the equality $\sigma(K \sharp L) = \sigma(K) + \sigma(L)$, where $\sigma(K) := (\nu_K)_* (\mathcal{H}^{n-1} \llcorner \partial K) \qquad \nu_K =$ Gauss map
- Kneser-Süss Theorem states that the above concavity inequality holds true for the volume functional.
- $S(K)$ is a Blaschke linear functional

Theorem (shapeopt under surface contraint).
[BUCUR-F.-LAMBOLEY '11]
Assume that $F : \mathcal{K}^n_0 \to \mathbb{R}^+$ satisfies a KS-type inequality, is invariant under rigid motions and continuous in the Hausdorff distance.

Consider the quotient $\mathcal{E}(K) := \dfrac{F^{(n-1)/\alpha}(K)}{S(K)}$. Then:

(i) the maximum of $\mathcal{E}$ over $\mathcal{K}^n_0$ is attained only on balls;

(ii) the minimum of $\mathcal{E}$ over $\mathcal{K}^n_0$ can be attained only on simplexes.

Counterexamples:

$F(\Omega) = \mathrm{Cap}(\Omega), \lambda_1(\Omega), \tau(\Omega)$ *do not satisfy* a KS-type inequality!

Concavity inequalities in different algebraic structures:
each of our model functional is *concave* with respect to
a *new* sum of convex bodies, which linearizes the first variation of F.

This leads to new "isoperimetric-like" inequalities.

4. Some results about PS conjecture

$$\inf_{K\in\mathcal{K}^3} \mathcal{E}(K) := \frac{\mathrm{Cap}^2(K)}{S(K)}$$

Theorem 1 (optimality of the disk among planar domains).
[Pólya-Szegö, '51]

Let D be a planar disk. For every planar convex domain with $\mathcal{H}^2(K) = \mathcal{H}^2(D)$, it holds

$$\mathrm{Cap}(K) \geq \mathrm{Cap}(D) \ .$$

Proof. By a cylindrical symmetrization. □

Theorem 2 (lower bound).
[PÓLYA-SZEGÖ, '51]
The infimum of $\mathcal{E}$ over $\mathcal{K}^3$ is strictly positive.

Proof. By using symmetrizations and monotonicity with respect to inclusions. □

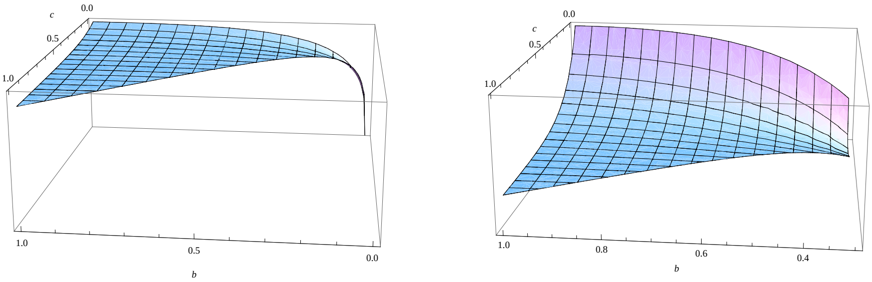
Theorem 3 (existence of a minimizer).
[CRASTA-F.-GAZZOLA, '05]
The infimum of $\mathcal{E}$ over $\mathcal{K}^3$ is attained.

Proof. By using Blaschke selection theorem, John Lemma, and the behaviour of thinning ellipsoids. □

Theorem 4 (optimality among ellipsoids).
[F.-Gazzola-Pierre '11]

The planar disk is optimal for $\mathcal{E}$ within the class of triaxial ellipsoids.

Proof. Plot of the map $(b, c) \mapsto \mathcal{E}^{-1}(E_{1,b,c})$ for (b, c) in the triangle $T = \{(b, c) \in \mathbb{R}^2 \,:\, 1 \geq b \geq c \geq 0\}$ and for (b, c) near $(1, 0)$.



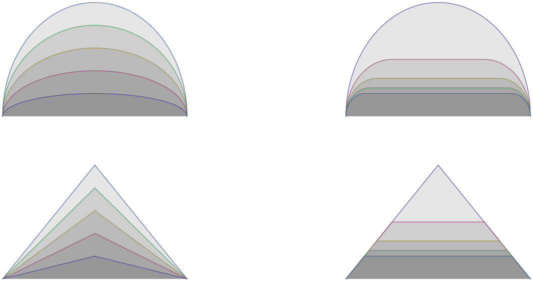
Remarks:
(i) There is no stationary ellipsoid different from a ball.
(ii) There exists b^* s.t. $c \mapsto \mathcal{E}(E(1, b^*, c))$ is *not* monotone. □

Theorem 5 (“local” optimality).
[F.-Gazzola-Pierre ’11]

For a large class of suitably defined one parameter families of convex domains D_t obtained by “fattening” the planar disk, it holds

$$\mathcal{E}(D) < \mathcal{E}(D_t) \qquad \text{for } 0 < t \ll 1 \, .$$

Proof. By a careful comparison of $\mathrm{Cap}'(0)$ and $S'(0)$.



Left: $\mathrm{Cap}'(0) > 0$, $S'(0) = 0$. *Right*: $\mathrm{Cap}'(0) = +\infty$, $S'(0) < +\infty$. $\square$

Theorem 6 (no smooth portions with positive Gauss curvature). [Bucur-F.-Lamboley '11]

Assume that $F : \mathcal{K}^n_0 \to \mathbb{R}^+$ is given by

$$F(K) = f(|K|, \lambda_1(K), \tau(K), \mathrm{Cap}(K)) \quad (\text{ with } f \in \mathcal{C}^2)\ .$$

Let K^* be a minimizer over $\mathcal{K}^n_0$ for the functional

$$\mathcal{E}(K) := \frac{F(K)}{S(K)}\ .$$

If ∂K^* contains a subset ω of class $\mathcal{C}^3$, then $G_{K^*} = 0$ on ω.

Proof.

$$\ell_2^S(K^*)\cdot(\varphi,\varphi) \geq c_1|\varphi|^2_{H^1(\omega)} + c_2\|\varphi\|^2_{L^2(\omega)}\ , \quad \left|\ell_2^F(K^*)\cdot(\varphi,\varphi)\right| \leq c_3\|\varphi\|^2_{H^{\frac{1}{2}}(\omega)}\ .$$

□

Lemma (local concavity entails local extremality).

Let $K^* \in \mathcal{K}_0^n$ be a minimizer for $J : \mathcal{K}_0^n \to \mathbb{R}^+$.
Let $\omega \subset \partial K^*$ of class $\mathcal{C}^3$, and assume that $t \mapsto J(K_t)$ is twice differentiable at $t = 0$ for any smooth V compactly supported on ω.

If the bilinear form $\ell_2^J(K^*)$ satisfies:

$$\forall \varphi \in \mathcal{C}_c^\infty(\omega), \quad \ell_2^J(K^*) \cdot (\varphi, \varphi) \leq -c_1 |\varphi|^2_{H^1(\omega)} + c_2 \|\varphi\|^2_{H^{\frac{1}{2}}(\omega)}$$

for some constants $c_1 > 0$, $c_2 \in \mathbb{R}$, then

$$G_{K^*} = 0 \ \text{ on } \ \omega$$

Proof. By contradiction, against the second order optimality condition. □

THE END. THANK YOU!