# Minimization of non-coercive integrals by means of convex rearrangement

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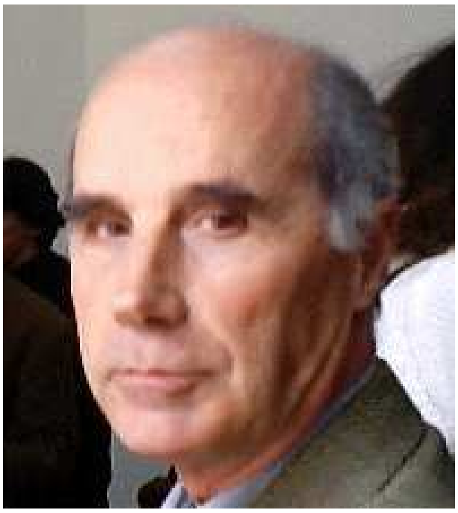
# Part I

# Introduction



My advisor's remark:

***“Nobody is teaching calculus of variations”***

## The problem of the brachistochrone

Find a path in a vertical plane connecting two given points so that a particle falling down from the first point along that path subject to the gravity and without friction will reach the second point in **the shortest possible time** (Johann Bernoulli, 1696)

$$F[u] = \int_a^b \sqrt{\frac{1 + (u'(x))^2}{v_0^2 - 2gu(x)}}\, dx$$

The time-of-transit functional in the problem of the brachistochrone

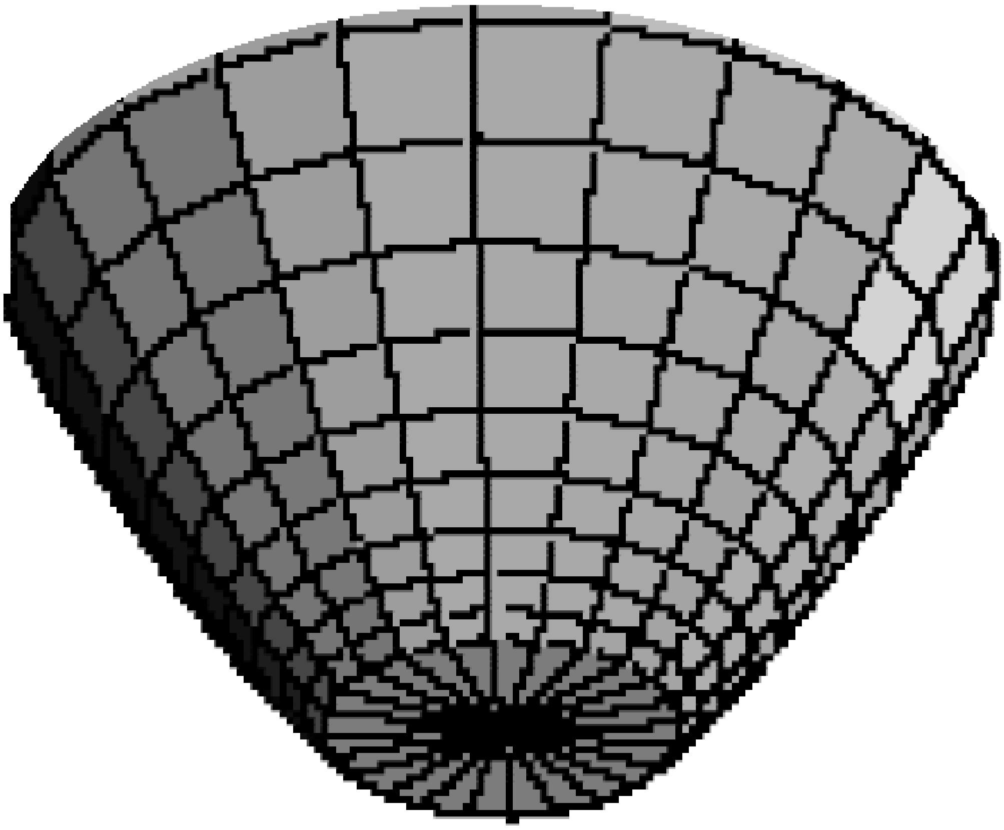
## Newton's problem of the body of minimal resistance

Find the shape of a rotationally symmetric body, with prescribed height and prescribed circular cross-section, that minimizes the resistance encountered when moving through a rarefied gas (Principia, 1687)

More precisely, find a monotone non-decreasing function $u$ over the interval $[0,R]$ with values in $[0,M]$ that minimizes the following functional

$$G[u] = \int_0^R \frac{x\,dx}{1 + (u'(x))^2}$$

# Newton's body of minimal resistance



C. H. Edwards, *Newton's nose-cone problem*, The Mathematica Journal **7** (1997), 64–71.

# The most difficult problem

The hardest problem is to prove that the functionals under consideration do possess a minimizer.

The **classical approach** to the brachistochrone problem has its roots in the work of Weierstrass and is found, for instance, in the following books:

- G. A. Bliss, Calculus of variations. The Carus Mathematical Monographs **1**. Washington: The Mathematical Association of America; La Salle, Ill.: The Open Court Publishing Company, 1925.

- M. Giaquinta, S. Hildebrandt, Calculus of variations. I. Grundlehren der Mathematischen Wissenschaften **310**. Springer-Verlag, Berlin, 1996.

- J. L. Troutman, Variational calculus with elementary convexity. Springer-Verlag, New York-Berlin, 1983.

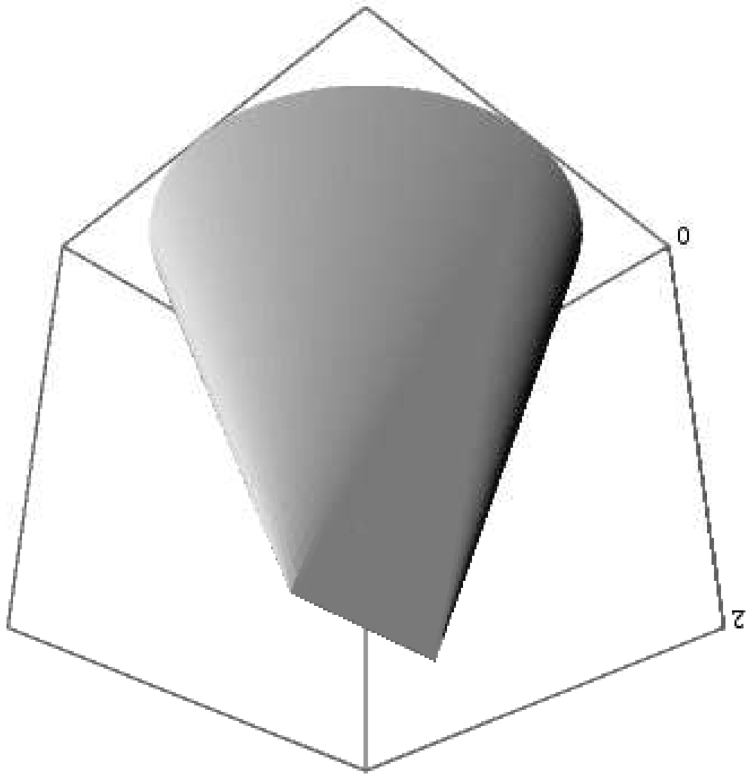
# Newton's problem revisited

Recent advances on Newton's problem include the existence proof of a non-radial minimizer in the class of bounded, convex functions:

- G. Buttazzo, *A survey on the Newton problem of optimal profiles*, in: Variational analysis and aerospace engineering, pp. 33–48, Springer Optim. Appl. **33**, Springer, New York, 2009.

- G. Buttazzo and B. Kawohl, *On Newton's problem of minimal resistance*, Math. Intelligencer **15** (1993), 7–12.

- G. Buttazzo, V. Ferone and B. Kawohl, *Minimum problems over sets of concave functions and related questions,* Math. Nachr. **173** (1995), 71–89.



# The present approach

The present approach is based on the following strategy:

1. The space of competing functions is the set of all functions in $W^{1,1}((a,b))$ attaining the prescribed boundary values. Let $(u_n)$ be a minimizing sequence.

2. Replace each $u_n$ with its *convex rearrangement* $u_n*$. Boundary values are preserved, as well as the graph length. Show that the functional is reduced (Pólya-Szegö inequality).

3. There exists a subsequence converging to some $u$ locally unifomly.

4. Give meaning to $F\,[u]$ even in case $u$ develops boundary singularities (no problem with $G[u]$).

5. Show that the functional $F$ is continuous under such convergence. Then $u$ is a minimizer.

6. Finally, verify that $u$ has no jumps at the endpoints. Thus, $u$ belongs to the admissible class and therefore it is a minimizer.

## Generalization

The present approach also applies to the following functionals:

$$F[u] = \int_a^b f(u(x))\,\sqrt{1 + (u'(x))^2}\,dx$$

$$G[u] = \int_a^b g(u'(x))\,\mu(x)\,dx$$

with $f$ non-decreasing, $g$ positive and non-increasing, and $\mu$ positive and non-decreasing.

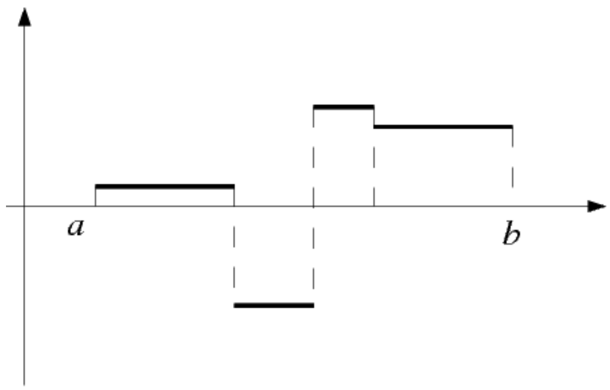
Isoperimetric problems as the problem of the catenary, which corresponds to $f(u) = u$, are included.

# Part II

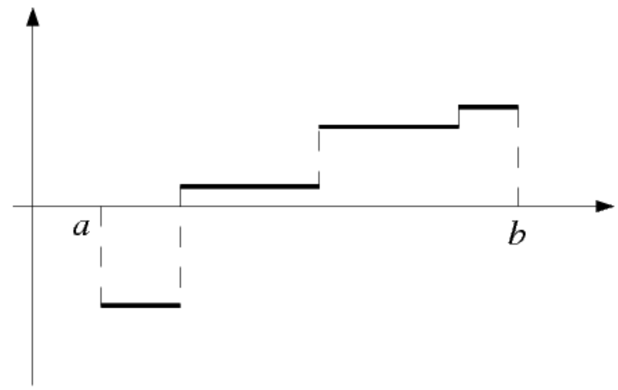
# The convex rearrangement

# The non-decreasing rearrangement

It is defined firstly on step functions, and then extended by continuity to the whole space $L^1((a,b))$:




**Before:** $v(x)$




**After:** $v^{\#}(x)$

## The convex rearrangement

The convex rearrangement of a function *u* in $W^{1,1}((a,b))$ is the function $u^*$ defined as follows:

$$u^*(x) = u(a) + \int_a^x (u')^\sharp(t)\,dt$$

**Equivalently:**

1. Take a function *u* in $W^{1,1}((a,b))$ whose derivative *u*' is a step function.
2. Rearrange the line segments in the graph of *u* preserving slope and continuity until slopes are non-decreasing: call $u^*$ the resulting function.
3. Extend to $W^{1,1}((a,b))$ by density.

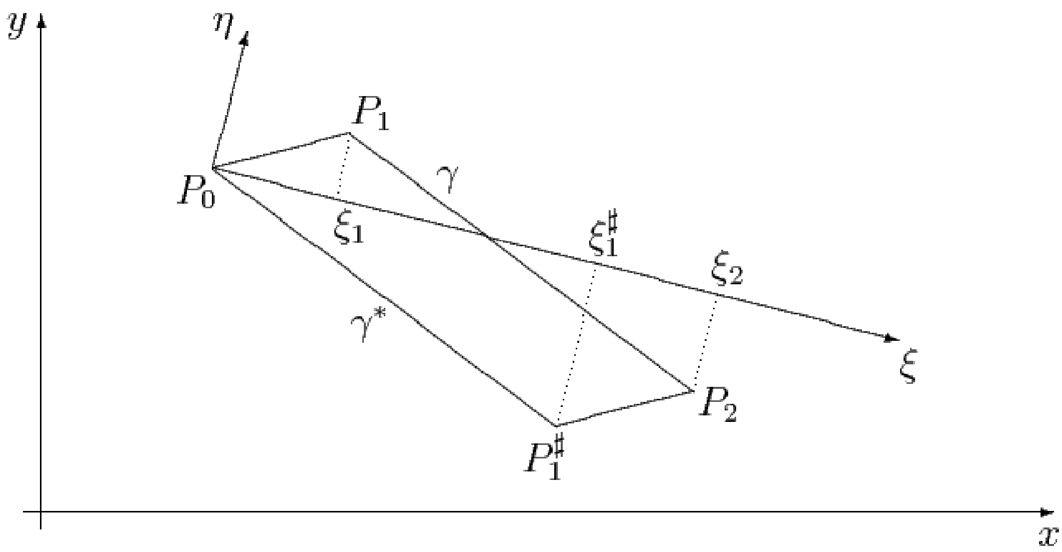



Figure 1: proving $F[u^*] \le F[u]$ for a piecewise-linear $u$

The change of variable $x = x(\xi)$ yields:

$$\int_a^b f(u(x))\,\sqrt{1+(u'(x))^2}\,dx = \int_0^{\xi_1} f(y(\xi))\,\sqrt{1+(\varphi'(\xi))^2}\,d\xi$$
$$+\int_{\xi_1}^{\xi_1^\sharp} f(y(\xi))\,\sqrt{1+(\varphi'(\xi))^2}\,d\xi + \int_{\xi_1^\sharp}^{\xi_2} f(y(\xi))\,\sqrt{1+(\varphi'(\xi))^2}\,d\xi.$$

By the change of variable $x = x^*(\xi)$ we obtain, instead:

$$\int_a^b f(u^*(x))\,\sqrt{1+((u^*)'(x))^2}\,dx = \int_0^{\xi_1} f(y^*(\xi))\,\sqrt{1+((\varphi^*)'(\xi))^2}\,d\xi$$
$$+\int_{\xi_1}^{\xi_1^\sharp} f(y^*(\xi))\,\sqrt{1+((\varphi^*)'(\xi))^2}\,d\xi + \int_{\xi_1^\sharp}^{\xi_2} f(y^*(\xi))\,\sqrt{1+((\varphi^*)'(\xi))^2}\,d\xi.$$

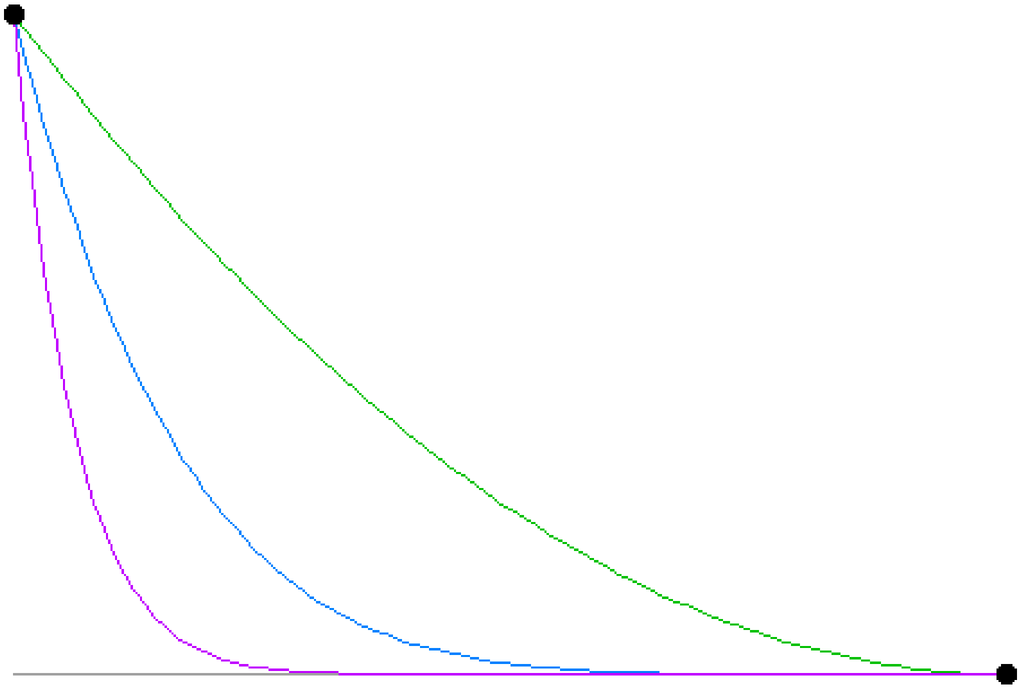
Equality holds if *f* is constant (graph length is preserved).

## Part III

# Managing boundary singularities

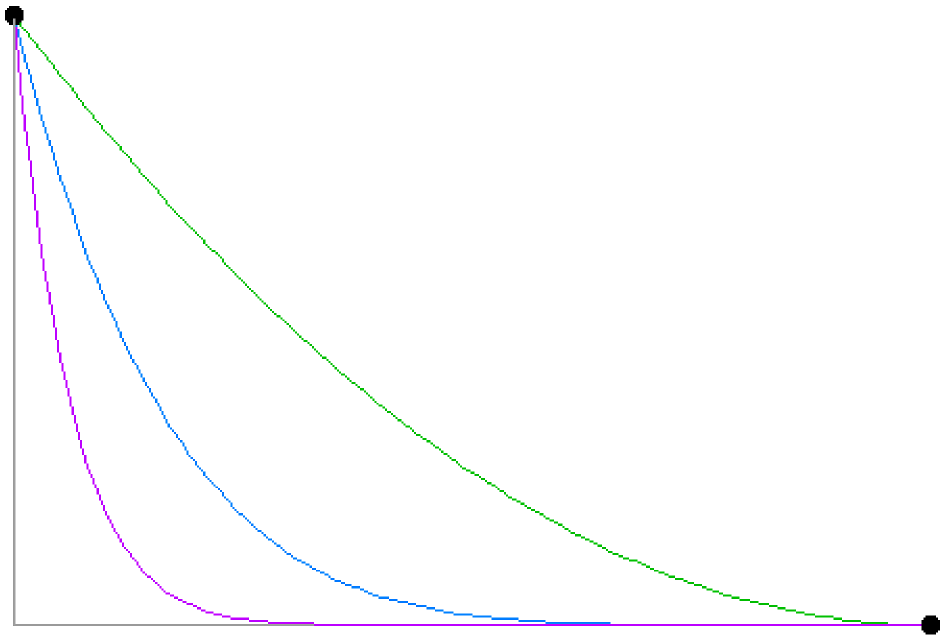
# Boundary singularities may appear

Since the convergence of $u_n$ is just locally uniform, the limit function $u$ may well have jump discontinuities at the endpoints:



## Extension of the functional *F*

The contribution of vertical segments at the endpoints is taken into account by suitably extending the functional $F$:



$$\overline{F}[u] = \int_{u(a)}^{u_a} f(y)\,dy \;+\; F[u] \;+\; \int_{u(b)}^{u_b} f(y)\,dy$$

The extended functional is continuous with respect to locally uniform convergence of convex functions.

## An unpleasant counterpart

The minimizer may, in principle, be a function with jump discontinuities at the endpoints.

## The minimizers have no boundary singularities

The proof is by contradiction. Suppose, for instance, that a minimizer *u* has a jump discontinuity at the first endpoint:

The function *u* (black), with a jump at the first endpoint, is replaced by $u_\varepsilon$ (red), which is continuous at the first endpoint and satisfies

$$\overline{F}[u_\varepsilon] < \overline{F}[u]$$

Hence a minimizer cannot have a jump at the first endpoint (and neither at the second one).

# Conclusion

- A new existence proof of the brachistochrone is available.

- The method also applies to the classical problem of the body of minimal resistance posed by Newton.

- Isoperimetric problems as the problem of the catenary are included.

- **Open problem:** it would be interesting to extend the method to dimension $N > 1$.

# Thank you

## for your attention