Andrea Malchiodi (SISSA, Trieste)

New improved Moser-Trudinger inequalities and singular Liouville equations on compact surfaces

Geometric Properties for Parabolic and Elliptic PDEs

Cortona, June 20-24, 2011

(1) A.M. - D.Ruiz: New improved Moser-Trudinger inequalities and singular Liouville equations on compact surfaces, GAFA, to appear.

(2) D.Bartolucci - F.De Marchis - A.M.: Supercritical conformal metrics on surfaces with conical singularities, IMRN, to appear.

(3) A.Carlotto - A.M.: Weighted barycentric sums and singular Liouville equations on compact surfaces, preprint, 2011.

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$$(E_\rho) \qquad -\Delta u = \rho \left(k(x) e^{2u} - 1 \right) - 2\pi \sum_{j=1}^{m} \alpha_j \left(\delta_{p_j} - 1 \right).$$

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• We assume also, without loss of generality, that $|\Sigma| = 1$.

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In the *static self-dual regime*, $\Gamma = -\frac{1}{m\kappa}$, $v = \log|\psi|$ satisfies

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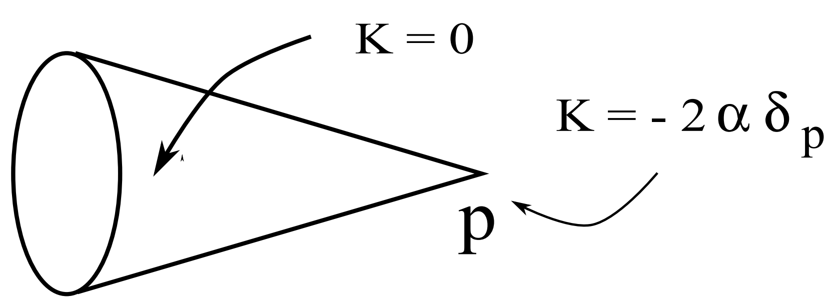
A Dirac delta on the right-hand side gives a singular metric with conical structure.



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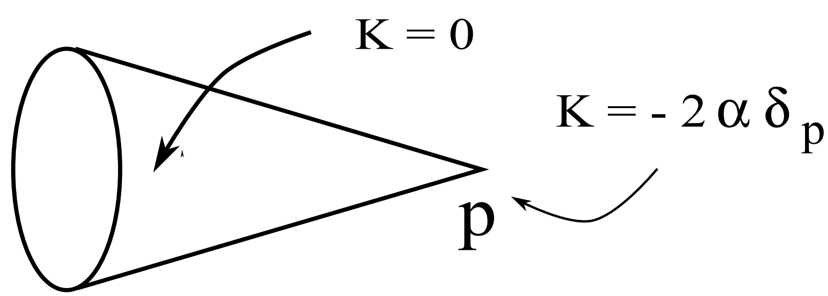

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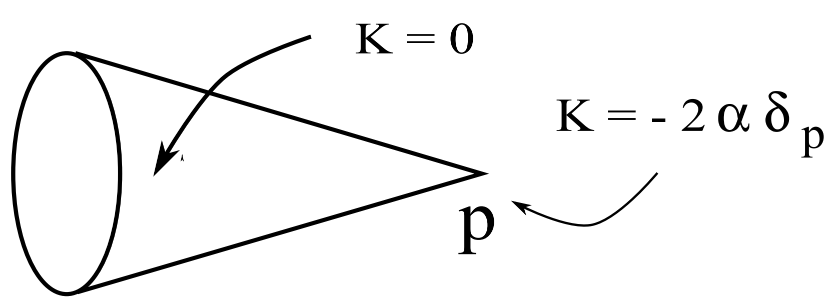

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$(\tilde{E}_\rho)$ is the Euler-Lagrange equation for $I_\rho : H^1(\Sigma) \to \mathbb{R}$ def. as

$$I_\rho(u) = \int_\Sigma |\nabla u|^2 + 2\rho \int_\Sigma u - \rho \log \int_\Sigma h(x) e^{2u}. \tag{2}$$

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Our goal is to develop a global variational strategy to find general critical points of saddle type. This could be more direct and more general than L-S theory, as degree cancelations may occur.

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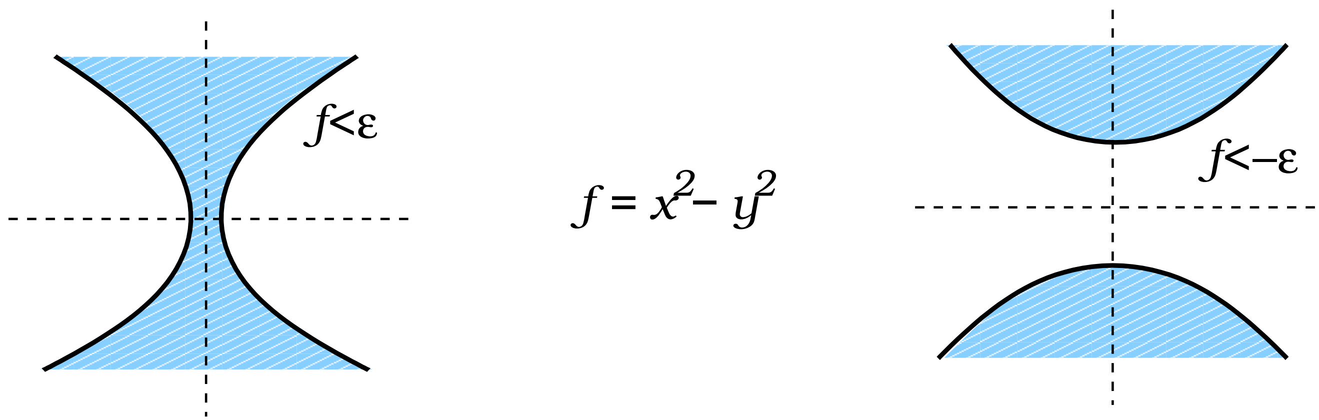

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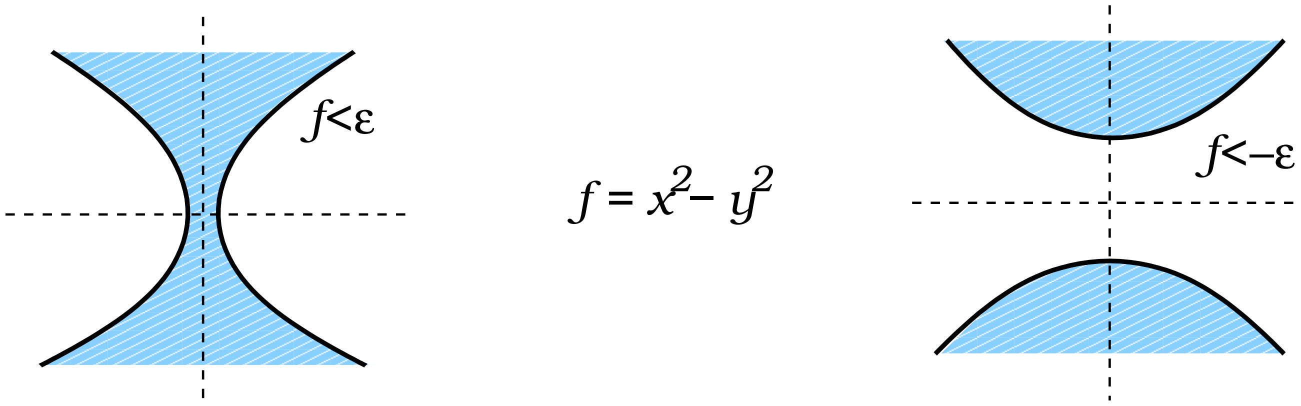

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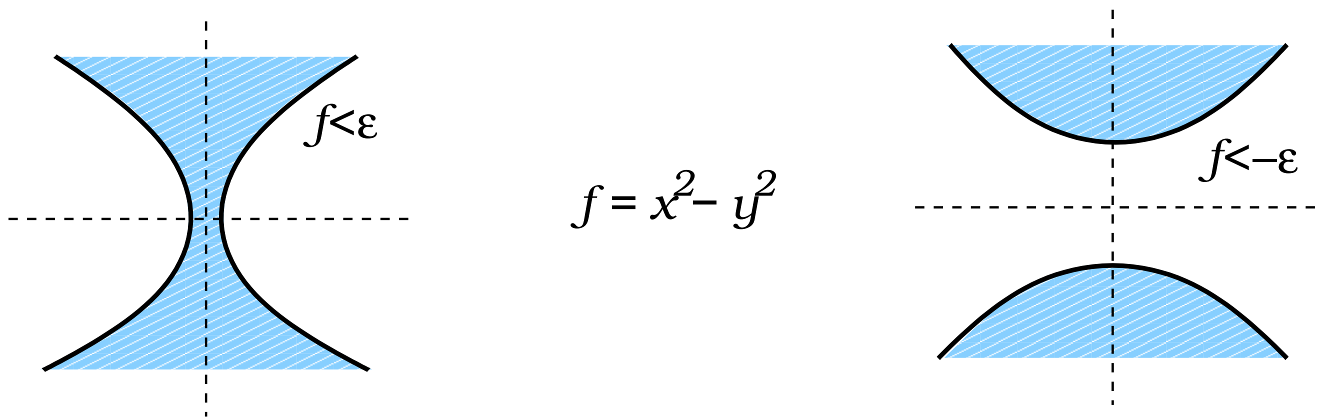

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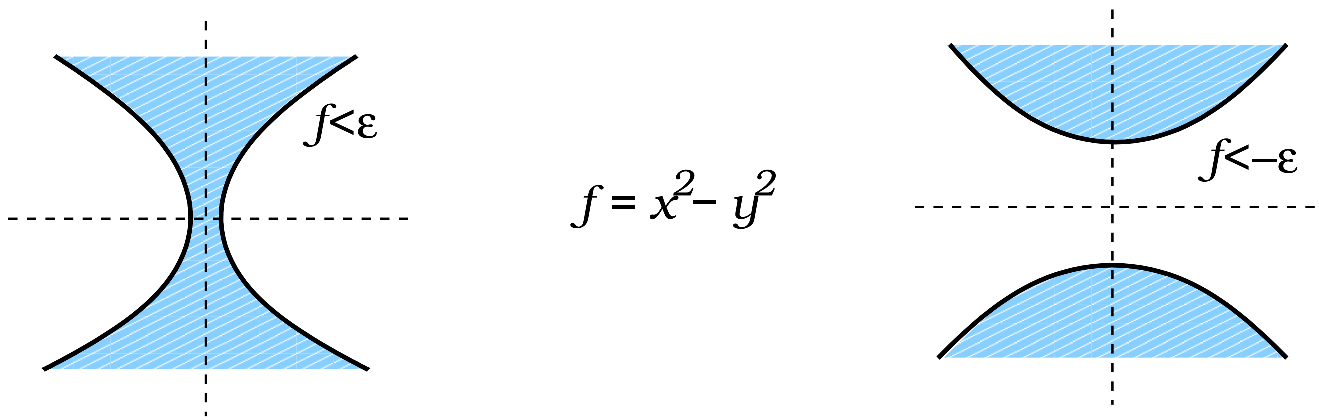

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Consequence:

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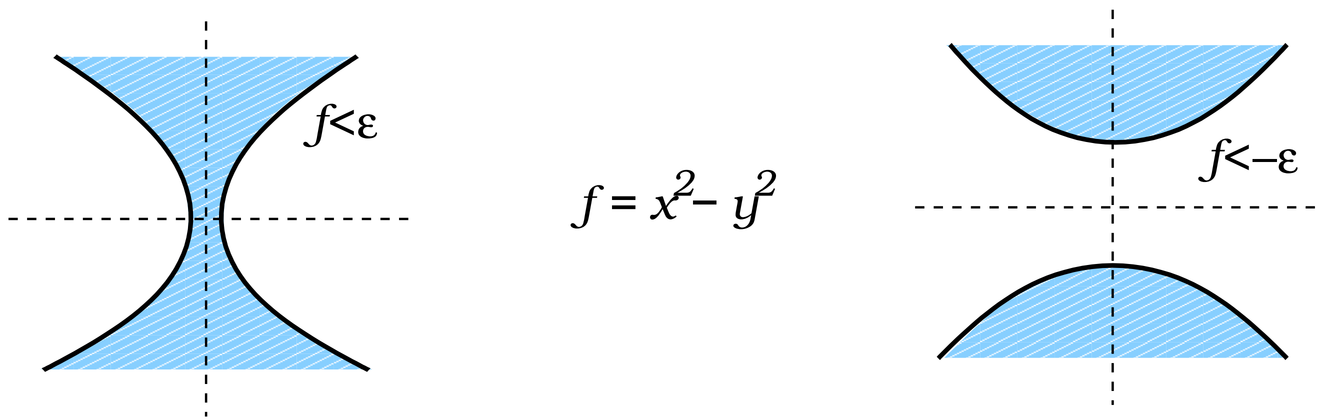

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Consequence: if one level does not deform into another, there should be a critical point between the two.

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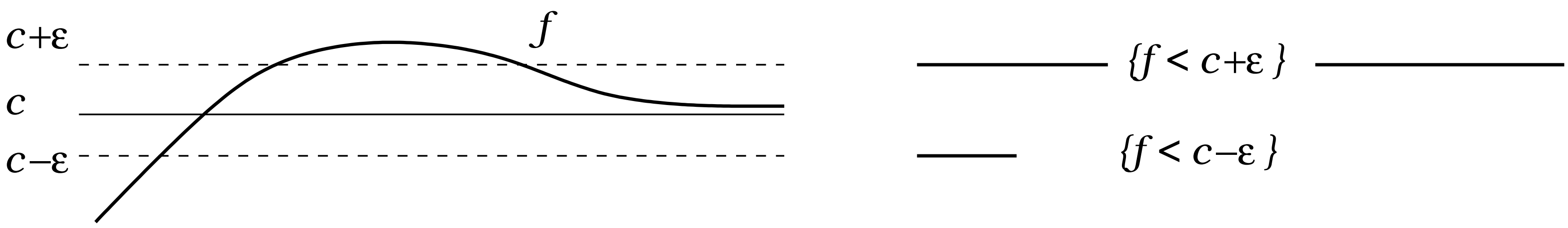

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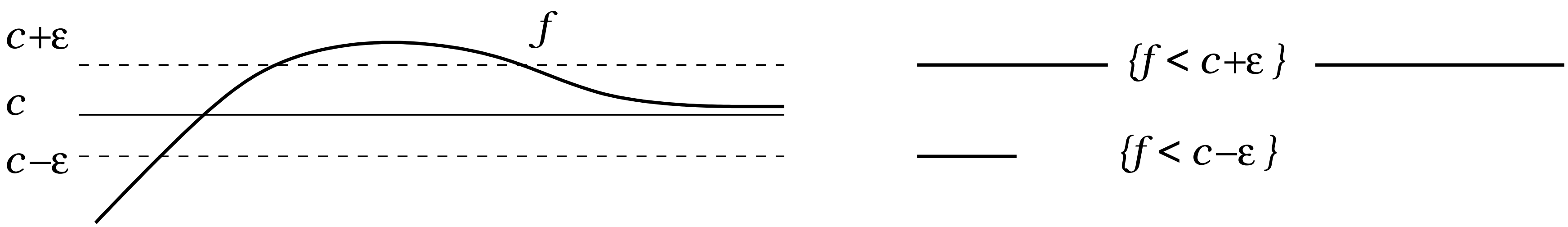

Some compactness criterion is needed (Palais-Smale condition or similar ones)

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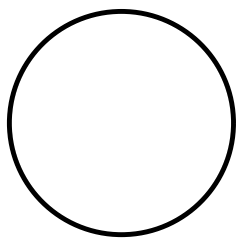

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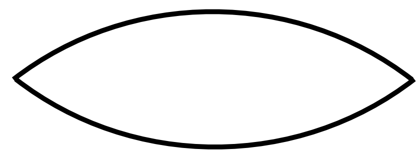

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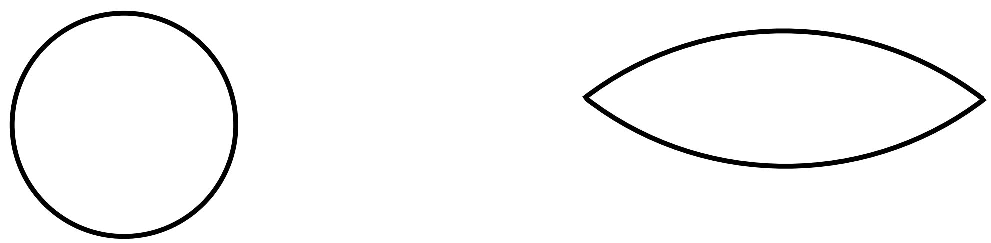

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Theorem ([Bartolucci-Tarantello, '02]) Suppose u_n are solutions of $(\tilde{E}_{\rho_n})$.

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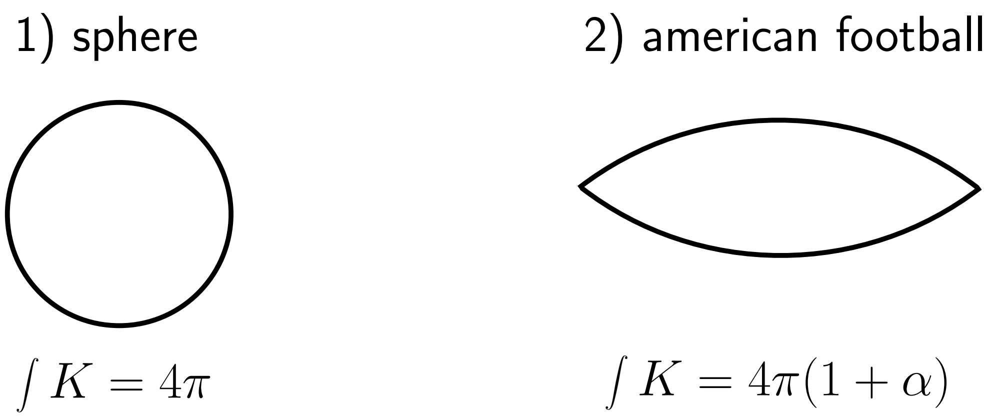

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Compactness and blow-up

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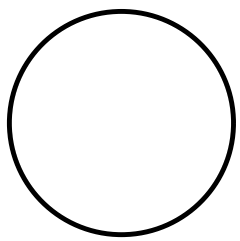

$\int K = 4\pi$

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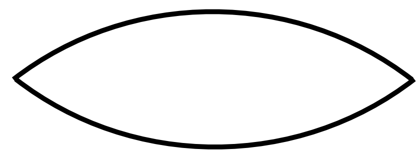

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For the regular case the above compactness theorem has previous counterparts by Y.Li, Shafrir, Ohtsuka, Suzuki, Ricciardi, C.Chen, C.S. Lin, . . .

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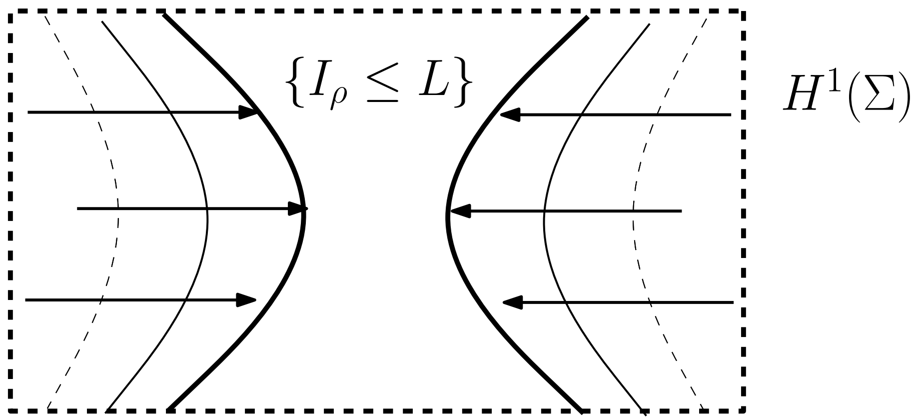

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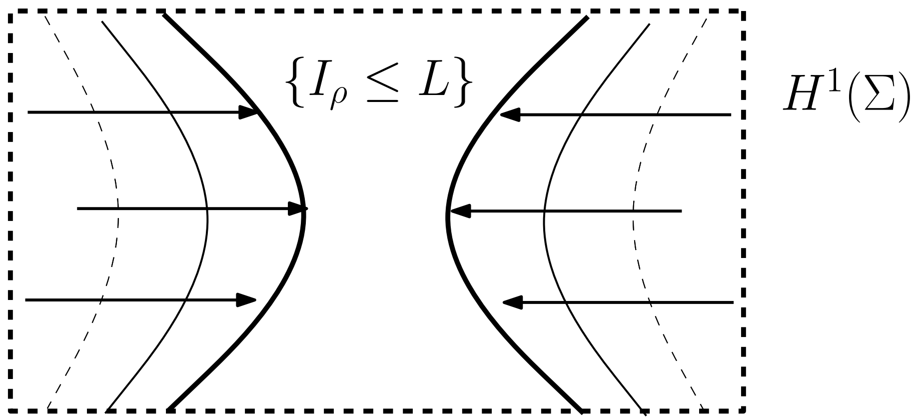

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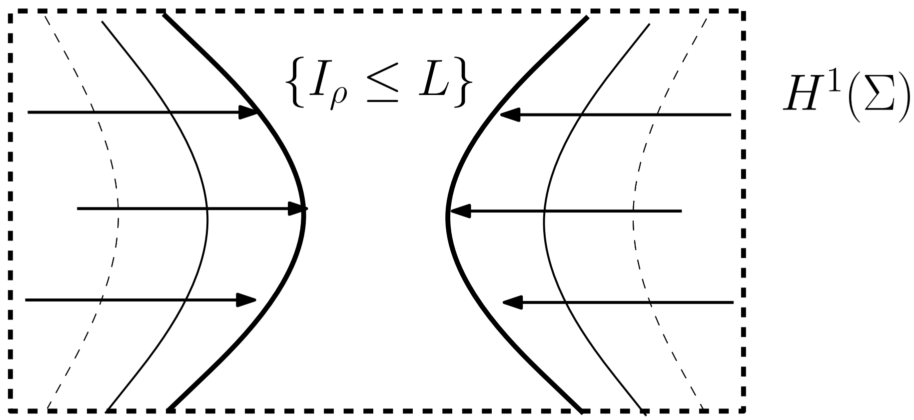

$$\Rightarrow \qquad \{I_\rho \leq L\} \text{ has trivial topology} \qquad (L \gg 0)$$

Corollary If for some large L the *low sublevel* $\{I_\rho \leq -L\}$ has non trivial topology, a solution to $(\tilde{E}_\rho)$ exists.

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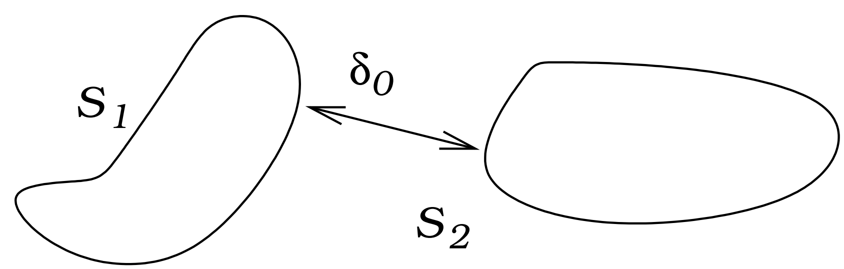

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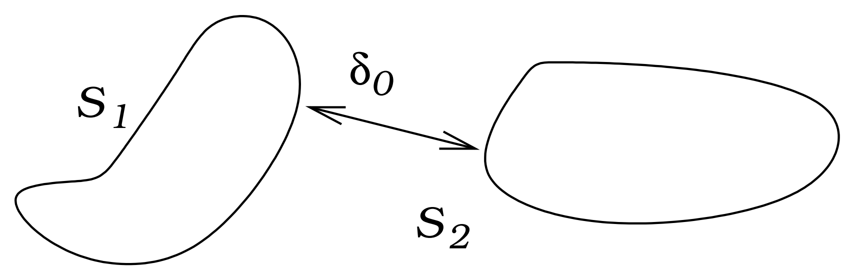

Then $\forall\, \tilde{\varepsilon} > 0 \quad \exists$ a constant $C = C(\tilde{\varepsilon}, \delta_0, \gamma_0)$ (indep. of u) s.t.

$$\log \int_{\Sigma} e^{2(u-\overline{u})} dV_g \leq C + \frac{1}{8\pi - \tilde{\varepsilon}} \int_{\Sigma} |\nabla u|^2 \, dV_g.$$

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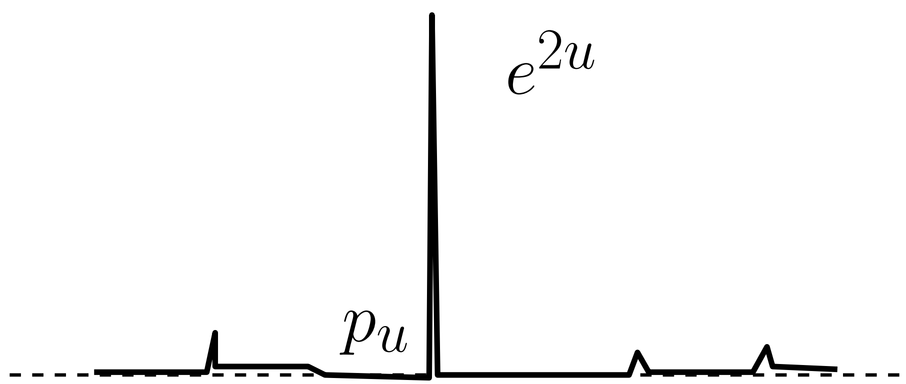

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• We want to understand next what is the role of the singularity.

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• The coefficient $\frac{1}{4\pi(1+\alpha)}$ is the one to aim for, but radiality assumption is rather restrictive. Our goal is to substitute it with a two-dimensional constraint, which is <u>much</u> more flexible.

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• The assumption $\alpha \leq 1$ is sharp. For $\alpha > 1$ *splitting* the mass gives a worse constant and lowers the functional.

Proof of Theorem 1: $\rho \in (4\pi, 4\pi(1+\alpha))$

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• In many geometrically relevant situations (prescription of the Gaussian curvature) ρ might belong to the critical set Λ, by the Gauss-Bonnet formula.

In this case one would need a more refined blow-up analysis, together with some detailed information on the Green's function or some linear combination of different ones.

Other physical models

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• The problem without the self-duality assumption seems for the moment completely out of reach.

Thank you for your attention