The location of the hot spot in a grounded convex conductor

ROLANDO MAGNANINI
joint paper with
Lorenzo BRASCO
and
Paolo SALANI

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In mathematical terms, we considere the IBVP:

$$\begin{cases} u_t = \Delta u & \text{in} \quad \Omega \times (0, \infty), \\ u = 1 & \text{on} \quad \Omega \times \{0\}, \\ u = 0 & \text{on} \quad \partial\Omega \times (0, \infty). \end{cases}$$

Here Ω — the *heat conductor* — is a bounded domain in the Euclidean space $\mathbb{R}^N$, $N \geq 2$, with Lipschitz boundary and $u = u(x, t)$ denotes the normalized temperature of the conductor at a point $x \in \Omega$ and time $t > 0$.

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A *hot spot* $x(t)$ is a point such that

$$u(x(t), t) = \max_{\overline{\Omega}} u(\cdot, t).$$

The hot spot of a convex conductor

If Ω is convex — in this case $\overline{\Omega}$ is said a *convex body* that we shall denote by $\mathcal{K}$ — results of Brascamp & Lieb (1976) and Korevaar () imply that

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$$\log u(x,t) \text{ is concave in } x \text{ for every } t > 0.$$

Based on this result and the analyticity of u in x, we have that for every $t > 0$

$$\exists! \text{ hot spot } x(t) \in \mathcal{K} \text{ and } \nabla u(x(t),t) = 0.$$

Evolution of the hot spot

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SHORT TIMES. Since, by a result of Varadhan,

$$-4t \log\{1 - u(x,t)\} \to \mathrm{dist}(x, \partial\Omega)^2$$

uniformly for $x \in \overline{\Omega}$ as $t \to 0^+$, we can claim that

$$\mathrm{dist}(x(t), \mathcal{M}) \to 0 \quad \text{as } t \to 0^+,$$
$$\mathrm{dist}(x(t), \partial\Omega) \to r_\Omega \quad \text{as } t \to 0^+,$$

where

$$\mathcal{M} = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) = r_\Omega\}$$

and

$$r_\Omega = \max\{\mathrm{dist}(y, \partial\Omega) \ : \ y \in \overline{\Omega}\}$$

is the **inradius** of Ω.

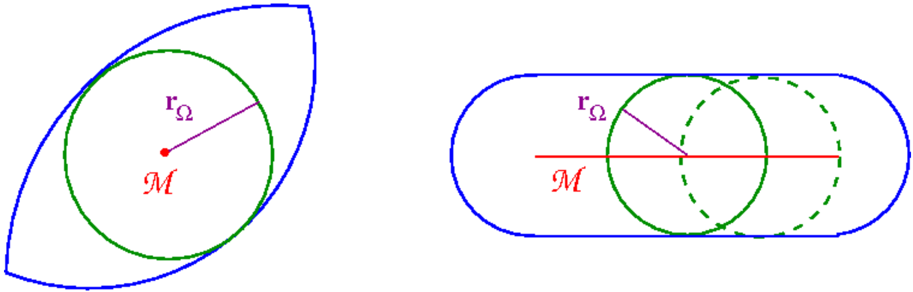

Figura: Two examples for the set $\mathcal{M}$.

Evolution of the hot spot

LARGE TIMES. Let ϕ_1 be the first Dirichlet eigenfunction of $-\Delta$ in Ω, i.e.

$$\Delta\phi_1 + \lambda_1\phi_1 = 0 \ \text{ and } \ \phi_1 > 0 \ \text{ in } \Omega\,, \quad \phi_1 = 0 \quad \text{on } \partial\Omega\,.$$

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Since $e^{\lambda_1 t}u(\cdot, t)$ converges to ϕ_1 locally uniformly in C^2 as $t \to \infty$, then, for a convex body $\mathcal{K}$,

$$x(t) \to x_\infty \text{ as } t \to \infty\,,$$

where x_∞ is the (unique) maximum point in $\mathcal{K}$ of ϕ_1.

The location of the hot spot

Remarks

1. It is relatively easy to locate the set $\mathcal{M}$ by geometrical means.
2. Saying that $x(t) \to x_\infty$ as $t \to \infty$ does not give much information: locating either $x(t)$ or x_∞ has more or less the same difficulty.

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Research proposal

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Known result: Grieser & Jerison, JAMS 1998

In the plane they estimate:

$$|x_\infty - \overline{x}| \leq C,$$

where $\overline{x}$ is the unique maximum point of a one-dimensional eigenfunction related to $-\Delta$ and $\mathcal{K}$. The estimate is uniform w.r.t. the ratio $r_\mathcal{K}/\delta_\mathcal{K}$ (sse figure).

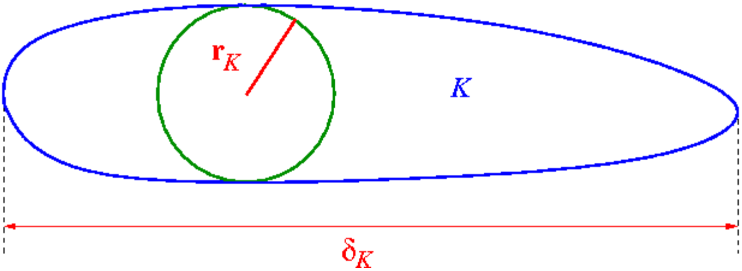

Figura: Totsu shuugou ippon.

Methods for locating the hot spot

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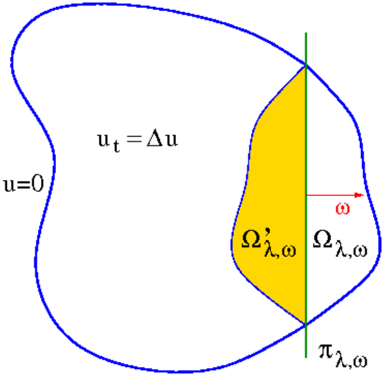
2. The latter is based on ideas related to the **Alexandrov-Bakelmann-Pucci maximum principle** and convex geometry.

Alexandrov's reflection principle

Fix a direction $\omega \in \mathbb{S}^{N-1}$ and a parameter $\lambda \in \mathbb{R}$ define the sets

$$\pi_{\lambda,\omega} = \{x \in \mathbb{R}^N : x \cdot \omega = \lambda\}, \quad \Omega_{\lambda,\omega} = \{x \in \Omega : x \cdot \omega > \lambda\},$$
$$\Omega'_{\lambda,\omega} = \text{ reflection of } \Omega_{\lambda,\omega} \text{ in the plane } \pi_{\lambda,\omega}.$$



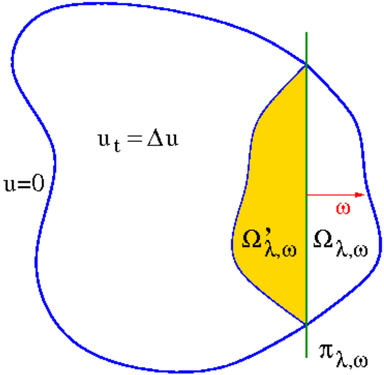
Alexandrov's reflection principle

Proposition

Let Ω be a bounded domain in $\mathbb{R}^N$ with Lipschitz continuous boundary $\partial\Omega$. As long as

$$\overline{\Omega_{\lambda,\omega} \cup \Omega'_{\lambda,\omega}} \subset \overline{\Omega},$$

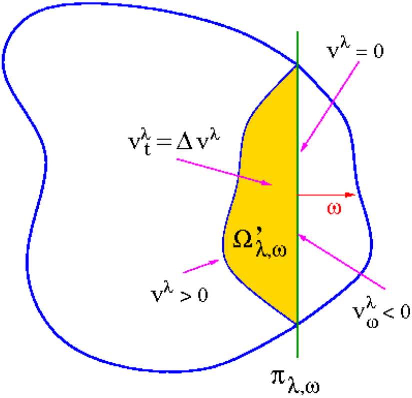
then $\pi_{\lambda,\omega} \cap \Omega$ cannot contain any critical point of *u*.



Alexandrov's reflection principle

Proof.

$$x^\lambda = \text{ reflection of } x \text{ in } \pi_{\lambda,\omega},$$
$$u^\lambda(x,t) = u(x^\lambda,t),$$
$$v^\lambda(x,t) = u(x,t) - u^\lambda(x,t),$$



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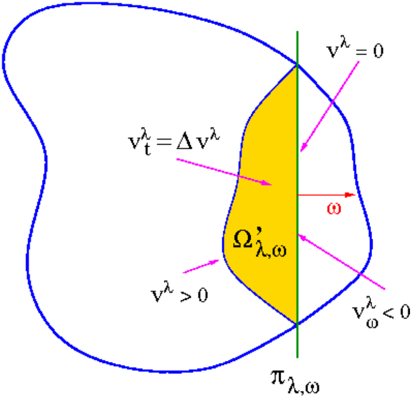

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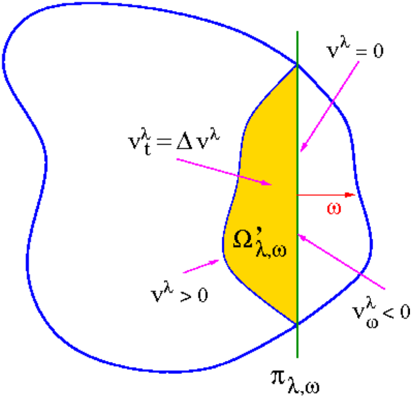

$$v_t^\lambda = \Delta v^\lambda \text{ in } \Omega'_{\lambda,\omega} \times (0,\infty),$$
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$$\Rightarrow \quad v^\lambda > 0 \text{ on } \Omega'_{\lambda,\omega} \times (0,\infty),$$

Hopf

$$\Rightarrow \quad 2u_\omega = v_\omega^\lambda < 0 \text{ on } \pi_{\lambda,\omega} \cap \Omega.$$

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These remarks motivate our interest in the set

$$\heartsuit(\Omega) = \bigcap \{\Omega \setminus \Omega_{\lambda,\omega} : \Omega'_{\lambda,\omega} \subset \Omega\},$$

that we call the **heart** of Ω.

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8. if $j = N$, then $\heartsuit(\mathcal{K})$ reduces to a single point and hence the hot spot **does not move.**

Stationary hot spot

When **8** occurs, we say that the **hot spot is stationary.**

PROBLEM (Klamkin, Siam Review 1994)

Can you characterize the convex conductors for which the hot spot does not move?

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4. hexagones circumscribed to a circle $\rightarrow$ hexagons invariant w.r.t. rotations of angles $\pi/3, 2\pi/3, \pi$;
5. general formula relating the (stationary) hot spot and the curvatures of certain subsets of $\partial\mathcal{K}$.

The maximal folding function

If we define the **maximal folding function** as

$$\mathcal{R}_{\mathcal{K}}(\omega) := \min\{\lambda \in \mathbb{R} \ : \ K'_{\lambda,\omega} \subseteq \mathcal{K}\}, \ \ \omega \in \mathbb{S}^{N-1},$$

then

$$\heartsuit(\mathcal{K}) = \{x \in \mathbb{R}^N : x \cdot \omega \leq R_{\mathcal{K}}(\omega) \ \text{ for every } \ \omega \in \mathbb{S}^{N-1}\}.$$

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Examples

1. $K = B(0, R) \Rightarrow R_{\mathcal{K}} \equiv 0$;
2. $\mathcal{K} =$ ellipse with semi-axes $a > b \Rightarrow$

$$R_{\mathcal{K}}(\omega) = \frac{a^2 - b^2}{\sqrt{b^2\omega_1^2 + a^2\omega_2^2}}\,|\omega_1\omega_2|;$$

the curve $\omega \mapsto R_{\mathcal{K}}(\omega)\,\omega$ is (an affine image of) a **rhodonea** with 4 petals.

The midpoint function

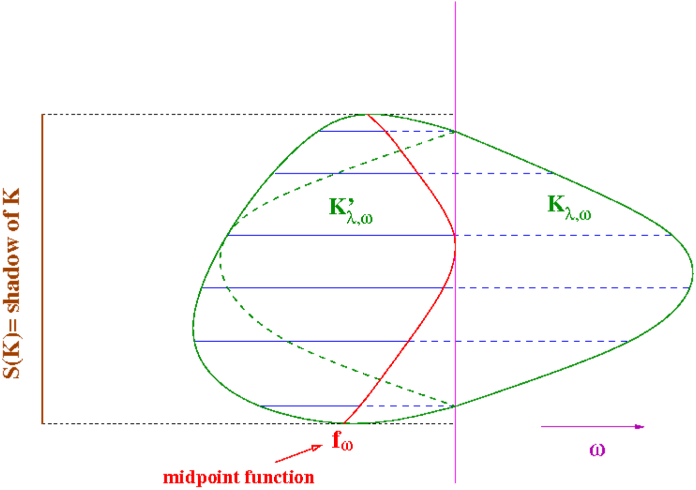

Figura: Definition of the midpoint function $f_\omega : \mathcal{S}(\mathcal{K}) \to \mathbb{R}$.

The midpoint function

A formula for f_ω using the Fourier transform of $1_\mathcal{K}$

$$f_\omega(x') = \frac{i \int_{\omega^\perp} \partial_\omega \hat{1}_\mathcal{K}(\eta)\, e^{ix'\cdot\eta}\, d\eta}{\int_{\omega^\perp} \hat{1}_\mathcal{K}(\eta)\, e^{ix'\cdot\eta}\, d\eta}, \ \ x' \in \mathcal{S}(\mathcal{K}).$$

Here, $\omega^\perp = \{\eta : \eta \cdot \omega = 0\}$ and $\mathcal{S}(\mathcal{K})$ is the **shadow** of $\mathcal{K}$.

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CHARACTERIZATION (Brasco - M. - Salani 2010)

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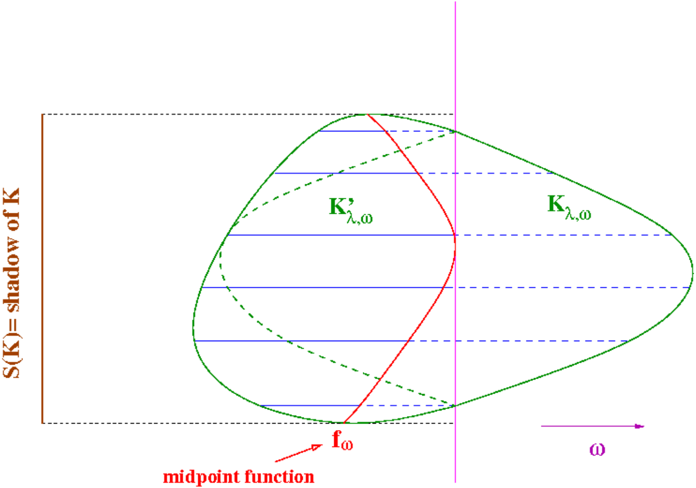

Figura: Proof.

An algorithm

CONVEX POLYHEDRON

For a convex polyhedron, we prove that the maximum in the characterization can be computed **only by visiting (the projections on $\mathcal{S}(\mathcal{K})$ of) the vertices of $\mathcal{K}$.**

This fact helps us to produce an algorithm to draw $\heartsuit(\mathcal{K})$ when $\mathcal{K}$ is a convex polyhedron:

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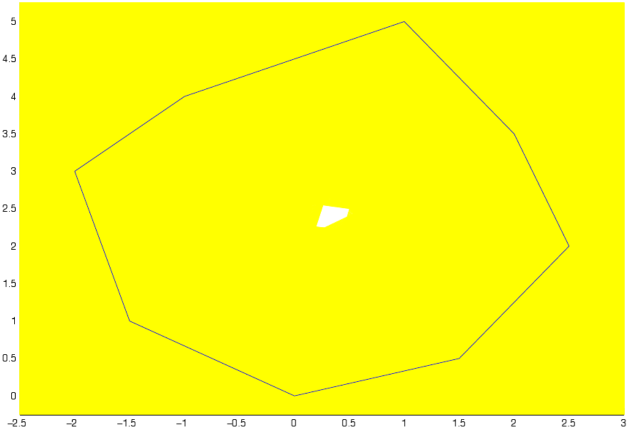

Figura: The heart of an octagon.

Example 2

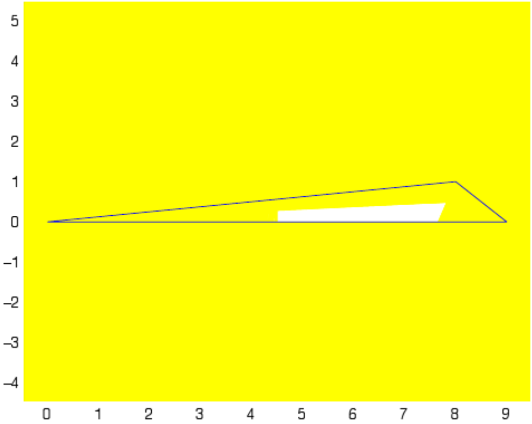

Figura: The heart of an obtuse triangle

Example 3

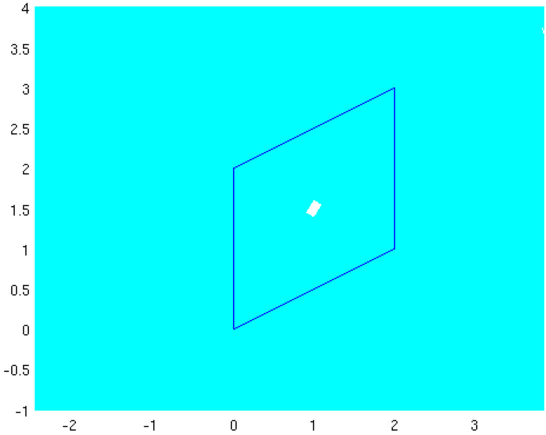

Figura: The heart of a parallelogram

Problems

Remark

In the case of the obtuse triangle, we observe that $\heartsuit(\mathcal{K}) \cap \partial\mathcal{K} \neq \varnothing$ (this is always the case when the circumcenter is not in the interior of $\mathcal{K}$), even if we are sure that $x(t)$ and x_∞ are not on $\partial\mathcal{K}$ (by Hopf lemma).

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Problems

1. If $\mathcal{K}$ is a polyhedron, can we choose only a finite number of directions ω to draw $\heartsuit(\mathcal{K})$?
2. Estimate the size of $\heartsuit(\mathcal{K})$. We know that

$$\mathrm{diam}[\heartsuit(\mathcal{K})] \geq \mathrm{diam}[\triangle(B, C, I)];$$

can we estimate the ratio

$$\frac{|\heartsuit(\mathcal{K})|}{|\mathcal{K}|}$$

from above?

Second method: using ABP principle

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where

$$C_N = \frac{(2^N N)^{N-1}}{\lambda_1(B_1)^N} \frac{\omega_{N-1}}{\omega_N} < 1,$$

$\lambda_1(B_1)$ is the first Dirichlet eigenvalue of the unit ball and

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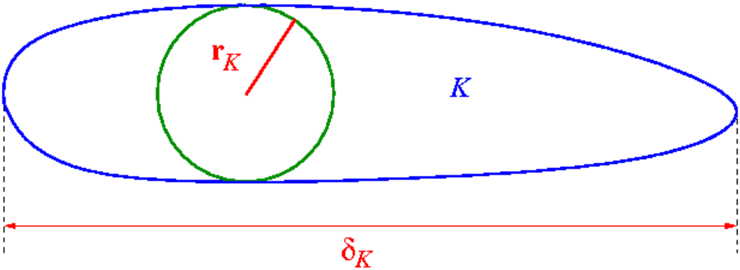

Figura: Totsu shuugou ippon.

Second method: using ABP principle

The idea is condensed il the following picture.

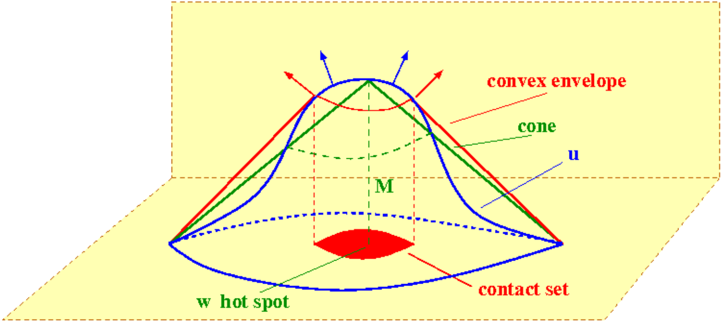

Figura: $u = u(x,t)$ or $\phi_1(x)$; $w = x(t)$ or x_∞; $M = u(w,t)$ or $\phi_1(w)$; $\mathcal{E} =$ convex envelope of u; $\mathcal{G} =$ cone with tip at the point (w, M); $\mathcal{C} =$ contact set (of points where $u = \mathcal{E}$.)

Subdifferential of a convex function and polar set

Define the **subgradient** of a function u at the point $z \in \mathcal{K}$ as the set

$$\partial u(z) = \{p \in \mathbb{R}^N : u(x) \geq u(z) + \langle p, x - z\rangle \text{ for every } x \in \mathcal{K}\}.$$

and

$$\partial u(\mathcal{K}) = \bigcup_{z \in \mathcal{K}} \partial u(z).$$

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1. Since $-\mathcal{E} \leq -\mathcal{G}$, we have that $\partial(-\mathcal{G})(\mathcal{K}) \subseteq \partial(-\mathcal{E})(\mathcal{K})$.
2. From convex geometry:

$$M\, w + \partial(-\mathcal{G})(\mathcal{K}) = \partial(-\mathcal{G})(w) = M\, \mathcal{K}_w^*,$$

where $\mathcal{K}_w^*$ is the **polar set of** $\mathcal{K}$ w.r.t. w :

$$\mathcal{K}_w^* = \left\{y \in \mathbb{R}^N \,:\, (x - w) \cdot (y - w) \leq 1 \text{ for every } x \in \mathcal{K}\right\}.$$

Estimating the polar set

Remarks

3 By the area formula and the arithmetic-geometric mean inequality, we have:

$$M^N |\mathcal{K}_w^*| = |\partial(-\mathcal{G})(\mathcal{K})| \le |\partial(-\mathcal{E})(\mathcal{K})| = \int_{\mathcal{C}} |\det(D^2 u)|\, dx \le N^{-N} \int_{\mathcal{C}} |\Delta u|^N\, dx.$$

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4 Finally, using the equations $u_t = \Delta u$ or $\Delta\phi_1 + \lambda_1 \phi_1 = 0$, we obtain the two bounds

$$|\mathcal{K}_{x(t)}^*| \le [N\,M(t)]^{-N} \int_{\mathcal{C}(t)} |u_t(x,t)|^N dx,$$

$$|\mathcal{K}_{x_\infty}^*| \le \left[\frac{\lambda_1}{N\,M_\infty}\right]^N \int_{\mathcal{C}} \phi_1(x)^N \, dx\,,$$

that is the **polar set cannot be too large.**

Hot spots and polar sets

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$$|\mathcal{K}^*_{x_\infty}| \le \left[\frac{\lambda_1}{N}\right]^N |\mathcal{K}|\,.$$

and

$$|\mathcal{C}| \ge \left[\frac{N}{\lambda_1}\right]^N |\mathcal{K}^*_{x_\infty}| \ge \left[\frac{N}{\lambda_1}\right]^N |\mathcal{K}^*_s|,$$

where $\mathcal{K}^*_s$ denotes the polar set of $\mathcal{K}$ with respect to the **Santalò point** s (the one that minimizes the function $w \mapsto |\mathcal{K}^*_w|$).

Estimating the volume of the polar set

Using the definition of the polar set, it is easy to see that $|\mathcal{K}_w^*|$ goes to ∞ as the point w approaches $\partial\mathcal{K}$. The following estimate gives a quantitative version of this fact and helps us to prove explicit estimates of the position of x_∞.

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$$|\mathcal{K}_w^*| \geq |E_w^*| \geq \frac{\omega_{N-1}/N}{R^{N-1}\, d}.$$

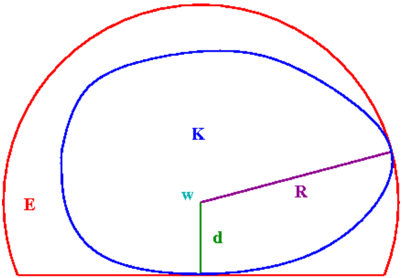

Figura:

Conclusion

Thus,

$$\frac{\omega_{N-1}/N}{R^{N-1}\, d} \le \left[\frac{\lambda_1}{N}\right]^N |\mathcal{K}|$$

and the bound

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follows by choosing $w = x_\infty$ and by using a standard inequality to bound λ_1 from above.

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Concluding remark

The two methods for locating the hot spot can be coupled. For example, in the case of the obtuse triangle, we know that its heart extends to part of the boundary; however, by the estimate we have just proved, we can quantitatively say how far x_∞ must be from the boundary.