Blow-up set for a semilinear heat equation with exponential nonlinearity

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2nd Italian-Japanese Workshop
Geometric properties for parabolic and elliptic PDE's

Consider the blow-up problem for a semilinear heat equation

$$(P) \quad \begin{cases} u_t = \epsilon \Delta u + e^u, & x \in \Omega, \quad t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = \varphi_\epsilon(x) \geq 0, & x \in \Omega, \end{cases}$$

where $0 < \epsilon << 1, \;\; N \geq 1, \;\; \Omega \subset \mathbf{R}^N$: b'dd domain,

and φ_ϵ : bounded nonnegative continuous function on Ω.

Aim

Location of the blow-up set for $0 < \epsilon << 1$

§1. Introduction

Consider

$$(P)\quad \begin{cases} u_t = \epsilon\Delta u + e^u, & x \in \Omega,\ \ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ \ t > 0, \\ u(x,0) = \varphi_\epsilon(x) \geq 0, & x \in \Omega, \end{cases}$$

where $\epsilon > 0,\ N \geq 1,\ \Omega$: bounded domain,

and $\varphi_\epsilon \geq 0$: bounded continuous function on Ω.

T_ϵ : maximal existence time of the classical sol. u of (P).

$$T_\epsilon < \infty \quad \Rightarrow \quad \limsup_{t \nearrow T_\epsilon} \|u(t)\|_{L^\infty(\Omega)} = \infty$$

(Blow-up time)

$$\partial_t u = \epsilon \Delta u + e^u \text{ in } \Omega \times (0, T_\epsilon), \quad u|_{\partial\Omega} = 0, \quad u(x,0) = \varphi_\epsilon(x).$$

T_ϵ : maximal existence time of the classical sol. u of (P).

$$T_\epsilon < \infty \quad \Rightarrow \quad \limsup_{t \nearrow T_\epsilon} \|u(t)\|_{L^\infty(\Omega)} = \infty$$

(Blow-up time)

B_ϵ : the blow-up set of the sol., that is,

$$B_\epsilon = \Big\{ x \in \overline{\Omega} : {}^\exists\{(x_n, t_n)\} \subset \overline{\Omega} \times (0, T_\epsilon) \text{ such that}$$
$$\lim_{n\to\infty} (x_n, t_n) = (x, T_\epsilon), \quad \lim_{n\to\infty} |u(x_n, t_n)| = \infty \Big\}.$$

(Blow-up set)

Known Results

$$\partial_t u = \epsilon \Delta u + e^u \text{ in } \Omega \times (0, T_\epsilon), \quad u|_{\partial\Omega} = 0, \quad u(x,0) = \varphi_\epsilon(x).$$

- For $\lambda > 0$, $u_\lambda(x,t) := u(\lambda x, \lambda^2 t) + \log \lambda^2$ satisfies

$$(u_\lambda)_t = \epsilon \Delta u_\lambda + e^{u_\lambda} \qquad \text{(Self-similarity)}$$

Behavior of the sol. of (P) near the blow-up point.

{ Bebernes-Bressan-Eberly '87, Bebernes-Eberly '88,
Fila-Pulkkinen '08.

- Friedman-McLeod '85

$$\Omega : \text{convex} \quad \Rightarrow \quad B_\epsilon \text{ is compact in } \Omega.$$

In general, it seems difficult to characterize the location of B_ϵ because of the interaction between Δu and e^u.

We consider the case $0 < \epsilon << 1$

Problem

$$(*) \quad \begin{cases} (P) \approx u_t = e^u \\ \Downarrow \\ B_\epsilon \approx \{x \in \overline{\Omega} : \varphi_\epsilon(x) = \|\varphi_\epsilon\|_{L^\infty(\Omega)}\} \quad ? \end{cases}$$

Known Results

- F.-Ishige '10 $\quad (u_t = \epsilon\Delta u + u^p,\ p > 1)$

Let $\varphi_\epsilon \in C^1(\overline{\Omega}) \cap L^\infty(\Omega)$ with $^\exists\lambda_1 \le \|\varphi_\epsilon\|_{L^\infty(\Omega)} \le^\exists \lambda_2$

satisfying

$$(A) \quad \epsilon^{1/2}\|\nabla\varphi_\epsilon\|_{L^\infty(\Omega)} \le C\epsilon^A \ \text{ for } {}^\exists C > 0,\ {}^\exists A \in (0, 1/2).$$

Assume $^\exists C_* > 0$ such that

$$\|u(t)\|_{L^\infty(\Omega)} \le C_*(T_\epsilon - t)^{-1/(p-1)}, \quad t \in (0, T_\epsilon).$$

⇒ $^\forall\delta > 0,\ {}^\exists\epsilon_\delta > 0$ such that

$$(*) \quad \begin{cases} (P) \approx u_t = e^u \\ \Downarrow \\ B_\epsilon \approx \{x \in \overline{\Omega} :\ \varphi_\epsilon(x) = \|\varphi_\epsilon\|_{L^\infty(\Omega)}\} \end{cases}$$

$$B_\epsilon \subset \{x \in \overline{\Omega} :\ \varphi_\epsilon(x) \ge \|\varphi_\epsilon\|_{L^\infty(\Omega)} - \delta\}, \quad \epsilon \in (0, \epsilon_\delta).$$

Known Results

- F.-Ishige, to appear $\quad (u_t = \epsilon\Delta u + u^p,\ p > 1)$

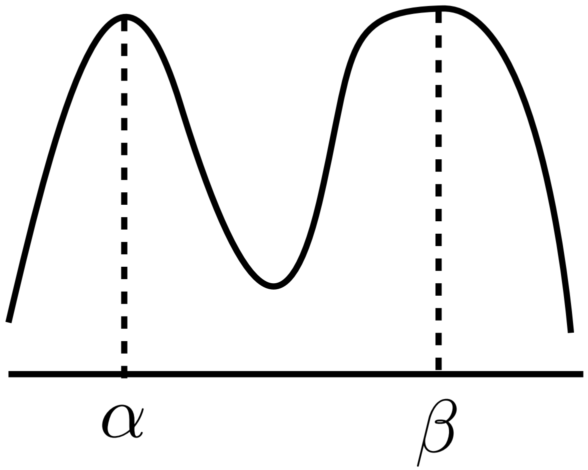
Let $\varphi_\epsilon \equiv \varphi, \quad \varphi \in C^2(\Omega) \cap C(\overline{\Omega}), \quad \varphi \geq 0.$

Assume ${}^\exists C_* > 0$ such that

$$\|u(t)\|_{L^\infty(\Omega)} \leq C_*(T_\epsilon - t)^{-1/(p-1)}, \quad t \in (0, T_\epsilon).$$

Assume ${}^\exists \alpha,\ \beta \in \{x \in \overline{\Omega} :\ \varphi(x) = \|\varphi\|_{L^\infty(\Omega)}\}$ s. t.

$$|\Delta\varphi(\alpha)| < |\Delta\varphi(\beta)|.$$

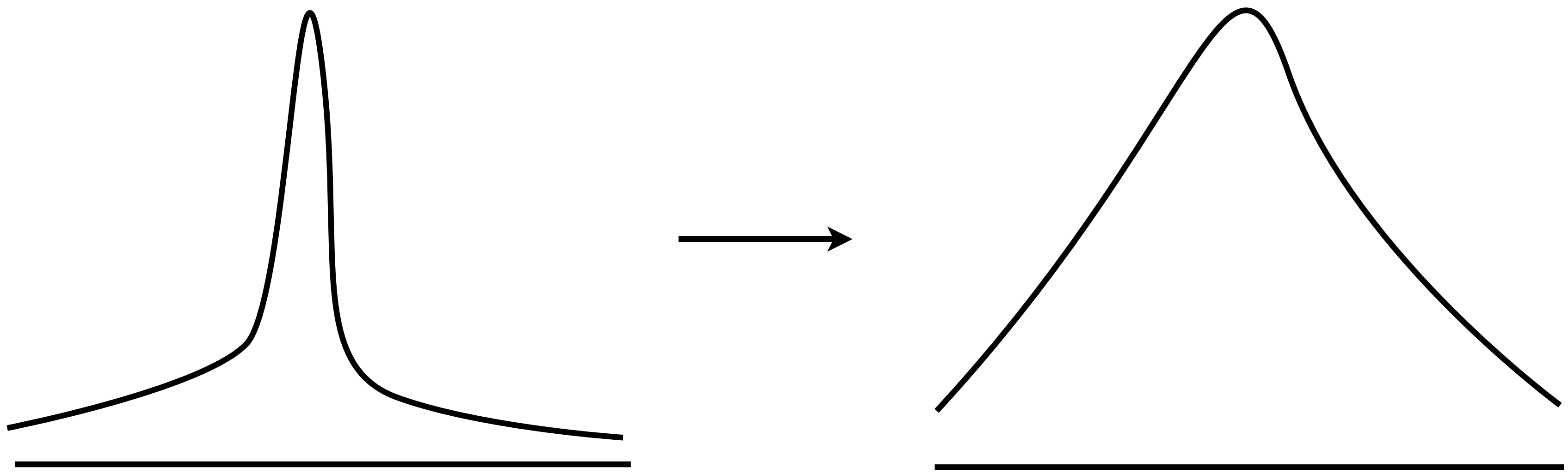


$\Rightarrow$ ${}^\exists \delta > 0,\ {}^\exists \epsilon_\delta > 0$ s. t.

$$B_\epsilon \cap \{y \in \overline{\Omega} :\ |y - \beta| < \delta\} = \emptyset, \qquad \epsilon \in (0, \epsilon_\delta).$$

About (A)

$(A) \quad \epsilon^{1/2}\|\nabla\varphi_\epsilon\|_{L^\infty(\Omega)} \le C\epsilon^A$ for $^\exists C>0,\ ^\exists A\in(0,1/2).$



$\|v(\cdot,0)\|_\infty \le C$

Scaling $\quad v(y,t) = u(\epsilon^{1/2-A}y,t)$

$$\Rightarrow \quad \partial_t v = \epsilon^{2A}\Delta_y v + v^p$$

Small diffusion

$(P) \approx u_t = u^p$

§2. Main Results

$$(P) \quad \begin{cases} u_t = \epsilon \Delta u + e^u, & x \in \Omega, \quad t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = \varphi_\epsilon(x) \geq 0, & x \in \Omega, \end{cases}$$

where $\epsilon > 0, \;\; N \geq 1, \;\; \Omega$: bounded domain.

<u>Assumptions</u>

(A_1) $$\lim_{\epsilon \to 0} \epsilon^{1/2} \|\nabla \varphi_\epsilon\|_{L^\infty(\Omega)} = 0.$$

(A_2) Let u_ϵ be the sol. of (P) and assume ${}^\exists C_* > 0$ s.t.

$$\|u_\epsilon(t)\|_{L^\infty(\Omega)} \leq \underbrace{\log \frac{C_*}{T_\epsilon - t}}_{\text{Blow-up rate for } u_t = e^u}, \qquad t \in (0, T_\epsilon).$$

$$(P) \quad \partial_t u = \epsilon \Delta u + e^u \ \text{ in } \ \Omega \times (0, T_\epsilon), \quad u|_{\partial\Omega} = 0, \quad u(x,0) = \varphi_\epsilon(x).$$

$$(A_1) \ \lim_{\epsilon \to 0} \epsilon^{1/2} \|\nabla \varphi_\epsilon\|_{L^\infty(\Omega)} = 0. \qquad (A_2) \ \|u_\epsilon(t)\|_{L^\infty(\Omega)} \le \log \frac{C_*}{T_\epsilon - t}.$$

Theorem

Assume (A_1) and (A_2).

Then, for any $\delta > 0$, there exists a constant $\epsilon_\delta > 0$ s.t.

$$B_\epsilon \subset \left\{ x \in \overline{\Omega} : \varphi_\epsilon(x) \ge \|\varphi_\epsilon\|_{L^\infty(\Omega)} - \delta \right\}, \quad \epsilon \in (0, \epsilon_\delta).$$

$$u_t = \epsilon \Delta u + f(u)$$

(Q) What is sufficient condition of f

such that this theorem holds?

Remark

- $\sigma_\epsilon := \epsilon^{1/2}\|\nabla\varphi_\epsilon\|_{L^\infty(\Omega)} \to 0$ as $\epsilon \to 0.$

  The function $v_\epsilon(y,t) := u_\epsilon(\epsilon^{1/2}\sigma_\epsilon y, t)$ satisfies

$$(v_\epsilon)_t = \sigma_\epsilon^2 \Delta_y v_\epsilon + e^{v_\epsilon}.$$

Small diffusion

- (Friedman-McLeod '85)

  Ω : convex, $\quad \epsilon\Delta\varphi_\epsilon + e^{\varphi_\epsilon} \geq 0$ in Ω

  $\Rightarrow \ (A_2) \ \|u_\epsilon(t)\|_{L^\infty(\Omega)} \leq \log \dfrac{C_*}{T_\epsilon - t}$ holds.

Remark

- Let u be the sol. of

$$\begin{cases} u_t = \Delta u + e^u & \text{in} \quad \Omega \times (0,T), \\ u(x,t) = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ u(x,0) = \phi(x) \geq 0 & \text{in} \quad \Omega, \end{cases}$$

where $\phi \in L^\infty(\Omega)$ and T : the blow-up time.

Assume $^\exists C_* > 0$ such that $\|u(t)\|_{L^\infty(\Omega)} \leq \log \dfrac{C_*}{T-t}$

and $\displaystyle\lim_{t \nearrow T} (T-t)^{1/2} \|\nabla u(t)\|_{L^\infty(\Omega)} = 0.$

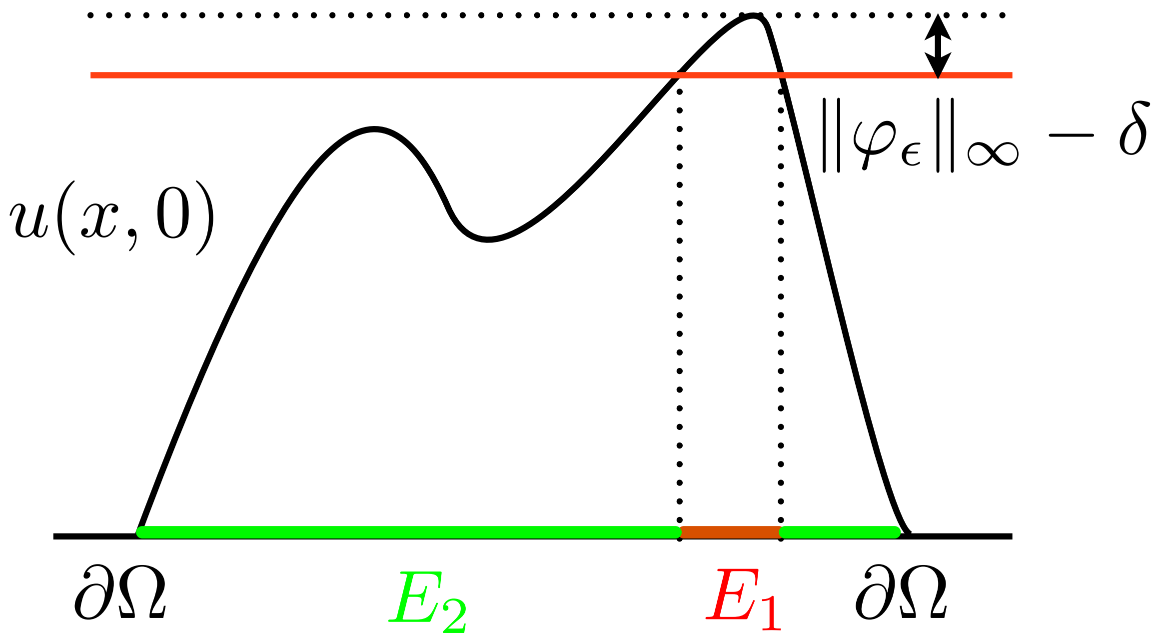
$$\Rightarrow \quad B_\epsilon \text{ is compact in } \Omega.$$

§3. Outline of the proof

We construct a super-solution $\overline{u}$ such that

- $\overline{u}$ exists on $(0, T_\epsilon)$.
- $\overline{u}$ does not blow-up in $\{x \in \overline{\Omega} : \varphi_\epsilon(x) < \|\varphi_\epsilon\|_\infty - \delta\}$.

$$v(x,t) = -\log\left[\exp(e^{t\epsilon\Delta}\varphi_\epsilon(x)) - t\right] \quad \Rightarrow \quad v_t \le \epsilon\Delta v + e^v$$



Supersolution

$$\overline{u}(x,t) = v(x,t) + \tilde{v}(x,t)$$

s. t.

$$\sup_{E_2 \times (0, T_\epsilon)} \overline{u} < \infty,$$

$$\overline{u} \ge \log \frac{2C_*}{T_\epsilon - t} \quad \text{in } E_1.$$

$$\overline{u}_t \ge \epsilon\Delta\overline{u} + e^{\overline{u}} \quad \text{in } E_2.$$

Instead of considering $u_t = \epsilon \Delta u + e^u$, we consider

$$u_t = \epsilon \Delta u + e^u + h(u) G_\epsilon(x,t),$$

where

$$h(u) = \begin{cases} 0 & \text{if} \quad u \le \log \frac{C_*}{T_\epsilon - t}, \\ 1 & \text{if} \quad u \ge \log \frac{2C_*}{T_\epsilon - t}, \end{cases} \qquad G_\epsilon = \overline{u}_t - \epsilon \Delta \overline{u} - e^{\overline{u}}.$$

- $u_t = \epsilon \Delta u + e^u + h(u) G_\epsilon(x,t)$
- $\overline{u}_t - \left[\epsilon \Delta \overline{u} + e^{\overline{u}} + h(\overline{u}) G_\epsilon(x,t)\right] = G_\epsilon(x,t)(1 - h(\overline{u})) \ge 0$

$(G_\epsilon(x,t) \ge 0$ in E_2, $\quad h(\overline{u}) = 1$, in E_1.)

$$\overline{u}_t \ge \epsilon \Delta \overline{u} + e^{\overline{u}} + h(\overline{u}) G_\epsilon(x,t) \text{ in } \Omega \times (0, T_\epsilon)$$