Decay estimates for a solution to the Laplace equation with a dynamical boundary condition and related topics

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Introduction 1. Decay estimates.

Let $N \geq 3$, $q \geq 1$.

• **The Laplace equation with a dynamical boundary condition:**
$\mathbb{R}^N_+ = \{(x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R};\, x_N > 0\}$, $\partial_j := \partial/\partial x_j$, $\varphi \in L^q(\partial\mathbb{R}^N_+)$,
$u = u(x,t) = u(x', x_N, t)$,

$$\text{(LD)} \quad \begin{cases} 0 = \Delta u & \text{in } \mathbb{R}^N_+ \times (0,\infty), \\ \partial_t u = \partial_N u & \text{on } \partial\mathbb{R}^N_+ \times (0,\infty), \\ u|_{t=0} = \varphi & \text{on } \partial\mathbb{R}^N_+ \end{cases}$$

• **Heat equation:** For $\psi \in L^q(\mathbb{R}^N)$,

$$\begin{cases} \partial_t u = \Delta u & \text{in } \mathbb{R}^N \times (0,\infty), \\ u|_{t=0} = \psi & \text{in } \mathbb{R}^N, \end{cases}$$

$$\|u(t)\|_p \leq \frac{C}{t^{\frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}} \|\psi\|_q \quad \text{for } p \geq q.$$

Question 1. L^p-L^q type estimates for (LD)?

Introduction 2. On the harmonic extension operator.

- **The harmonic extention:** For $\varphi : \partial\mathbb{R}^N_+ \to \mathbb{R}$, let u be

$$\text{(HE)} \quad \begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^N_+, \\ u|_{\partial\Omega} = \varphi & \text{on } \partial\mathbb{R}^N_+. \end{cases}$$

- **The harmonic extention operator:** For $P\varphi(x) := u(x)$,

$$\|P\varphi\|_{p;\mathbb{R}^N_+} \le C\|\varphi\|_{\frac{(N-1)p}{N};\partial\mathbb{R}^N_+}, \qquad p > \frac{N}{N-1}. \qquad \text{(HEI)}$$

i.e.,

$$P : L^{\frac{(N-1)p}{N}}(\partial\mathbb{R}^N_+) \to L^p(\mathbb{R}^N_+), \quad \text{bounded.}$$

Question 2. $\|P\varphi\|_{p;\mathbb{R}^N_+} \le C\|\varphi\|^{\sigma}_{s;\partial\mathbb{R}^N_+} \times \boxed{?}$ for $s < \frac{(N-1)p}{N}$?

cf. The Sobolev and the Gagliardo-Nirenberg-Sobolev inequality:

$$\|u\|_\alpha \le C\|\nabla u\|_{\frac{N\alpha}{N+\alpha}}, \qquad \text{(Sobolev)}$$

$$\|u\|_\alpha \le C\|\nabla u\|^{\sigma}_{\beta}\|u\|^{1-\sigma}_{\gamma}, \quad \beta < \frac{N\alpha}{N+\alpha} \qquad \text{(GNS)}$$

Related Results.

- Amann-Fila *Acta Math. Univ. Commenicae Vol. LXVI 2 (1997).*

$N \geq 2$, $q \in (1, \infty)$, $\varphi \geq 0$, bounded, unif. conti.,

$$\begin{cases} 0 = \Delta u & \text{in} \quad \mathbb{R}^N_+ \times (0, T_m), \\ \partial_t u - \partial_N u = u^q & \text{on} \quad \partial\mathbb{R}^N_+ \times (0, T_m), \\ u|_{t=0} = \varphi & \text{on} \quad \partial\mathbb{R}^N_+ \end{cases}$$

$\Rightarrow$ Fujita exponent $q_* = N/(N-1)$

$(T_m < \infty$ if $q \leq q_*$, $T_m \leq \infty$ if $q > q_*)$.

- Hang-Wang-Yan *CPAM Vol. LXI, 54-95 (2008)*

$$\boxed{\|P\varphi\|_{p;\mathbb{R}^N_+} \leq C\|\varphi\|_{\frac{(N-1)p}{N};\partial\mathbb{R}^N_+}, \quad p > \frac{N}{N-1}} \qquad \text{(HEI)}$$

  - $\exists$maximizer for any p (lack of the compactness).
  - $p = 2N/(N-2) \Rightarrow$
    - all maximizers are of the form $f(x') = \frac{c}{[\lambda^2 + |x' - x'_0|^2]^{\frac{N-2}{2}}}$ for some $\lambda > 0$ and $x'_0 \in \partial\Omega$.
    - best contant.

Main Theorems.

Let $N \geq 3$, $\Omega := \mathbb{R}^N_+$.

Theorem 1.
(i) $q \geq 1$ and $\frac{N-1}{q} - \frac{N}{r} > 0$, or (ii) $q > 1$ and $\frac{N-1}{q} - \frac{N}{r} = 0$
$\Rightarrow \quad \exists C > 0 \quad$ s.t. $\quad \|u(t)\|_{r;\Omega} \leq \frac{C}{t^{\frac{N-1}{q} - \frac{N}{r}}} \|\varphi\|_{q;\partial\Omega}$.

($\Leftarrow$) Direct estimates for a fundamental sol. for (LD) *basically....*

Theorem 2.
$p > \frac{N}{N-1}$, $s \in [1, \frac{(N-1)p}{N}]$, $A := -(-\Delta_{N-1})^{1/2}$
$\Rightarrow \quad \exists C > 0 \quad$ s.t. $\quad \|P\varphi\|_{p;\Omega} \leq C \|\varphi\|^{\frac{1}{1+\delta}}_{s;\partial\Omega} \|A\varphi\|^{1-\frac{1}{1+\delta}}_{\frac{(N-1)p}{N};\partial\Omega}$
for $\varphi \in L^s(\partial\Omega)$, $A\varphi \in L^{\frac{(N-1)p}{N}}(\partial\Omega)$, where $\delta := \frac{N-1}{s} - \frac{N}{p}$.

($\Leftarrow$) Application of (HEI) and Theorem 1.
cf. $\quad s :, 1 \uparrow \frac{(N-1)p}{N} \quad \Leftrightarrow \quad \frac{1}{1+\delta} : \frac{p}{N(p-1)} \uparrow 1$.

Proof of Theorem 1.

- **Solution formula:** For $P_\sigma(x') := \frac{1}{\sigma^{N-1}} c_N [\frac{1}{1+|\frac{x'}{\sigma}|^2}]^{\frac{N}{2}}$, $x' \in \partial\Omega$,

$$u(x', x_N, t) = P_{x_N+t} * \varphi = \int_{\partial\Omega} dy' P_{x_N+t}(x'-y')\varphi(y').$$

Amann-Fila *Acta Math. Univ. Commenicae Vol. LXVI 2 (1997)*.

- **Decay of norm on the hyperplane** $x_N = a > 0$**:**

$$\|u(t)\|^r_{r,\mathbb{R}^N_+} = \int_{\mathbb{R}+} dx_N \int_{\mathbb{R}^{N-1}} dx' |u(x', x_N, t)|^r$$

$$= \int_{\mathbb{R}+} dx_N \|u(\cdot, x_N, t)\|^r_{r;\mathbb{R}^{N-1}}.$$

Lemma 1. For $r \geq q \geq 1$, $\exists L > 0$ s.t.

$$\|u(\cdot, a, t)\|_{r;\mathbb{R}^{N-1}} \leq \frac{L}{(a+t)^{(N-1)\left(\frac{1}{q}-\frac{1}{r}\right)}} \|\varphi\|_{q;\partial\Omega}, \quad \forall a > 0.$$

cf. Propagation of "decay" from the hyperplane $x_N = a$ to $x_N = 0$, i.e.,

The situation at $(x_N = a,\ t = 0) \simeq (x_N = 0,\ t = a)$.

Aim: $\|u(t)\|_{r;\Omega} \leq \frac{C}{t^{\frac{N-1}{q}-\frac{N}{r}}} \|\varphi\|_{q;\partial\Omega}$.

- **Case (i):** $q \geq 1,\ \frac{N-1}{q} - \frac{N}{r} > 0 \overset{\text{Lemma1}}{\Longrightarrow}$

$$\|u(t)\|_r^r \leq L \int_{\mathbb{R}^+} dx_N \frac{L^r}{(x_N+t)^{(N-1)\left(\frac{1}{q}-\frac{1}{r}\right)r}} \|\varphi\|_{q,\partial\Omega}^r \leq \frac{C}{t^{r\left(\frac{N-1}{q}-\frac{N}{r}\right)}} \|\varphi\|_{q;\partial\Omega}.$$

$$cf.\ 1-(N-1)\left(\frac{1}{q}-\frac{1}{r}\right)r = 1-(N-1)\frac{r}{q}+(N-1)$$
$$= N-(N-1)\frac{r}{q} = -r\left(\frac{N-1}{q}-\frac{N}{r}\right).$$

- **Case (ii):** $q > 1,\ \frac{N-1}{q} - \frac{N}{r} = 0 \Rightarrow$

$$\int_{\mathbb{R}^+} \frac{dx_N}{(x_N+t)^{(N-1)\left(\frac{1}{q}-\frac{1}{r}\right)r}} = \int_{\mathbb{R}^+} \frac{dx_N}{x_N+t} = \infty.$$

- L^1 **and** $L^{1,\infty}$**:** $\left\|\frac{1}{\cdot+t}\right\|_1 = \int_{\mathbb{R}^+} \frac{dx_N}{x_N+t} = \infty$, but $\left\|\frac{1}{\cdot+t}\right\|_{1,\infty} \leq 1$.

  $\Rightarrow$ Marcinkiewicz interpolation theorem? No...?

Abstract structure of (LD).

$$Ff(\xi') := \left(\frac{1}{2\pi}\right)^{\frac{N-1}{2}} \int_{\mathbb{R}^{N-1}} dx' e^{-i\xi' x'} f(x'),$$
$$F^{-1}g(x') := \left(\frac{1}{2\pi}\right)^{\frac{N-1}{2}} \int_{\mathbb{R}^{N-1}} d\xi' e^{i\xi' x'} g(\xi')$$

:Fourier transformation in the variable $x' \in \mathbb{R}^{N-1} (\simeq \partial\Omega)$.

$$\begin{aligned} 0 &= \Delta u \quad \text{in} \quad \mathbb{R}^N_+ \times (0,\infty), && (1)\\ \partial_t u &= \partial_N u \quad \text{on} \quad \partial\mathbb{R}^N_+ \times (0,\infty), && (2)\\ u|_{t=0} &= \varphi \quad \text{on} \quad \partial\mathbb{R}^N_+ \quad . \end{aligned}$$

- **Solving (1), (2) by the Fourier transformation:**
  - (1)

$$\Rightarrow \frac{\partial^2}{\partial x_N^2} u(x', x_N, t) = -\sum_{j=1}^{N-1} \frac{\partial^2}{\partial x_j^2} u(x', x_N, t).$$
$$\overset{F.tr.}{\Rightarrow} \frac{\partial^2}{\partial x_N^2} Fu(\xi', x_N, t) = -|\xi'|^2 Fu(\xi', x_N, t)$$
$$\Rightarrow Fu(\xi', x_N, t) = e^{-|\xi'|x_N} Fu(\xi', 0, t) \quad (3)$$

- (3)
$$\Rightarrow \frac{\partial}{\partial x_N} Fu(\xi', x_N, t) = -|\xi'| e^{-|\xi'| x_N} Fu(\xi', 0, t)$$
$$\Rightarrow \frac{\partial}{\partial x_N} Fu(\xi', x_N, t)\Big|_{x_N=0} = -|\xi'| Fu(\xi', 0, t). \tag{4}$$
- (2)
$$\Rightarrow \frac{\partial}{\partial t} Fu(\xi', 0, t) = \frac{\partial}{\partial x_N} Fu(\xi', x_N, t)\Big|_{x_N=0}$$
$$\overset{(4)}{\Rightarrow} \boxed{\frac{\partial}{\partial t} Fu(\xi', 0, t) = -|\xi'| Fu(\xi', 0, t)} \tag{5}$$
$$\Rightarrow Fu(\xi', 0, t) = e^{-t|\xi'|} Fu(\xi', 0, 0) = e^{-t|\xi'|} F\varphi(\xi') \tag{6}$$
- (3), (6) $\Rightarrow$
$Fu(\xi', x_N, t) = e^{-|\xi'| x_N} e^{-|\xi'| t} F\varphi(\xi') = e^{-|\xi'|(x_N + t)} F\varphi(\xi')$.
- **Solution formula:** For $P_\sigma(x') := \frac{1}{\sigma^{N-1}} c_N [\frac{1}{1+|\frac{x'}{\sigma}|^2}]^{\frac{N}{2}}$, $x' \in \partial\Omega$,
$$u(x', x_N, t) = P_{x_N+t} * \varphi = \int_{\partial\Omega} dy' P_{x_N+t}(x' - y')\varphi(y').$$

- **Equation on the boundary:** On the boundary $\partial\Omega$ ($x_N = 0$), (5)

$\Rightarrow u_t(x', 0, t) = -F^{-1}|\xi'|Fu(x', 0, t)$

$\Rightarrow \boxed{u_t = -(-\Delta_{N-1})^{1/2} u}$: evolution equation by $-(-\Delta_{N-1})^{1/2}$.

- **The structure of (LD):** (LD) can be solved by
  - **Step 1.** Solve the problem on $\partial\Omega$.
    Find a solution $v(t)$ for the (IVP) on $\partial\Omega$:

$$\begin{cases} \partial_t v &= -(-\Delta_{N-1})^{\frac{1}{2}} v \quad \text{in} \quad \partial\Omega \times (0, \infty), \\ v|_{t=0} &= \varphi \quad \text{in} \quad \partial\Omega. \end{cases}$$

  - **Step 2.** Take a harmonic extension from $\partial\Omega$ to Ω.

$$u(t) := Pv(t)$$

$$\Downarrow$$

$$u(t) = Pe^{-t(-\Delta_{N-1})^{\frac{1}{2}}}\varphi =: Pe^{tA}\varphi$$

Proof of Theorem 1, suite.

Aim: $q > 1$, $\frac{N-1}{q} - \frac{N}{r} = 0 \Rightarrow \|u(t)\|_{r;\Omega} \leq C\|\varphi\|_{q;\partial\Omega}$.

- **Representation of $u(t)$:** $u(t) = Pe^{tA}\varphi$, $A = -(-\Delta_{N-1})^{\frac{1}{2}}$.
- **Use of the (HEI):**
  $q > 1$ and $r = Nq/(N-1) \Rightarrow r > N/(N-1)$. Hence
  $\|u(t)\|_{r;\Omega} = \|Pe^{tA}\varphi\|_{r;\Omega} \leq C_1\|e^{tA}\varphi\|_{r\frac{N-1}{N};\partial\Omega}$.
- **Use of the decay estimate for e^{tA}:**
  $\|e^{tA}\varphi\|_{r\frac{N-1}{N};\partial\Omega} \leq \frac{C_2}{t^{(N-1)\left(\frac{1}{q}-\frac{N}{N-1}\frac{1}{r}\right)}}\|\varphi\|_{q;\partial\Omega} = \frac{C_2}{t^{\frac{N-1}{q}-\frac{N}{r}}}\|\varphi\|_{q;\partial\Omega}$.
- **Conclusion:** Relations above yield

$$\|u(t)\|_{r;\Omega} \leq \frac{C}{t^{\frac{N-1}{q}-\frac{N}{r}}}\|\varphi\|_{q;\partial\Omega} = C\|\varphi\|_{q;\partial\Omega}.$$

Also works for Case (i)...

Proof of Theorem 2.

- Aim: $p > \frac{N}{N-1}$, $s \in [1, \frac{(N-1)p}{N}]$, $\delta := \frac{N-1}{s} - \frac{N}{p}$,
$$\|P\varphi\|_{p;\Omega} \le C\|\varphi\|_{s;\partial\Omega}^{\frac{1}{1+\delta}} \|A\varphi\|_{\frac{(N-1)p}{N};\partial\Omega}^{1-\frac{1}{1+\delta}}.$$
- Tool: (LD) $\Leftrightarrow$
$$\begin{cases} \partial_t v = -(-\Delta_{N-1})^{\frac{1}{2}} v, \ v|_{t=0} = \varphi & \text{on} \quad \partial\Omega \times (0,\infty), \\ u = Pv & \text{in} \quad \Omega \times (0,\infty). \end{cases}$$

• The characterization of $\|\cdot\|_p$ by duality:

$$\|P\varphi\|_{p;\Omega} = \sup\left\{ \int_\Omega dx P\varphi f; \ f \in L^{p'}(\Omega), \ \|f\|_{p';\Omega} \le 1 \right\}.$$

$\forall f \in L^{p'}(\Omega)$ with $\|f\|_{p';\Omega} \le 1$: fix. $\boxed{\int_\Omega dx P\varphi f = ?}$

• Representation of $P\varphi$: $u(t) = Pe^{-t(-\Delta_{N-1})^{\frac{1}{2}}}\varphi =: Pe^{tA}\varphi$

$\Rightarrow u(t) - u(0) = \int_0^t ds \frac{d}{ds} Pe^{sA}\varphi$

$\Rightarrow u(t) - P\varphi = \int_0^t ds PAe^{sA}\varphi$, i.e., $\boxed{P\varphi = u(t) - \int_0^t ds PAe^{sA}\varphi}$

$$\Rightarrow \int_\Omega dxP\varphi f = \int_\Omega dxu(t)f - \int_\Omega dxf \int_0^t dsPAe^{sA}\varphi.$$

- **For the first term:** $\|u(t)\|_{r;\Omega} \le \frac{C}{t^{\frac{N-1}{q}-\frac{N}{r}}}\|\varphi\|_{q;\partial\Omega} \Rightarrow \delta := \frac{N-1}{s} - \frac{N}{p}$,

$$\left|\int_\Omega dxu(t)f\right| \le \|u(t)\|_{p;\Omega}\|f\|_{p';\Omega} \le \frac{C_1}{t^\delta}\|\varphi\|_{s;\partial\Omega}\|f\|_{p';\Omega} \le \frac{C_1}{t^\delta}\|\varphi\|_{s;\partial\Omega}$$

- **For the second term:** $\|P\varphi\|_{p;\mathbb{R}^N_+} \le C\|\varphi\|_{\frac{(N-1)p}{N};\partial\mathbb{R}^N_+} \Rightarrow$

$$\left|\int_\Omega dxfPAe^{sA}\varphi\right| \le \|PAe^{sA}\varphi\|_{p;\Omega}\|f\|_{p';\Omega}$$
$$\le \|PAe^{sA}\varphi\|_{p;\Omega} \le C_2\|Ae^{sA}\varphi\|_{\frac{(N-1)p}{N};\partial\Omega}$$
$$= C_2\|e^{sA}A\varphi\|_{\frac{(N-1)p}{N};\partial\Omega} \le C_2\|A\varphi\|_{\frac{(N-1)p}{N};\partial\Omega}.$$

$\Rightarrow$

$$\left|\int_\Omega dxf \int_0^t dsPAe^{sA}\varphi\right| = \left|\int_0^t ds \int_\Omega dxfPAe^{sA}\varphi\right| \le tC_2\|A\varphi\|_{\frac{(N-1)p}{N};\partial\Omega}.$$

- **Conclusion:** Consequently,

$$\|P\varphi\|_{p;\Omega} = \left|\int_\Omega dx P\varphi f\right| \leq C\left(\frac{1}{t^\delta}\|\varphi\|_{s;\partial\Omega} + t\|A\varphi\|_{\frac{(N-1)p}{N};\partial\Omega}\right).$$

Take t s.t.

$$\frac{1}{t^\delta}\|\varphi\|_{s;\partial\Omega} = t\|A\varphi\|_{\frac{(N-1)p}{N};\partial\Omega} \Leftrightarrow t = \left(\frac{\|\varphi\|_{s;\partial\Omega}}{\|A\varphi\|_{\frac{(N-1)p}{N};\partial\Omega}}\right)^{\frac{1}{1+\delta}}.$$

$\Rightarrow$

$$\begin{aligned}
\frac{1}{t^\delta}\|\varphi\|_{s;\partial\Omega} + t\|A\varphi\|_{\frac{(N-1)p}{N};\partial\Omega} &\leq 2t\|A\varphi\|_{\frac{(N-1)p}{N};\partial\Omega} \\
&= 2\left(\frac{\|\varphi\|_{s;\partial\Omega}}{\|A\varphi\|_{\frac{(N-1)p}{N};\partial\Omega}}\right)^{\frac{1}{1+\delta}} \|A\varphi\|_{\frac{(N-1)p}{N};\partial\Omega} \\
&= 2\|\varphi\|_{s;\partial\Omega}^{\frac{1}{1+\delta}}\|A\varphi\|_{\frac{(N-1)p}{N};\partial\Omega}^{1-\frac{1}{1+\delta}}.
\end{aligned}$$

Open problems.

- **On Theorem 1:**
  - Find direct and simple proof for Case (ii) which does not use the decay estimates for $e^{-t(-\Delta_{N-1})^{1/2}}$.
  - Prove the anarogous result for unbounded domains different from $\mathbb{R}^N_+$.
  - p-harmonic extension case? ($\exists$abstract framework?)
  - Apply Theorem 1 to some nonlinear problem.
- **On Theorem 2:**
  - Prove $\|P\varphi\|_{p;\Omega} \leq C\|\varphi\|^{\sigma}_{s;\partial\Omega}\|A\varphi\|^{1-\sigma}_{q;\partial\Omega}$ for $q \neq (N-1)p/N$.
  - Find a maximizer $\psi : \partial\Omega \to \mathbb{R}$ which attains $\sup_{\varphi} \frac{\|P\varphi\|_{p;\Omega}}{\|\varphi\|^{\sigma}_{s;\partial\Omega}\|A\varphi\|^{1-\sigma}_{q;\partial\Omega}}$.

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$$\sup_{\varphi} \frac{\|P\varphi\|_{p;\Omega}}{\|\varphi\|^{\sigma}_{s;\partial\Omega}\|A\varphi\|^{1-\sigma}_{q;\partial\Omega}}.$$

Thank you for your kind attention!