Existence of minimizers for some coupled nonlinear Schrödinger equations

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Cortona, June 21st, 2011

Second Italian-Japanese Workshop
Geometric properties for Parabolic and elliptic PDE's

§1 Introduction

(Kartashov-Torner-Vysloukh-Mihalache ('06))

$$(1) \quad \begin{cases} i\dfrac{\partial \psi_1}{\partial t} + \Delta\psi_1 + \mu_1\psi_1\psi_3 = 0, \\ i\dfrac{\partial \psi_2}{\partial t} + \Delta\psi_2 + \mu_2\psi_2\psi_3 = 0 \\ -\varepsilon^2\Delta\psi_3 + \psi_3 = \mu_1|\psi_1|^2 + \mu_2|\psi_2|^2. \end{cases} \quad \text{in } (0,\infty)\times\mathbf{R}^N,$$

$0 < \mu_2 \le \mu_1, 0 < \varepsilon$: constants, $1 \le N \le 3$,
$\psi_1, \psi_2 : (0,\infty)\times\mathbf{R}^N \to \mathbf{C}$, $\psi_3 : (0,\infty)\times\mathbf{R}^N \to \mathbf{R}$: Unknown.

standing wave sol. $\psi_j(t,x) = e^{i\omega_j t}u_j(x)$ $(j = 1,2)$, $\psi_3(t,x) = u_3(x)$.

$$(2) \quad \begin{cases} -\Delta u_1 + \omega_1 u_1 = \mu_1 u_1 u_3, \\ -\Delta u_2 + \omega_2 u_2 = \mu_2 u_2 u_3 \\ -\varepsilon^2\Delta u_3 + u_3 = \mu_1|u_1|^2 + \mu_2|u_2|^2. \end{cases} \quad \text{in } \mathbf{R}^N,$$

$U = (u_1, u_2, u_3)$: positive sol. $\Leftrightarrow$ $u_1, u_2, u_3 > 0$ in $\mathbf{R}^N$.

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$U = (u_1, u_2, u_3)$: positive sol. $\Leftrightarrow$ $u_1, u_2, u_3 > 0$ in $\mathbf{R}^N$.

Remark

(i) If (2) has a pos. sol. $\Rightarrow$ $\omega_2 \le \omega_1$.
$\therefore (\omega_1 - \omega_2) \int_{\mathbf{R}^N} u_1 u_2 dx = (\mu_1 - \mu_2) \int_{\mathbf{R}^N} u_1 u_2 u_3 dx.$

(ii) (2) has a variational structure:
$I(U) = \sum_{j=1}^{2} (\|\nabla u_j\|_{L^2}^2 + \omega_j \|u_j\|_{L^2}^2)/2 + (\varepsilon^2 \|u_3\|_{L^2}^2 + \omega \|u_3\|_{L^2}^2)/4$
$\quad - \int_{\mathbf{R}^N} (\mu_1 |u_1|^2 + \mu_2 |u_2|^2) u_3 dx/2$

- I satisfies MP geometry.
- Every (PS) sequence is bounded.

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Conserved quantity of (1)

$$Q_1(u_1) := \|u_1\|_{L^2}^2,\ Q_2(u_2) := \|u_2\|_{L^2}^2,$$

$$E_1(U) := \frac{1}{2}(\|\nabla u_1\|_{L^2}^2 + \|\nabla u_2\|_{L^2}^2) + \frac{\|u_3\|_\varepsilon^2}{4} - \frac{1}{2}\int_{\mathbf{R}^N}(\mu_1|u_1|^2 + \mu_2|u_2|^2)u_3 dx$$

where $U = (u_1, u_2, u_3),\ \|u_3\|_\varepsilon^2 := \varepsilon^2\|\nabla u_3\|_{L^2}^2 + \|u_3\|_{L^2}^2$

Minimizing problem:

Let $H := (H^1(\mathbf{R}^N, \mathbf{C}))^2 \times H^1(\mathbf{R}^N, \mathbf{R})$ and $\alpha_1, \alpha_2 > 0$.

$$c_1(\varepsilon, \alpha_1, \alpha_2) := \inf\{E_1(U) : U \in H,\ \|u_1\|_{L^2}^2 = \alpha_1,\ \|u_2\|_{L^2}^2 = \alpha_2\},$$

$$\mathcal{M}(\varepsilon, \alpha_1, \alpha_2) := \{U \in H :\ U \text{ is a minimizer for } c_1(\varepsilon, \alpha_1, \alpha_2)\}$$

Note:

If $U \in \mathcal{M}(\varepsilon, \alpha_1, \alpha_2)$, then U is a sol. of (2) for some $\omega_1, \omega_2 \geq 0$.

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§2 Main results

Theorem 1 (Existence and nonexistence)

(i) (N=1) For any $\alpha_1, \alpha_2 > 0$, $\mathcal{M}(\varepsilon, \alpha_1, \alpha_2) \neq \emptyset$.

(ii) (N=2) There are $0 < \underline{\alpha} \leq \overline{\alpha} < \infty$ such that

$$\max\{\alpha_1, \alpha_2\} < \underline{\alpha} \quad \Rightarrow \quad \mathcal{M}(\varepsilon, \alpha_1, \alpha_2) = \emptyset.$$
$$\overline{\alpha} < \max\{\alpha_1, \alpha_2\} \quad \Rightarrow \quad \mathcal{M}(\varepsilon, \alpha_1, \alpha_2) \neq \emptyset.$$

(iii) (N=3) There are $0 < \underline{\alpha} \leq \overline{\alpha} < \infty$ such that

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Lemma (cf. Montefusco-Pellaci-Squassina ('10))

$$\mathcal{M}(\varepsilon, \alpha_1, \alpha_2) = \Big\{ \Big(e^{i\theta_1} w_1(x-y), e^{i\theta_2} w_2(x-y), w_3(x-y) \Big) :$$

$\theta_1, \theta_2 \in \mathbf{R}, y \in \mathbf{R}^N$, (w_1, w_2, w_3) : sol. of (2) for some $\omega_1 \geq \omega_2 > 0$,

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Let $\alpha = \alpha_1 + \alpha_2$ and consider

$$c_2(\varepsilon, \alpha) := \inf\{E_1(U) : \ U \in H, \ \|u_1\|_{L^2}^2 + \|u_2\|_{L^2}^2 = \alpha\},$$

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Note: $c_2(\varepsilon, \alpha) \leq c_1(\varepsilon, \alpha_1, \alpha_2)$ for $^\forall \alpha_1, \alpha_2 > 0$.

Remark

- $(N = 1)$ $\mathcal{M}(\varepsilon, \alpha) \neq \emptyset$ for $^\forall \alpha > 0$.
- $(N = 2)$ $^\exists \alpha_0 > 0$ $(\alpha_0 \leq \overline{\alpha})$ s.t. $\alpha_0 < \alpha \iff \mathcal{M}(\varepsilon, \alpha) \neq \emptyset$.
- $(N = 3)$ $^\exists \alpha_0 > 0$ $(\alpha_0 < \overline{\alpha})$ s.t. $\alpha_0 < \alpha \Rightarrow \mathcal{M}(\varepsilon, \alpha) \neq \emptyset$.

(i) $\mu_2 < \mu_1$, $\mathcal{M}(\varepsilon, \alpha) \neq \emptyset \Rightarrow$ all elements of $\mathcal{M}(\varepsilon, \alpha)$ have the form $(u_1, 0, u_3)$ and $c_2(\varepsilon, \alpha) < c_1(\varepsilon, \alpha_1, \alpha_2)$ for any $\alpha_1, \alpha_2 > 0$.

(ii) $\mu_1 = \mu_2$, $\mathcal{M}(\varepsilon, \alpha) \neq \emptyset \Rightarrow c_2(\varepsilon, \alpha) = c_1(\varepsilon, \alpha_1, \alpha_2)$ for all α_1, α_2. Indeed, $E_1(\varphi \cos\theta, \varphi \sin\theta, \psi) = E_1(\varphi, 0, \psi) = E_1(0, \varphi, \psi)$ holds.

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We consider the behavior of $\mathcal{M}(\varepsilon, \alpha_1, \alpha_2)$ as $\varepsilon \to 0$ (when $N = 1$).

(3) $$\begin{cases} -v_1'' + \omega_1 v_1 = \tilde{\mu}_1 |v_1|^2 v_1 + \beta |v_2|^2 v_1 & \text{in } \mathbf{R}, \\ -v_2'' + \omega_2 v_2 = \beta |v_1|^2 v_2 + \tilde{\mu}_2 |v_2|^2 v_2 & \text{in } \mathbf{R}. \end{cases}$$

(3) comes from the following equations:

$$\begin{cases} i\dfrac{\partial \psi_1}{\partial t} + \psi_1'' + (\tilde{\mu}_1 |\psi_1|^2 + \beta |\psi_2|^2)\psi_1 = 0, \\ i\dfrac{\partial \psi_2}{\partial t} + \psi_2'' + (\beta |\psi_1|^2 + \tilde{\mu}_2 |\psi_2|^2)\psi_2 = 0. \end{cases}$$

(Formally) (3) appears when $\varepsilon = 0$ in (2) with $\tilde{\mu}_1 = \mu_1^2$, $\tilde{\mu}_2 = \mu_2^2$, $\beta = \mu_1 \mu_2$.

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Minimizing Problem

$$d(\alpha_1, \alpha_2) := \inf\{E_2(V) : \ V \in (H^1(\mathbf{R}, \mathbf{C}))^2, \ \|v_1\|_{L^2}^2 = \alpha_1, \ \|v_2\|_{L^2}^2 = \alpha_2\},$$
$$E_2(V) := \frac{1}{2}\|v_1'\|_{L^2}^2 + \frac{1}{2}\|v_2'\|_{L^2}^2 - \frac{1}{4}\int_{\mathbf{R}} \tilde{\mu}_1|v_1|^4 + 2\beta|v_1|^2|v_2|^2 + \tilde{\mu}_2|v_2|^4 dx,$$
$\mathcal{N}(\alpha_1, \alpha_2) := \{V \in (H^1(\mathbf{R}^N, \mathbf{C}))^2 : \ V$ is a minimizer of $d(\alpha_1, \alpha_2)\}.$

Achievement of $d(\alpha_1, \alpha_2)$ (Cao-Chern-Wei, *Nodea*, Online-First)

- For each $\alpha_1, \alpha_2 > 0$, $d(\alpha_1, \alpha_2)$ is attained.

Note:
By Thm. 1, $\mathcal{M}(\varepsilon, \alpha_1, \alpha_2) \neq \emptyset$ for all $\alpha_1, \alpha_2 > 0$.

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$\mathcal{N}(\alpha_1,\alpha_2) := \{V \in (H^1(\mathbf{R}^N,\mathbf{C}))^2 :\ V$ is a minimizer of $d(\alpha_1,\alpha_2)\}$.

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§2 Main results

Theorem 2

Let $\tilde{\mu}_1 = \mu_1^2, \tilde{\mu}_2 = \mu_2^2, \beta = \mu_1\mu_2$. Then for each $\alpha_1, \alpha_2 > 0$,

(i) $c_1(\varepsilon, \alpha_1, \alpha_2) \to d(\alpha_1, \alpha_2)$ as $\varepsilon \to 0$.

(ii) For any $\eta > 0$, there is an $\varepsilon_\eta > 0$ such that if $0 < \varepsilon \leq \varepsilon_\eta$, then

$$\mathrm{dist}_{H^1 \times H^1}((u_1, u_2), \mathcal{N}(\alpha_1, \alpha_2)) \leq \eta,$$
$$||u_3 - \mu_1|u_1|^2 - \mu_2|u_2|^2||_{H^1} \leq \eta$$

for all $(u_1, u_2, u_3) \in \mathcal{M}(\varepsilon, \alpha_1, \alpha_2)$.

§3 Idea of Proof (Existence Result)

First we consider

$$c_{k,1}(\varepsilon, \alpha_1, \alpha_2) := \inf \{E_1(U) : \ U \in (H_0^1(B_k(0)))^2 \times H_0^1(B_k(0)), \|u_j\|_{L^2}^2 = \alpha_j\}.$$

By the compactness of Sobolev's inequality $\Rightarrow$ $c_{k,1}(\varepsilon, \alpha_1, \alpha_2)$ is always attained.

Let $U_k = (u_{k,1}, u_{k,2}, u_{k,3})$ be a minimizer. Moreover, we can show:

(i) $c_{k,1}(\varepsilon, \alpha_1, \alpha_2) \to c_1(\varepsilon, \alpha_1, \alpha_2)$.

(ii) $u_{k,j}$ $(j = 1, 2, 3)$ is positive, radially symmetric and monotone.

(iii) $^{\exists}\lambda_{k,j} \in \mathbf{R}$ such that

$$\begin{cases} -\Delta u_{k,1} + \lambda_{k,1} u_{k,1} = \mu_1 u_{k,1} u_{k,3}, \\ -\Delta u_{k,2} + \lambda_{k,2} u_{k,2} = \mu_2 u_{k,2} u_{k,3} & \text{in } B_k(0), \\ -\varepsilon^2 \Delta u_{k,3} + u_{k,3} = \mu_1 u_{k,1}^2 + u_{k,2}^2. \end{cases}$$

If $\lambda_{k,j} \to \lambda_{0,j} > 0$, then $(u_{k,j})$ converges $u_{0,j}$ strongly in $L^2(\mathbf{R}^N)$.

The largeness of α_1, α_2 (when $N = 2, 3$) $\Rightarrow$ $\lambda_{0,j} > 0$.

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$$\begin{cases} -\Delta u_{k,1} + \lambda_{k,1}u_{k,1} = \mu_1 u_{k,1}u_{k,3}, & \\ -\Delta u_{k,2} + \lambda_{k,2}u_{k,2} = \mu_2 u_{k,2}u_{k,3} & \text{in } B_k(0), \\ -\varepsilon^2\Delta u_{k,3} + u_{k,3} = \mu_1 u_{k,1}^2 + u_{k,2}^2. & \end{cases}$$

If $\lambda_{k,j} \to \lambda_{0,j} > 0$, then $(u_{k,j})$ converges $u_{0,j}$ strongly in $L^2(\mathbf{R}^N)$.

The largeness of α_1, α_2 (when $N = 2, 3$) $\Rightarrow$ $\lambda_{0,j} > 0$.

§3 Idea of Proof (Existence Result)

First we consider

$$c_{k,1}(\varepsilon,\alpha_1,\alpha_2) := \inf\{E_1(U) : \ U \in (H_0^1(B_k(0)))^2 \times H_0^1(B_k(0)), \|u_j\|_{L^2}^2 = \alpha_j\}.$$

By the compactness of Sobolev's inequality $\Rightarrow$ $c_{k,1}(\varepsilon,\alpha_1,\alpha_2)$ is always attained.

Let $U_k = (u_{k,1}, u_{k,2}, u_{k,3})$ be a minimizer. Moreover, we can show:

(i) $c_{k,1}(\varepsilon,\alpha_1,\alpha_2) \to c_1(\varepsilon,\alpha_1,\alpha_2)$.

(ii) $u_{k,j}$ $(j = 1,2,3)$ is positive, radially symmetric and monotone.

(iii) ${}^{\exists}\lambda_{k,j} \in \mathbf{R}$ such that

$$\begin{cases} -\Delta u_{k,1} + \lambda_{k,1} u_{k,1} = \mu_1 u_{k,1} u_{k,3}, & \\ -\Delta u_{k,2} + \lambda_{k,2} u_{k,2} = \mu_2 u_{k,2} u_{k,3} & \text{in } B_k(0), \\ -\varepsilon^2 \Delta u_{k,3} + u_{k,3} = \mu_1 u_{k,1}^2 + u_{k,2}^2. & \end{cases}$$

If $\lambda_{k,j} \to \lambda_{0,j} > 0$, then $(u_{k,j})$ converges $u_{0,j}$ strongly in $L^2(\mathbf{R}^N)$.

The largeness of α_1, α_2 (when $N = 2,3$) $\Rightarrow$ $\lambda_{0,j} > 0$.