Imperfect bifurcation for the Liouville-Gel'fand equation on a perturbed annulus

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The Liouville-Gel'fand equation

$$\text{(LGE)} \quad \begin{cases} \Delta u + \lambda\, e^{u} = 0 & \text{in } \ \Omega, \\ u = 0 & \text{on } \ \partial\Omega. \end{cases}$$

- $\lambda > 0$: bifurcation parameter
- $\Omega \subset \mathbb{R}^2$: bounded domain

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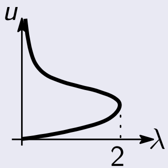
Theorem (Gel'fand '63)

Let Ω be the unit disc. Then the number of positive (radially symmetric) solutions of (LGE) is

exactly two if $0 < \lambda < 2$,

one if $\lambda = 2$,

zero if $\lambda > 2$.



The Liouville-Gel'fand equation

Theorem (Song-Sun Lin '89)

If Ω is an annulus, then the followings hold.

(i) $\exists\lambda^* > 0$ s.t. the number of positive radially symmetric solutions of (LGE) is

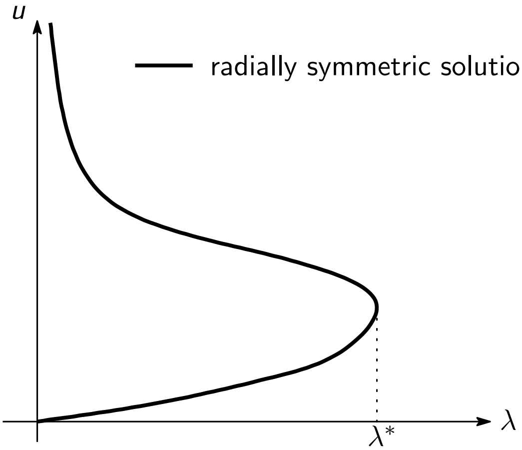
$$\text{exactly two if } 0 < \lambda < \lambda^*,$$
$$\text{one if } \lambda = \lambda^*,$$
$$\text{zero if } \lambda > \lambda^*.$$

(ii) $\exists\{\lambda_k\}_{k=1}^{\infty} \subset (0, \lambda^*)$ with $\lambda_1 > \lambda_2 > \lambda_3 > \cdots \to 0$ s.t.

$$(\lambda_k, u_{\lambda_k}) \text{ is a symmetry-breaking bifurcation point.}$$

u_λ $(0 < \lambda < \lambda^*)$: Bigger one of positive radially symmetric solutions

Bifucation diagram (the case of an annulus)



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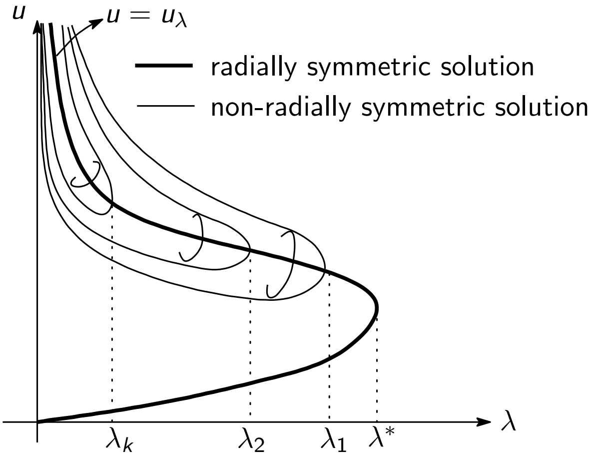

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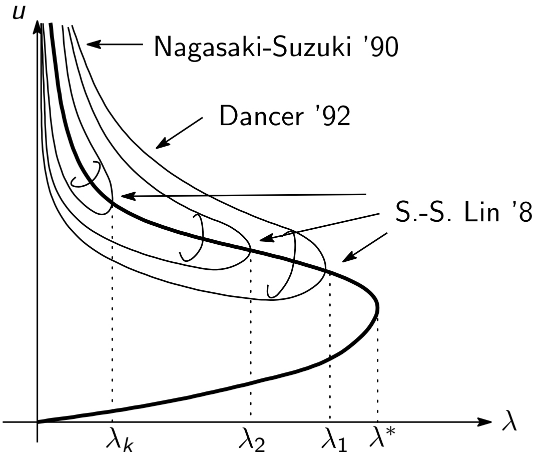
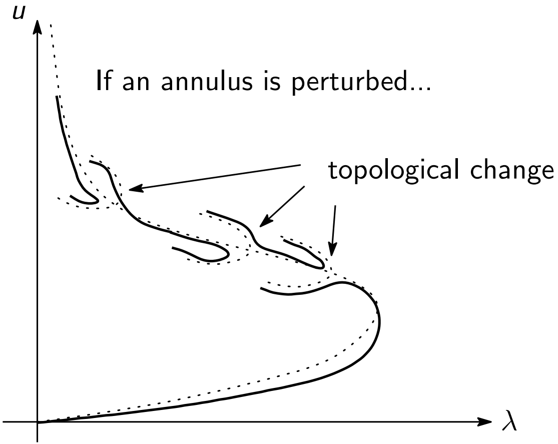


Bifucation diagram (the case of a perturbed annulus)



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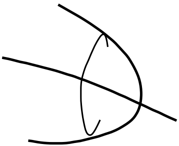
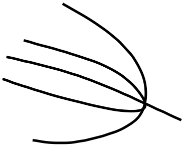
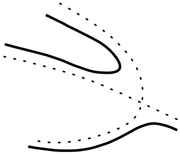


persistence



parasol-like structure



break-up

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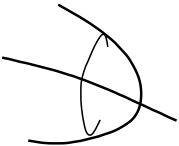

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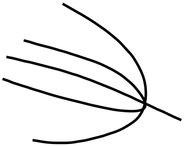

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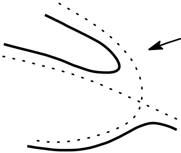

break-up

What is the domain perturbation for the break-up?

Perturbed annular domain

$$\Omega_\varepsilon := \{(r\cos\theta, r\sin\theta); a + \varepsilon g(\theta) < r < b + \varepsilon h(\theta), \theta \in \mathbb{R}/2\pi\mathbb{Z}\}$$

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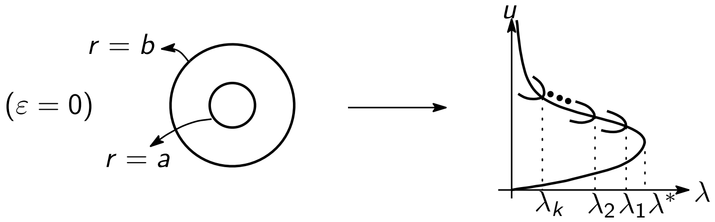

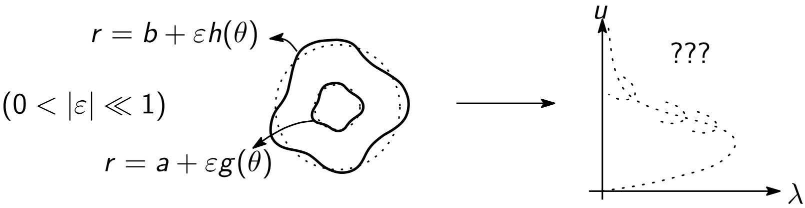

Main result

Theorem (Imperfect bifurcation caused by a domain perturbation)

- $h(\theta) = g(\theta)$, $g(\theta) = g(-\theta)$ $(\forall \theta \in \mathbb{R}/2\pi\mathbb{Z})$

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- $h(\theta) = g(\theta)$, $g(\theta) = g(-\theta)$ $(\forall\theta \in \mathbb{R}/2\pi\mathbb{Z})$

$\Rightarrow$ If an integer k satisfies $k \geq 2$,

$$\int_0^{\pi} g(\theta) \cos k\theta d\theta \neq 0,$$

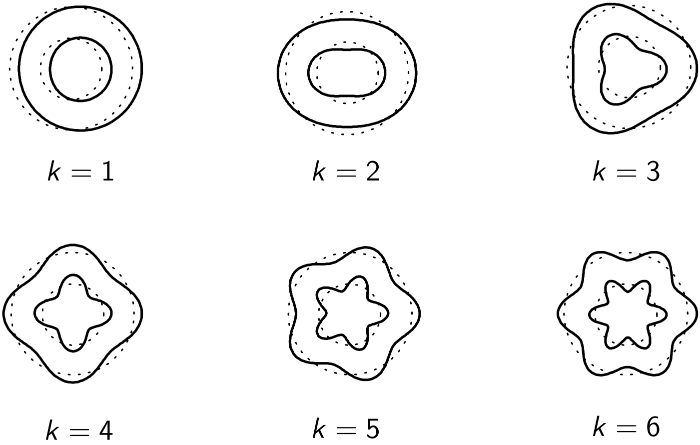
and

$$(k+1) \log \frac{b}{a} \ll 1 \quad \text{or} \quad (k+1) \log \frac{b}{a} \gg 1,$$

the solution set near $(\lambda_k, u_{\lambda_k})$ breaks up. More precisely, $\exists\varepsilon_{a,b,k} > 0$ such that for $0 < |\varepsilon| < \varepsilon_{a,b,k}$,

solutions which are symmetric about x-axis and are in a neighborhood of $(\lambda_k, u_{\lambda_k})$ consist only of two curves which do not intersect.

Domains for the case where $h(\theta) = g(\theta) = \cos k\theta$



Domains $a + \varepsilon \cos k\theta < r < b + \varepsilon \cos k\theta$

Sketch of the proof

- Lyapunov-Schmidt reduction

  ϕ ($\|\phi\|_{L^2} = 1$) : The nontrivial solution of

$$\begin{cases} \Delta\phi + \lambda_k e^{u_{\lambda_k}} \phi = 0 & \text{in } \Omega_0, \\ \phi = 0 & \text{on } \partial\Omega_0, \\ \phi(x,y) = \phi(x,-y). \end{cases}$$

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$$\lambda = \lambda_k + s,\ u = u_{\lambda_k} + t\phi + v\ (s,t \in \mathbb{R}, \langle v, \phi\rangle_{L^2} = 0)$$

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$\longrightarrow \exists_1 v = v(x, y\,; s, t, \varepsilon)$ s.t.

$$v(\cdot\,; s, t, \varepsilon) \text{ is a solution of } (*),$$
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$$\Downarrow$$

$$\underline{\langle \Delta(u_k + t\phi + v(x, y\,; s, t, \varepsilon)) + (\lambda_k + s)\, e^{u_k+t\phi+v(x,y\,;s,t,\varepsilon)}, \ \phi\rangle_{L^2}} = 0$$
$$=: G(s, t, \varepsilon)$$

Sketch of the proof

$$G(s,t,\varepsilon) = \alpha s^2 + 2\beta st + \gamma t^2 + \delta\varepsilon + (\text{higher order terms})$$

$$\left(\begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} : \textit{indefinite}\right)$$

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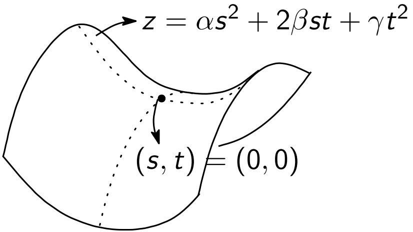

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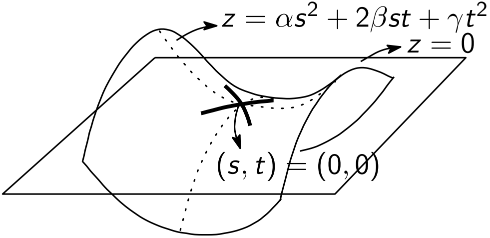

heavy line : solution curve for $\varepsilon = 0$

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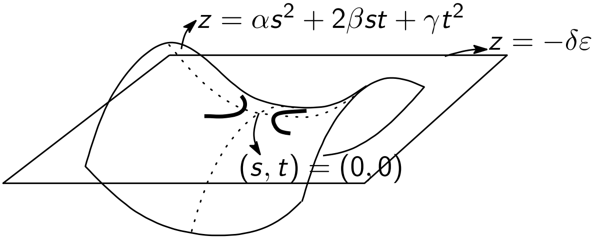

heavy line : solution curve for $\varepsilon \neq 0$ (when $\delta \neq 0$)