DIFFERENTIABILITY OF SOLUTIONS TO SECOND-ORDER ELLIPTIC EQUATIONS VIA DYNAMICAL SYSTEMS

Vladimir Maz'ya

University of Liverpool and Linköping University

Geometric properties for parabolic and elliptic PDE's
Cortona, June 19 –24, 2011

This talk is based on the joint papers by

V. Maz'ya and R. McOwen

Differentiability of solutions to second order elliptic equations via dynamical systems,

Journal of Differential Equations, **250**, 2011

and

T. Jin, V. Maz'ya, J. Van Schaftingen

Pathological solutions to elliptic problems in divergence form with continuous coefficients,

C.R. Math. Acad. Sci. Paris, **347**, 2009

Consider a weak solution of a linear uniformly elliptic equation in divergence form in an open set U of $\mathbb{R}^n$ for $n \geq 2$:

$$\mathcal{L}u := \partial_i(a_{ij}(x)\partial_j u) = 0 \quad \text{in } U, \tag{1}$$

where $a_{ij} = a_{ji}$ are bounded, measurable, real-valued functions.

By a *weak solution* of (1) we mean that $u \in H^{1,2}_{\ell oc}(U)$, i.e. ∇u is locally square-integrable, and satisfies

$$\int_U a_{ij}(x)\,\partial_j u\,\partial_i \eta\,dx = 0 \quad \text{for all } \eta \in C_0^\infty(U). \tag{2}$$

The classical results of De Giorgi (1957) and Nash (1957) show that u is locally Hölder continuous in U. When the coefficients are continuous in U, then it is well-known that $\nabla u \in L^p_{\ell oc}(U)$ for $1 < p < \infty$.

This is even true when the coefficients are in VMO (Di Fazio, 1996).

If the coefficients are Dini-continuous in U, then u is known to be continuously differentiable (P. Hartman, A. Wintner, 1955; M.Taylor, 2000).

We find conditions on the coefficients a_{ij}, milder than Dini-continuity, under which u must be Lipschitz continuous, or even differentiable, at a given point.

Square-Dini condition

Fix an interior point of U, which for convenience we shall assume is the origin, $x = 0$. Using a change of independent variables, we may assume that $a_{ij}(0) = \delta_{ij}$. Suppose that

$$\sup_{|x|=r} |a_{ij}(x) - \delta_{ij}| \leq \omega(r) \quad \text{as } r \to 0, \tag{3}$$

where $\omega(r)$ is a continuous, nondecreasing function for $0 \leq r < 1$ satisfying $\omega(0) = 0$. We do not require the Dini condition on ω, i.e. $r^{-1}\omega(r) \in L^1(0,1)$.

Instead we assume that ω satisfies the *square-Dini condition*:

$$\int_0^1 \omega^2(r) \, \frac{dr}{r} < \infty. \tag{4}$$

Our additional conditions for regularity are derived from a dynamical system that we shall now describe. Let

$$R(r) := \fint_{S^{n-1}} \left(a_{ij}(r\theta) - n a_{ik}(r\theta)\theta_k\theta_j\right) ds, \tag{5}$$

where the slashed integral denotes mean value, $r = |x|$, $\theta = x/|x| \in S^{n-1}$. Note that $|R(r)| \leq c\,\omega(r)$, where we use $|\cdot|$ to denote the matrix norm. Also note that R need not be symmetric.

Dynamical system

Consider the dynamical system

$$\frac{d\phi}{dt} + R(e^{-t})\,\phi = 0 \quad \text{for } T < t < \infty, \tag{6}$$

where $t = -\log r$ and T is sufficiently large. We shall find that the regularity of weak solutions of (1) is determined by the asymptotic behavior as $t \to \infty$ of solutions of (6).

We say that (6) is uniformly stable as $t \to \infty$ if for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that any solution ϕ of (6) satisfying

$$|\phi(t_1)| < \delta \quad \text{for some } t_1 > 0$$

satisfies

$$|\phi(t)| < \varepsilon \quad \text{for all } t \geq t_1.$$

In addition, we are interested in the condition that every solution of (6) is asymptotically constant, i.e.

$$\phi(t) \to \phi_\infty \quad \text{as} \;\; t \to \infty.$$

These two stability conditions are independent of each other.

On the other hand, it is easy to see that $r^{-1}R(r) \in L^1(0, \varepsilon)$ implies that (6) is uniformly stable and every solution is asymptotically constant as $t \to \infty$.

In particular, if ω satisfies the Dini condition, then these conditions are met.

Main Theorem

We are only concerned with regularity at $x = 0$, so the coefficients are not required to be continuous elsewhere.

Theorem Suppose that

$$\sup_{|x|=r} |a_{ij}(x) - \delta_{ij}| \leq \omega(r) \quad \text{as } r \to 0,$$

where ω obeys the square-Dini condition and that the dynamical system (6) is uniformly stable. Then every weak solution $u \in H^{1,2}_{\ell oc}(U)$ of $\partial_i(a_{ij}(x)\partial_j u) = 0$ in U is Lipschitz continuous at $x = 0$ and

$$|u(x) - u(0)| \leq \frac{c\,|x|}{r} \left(\fint_{|y|<r} |u(y)|^2 \, dy \right)^{1/2} \quad \text{for } |x| < r/2,$$

where r is sufficiently small. In addition, if every solution of (6) is asymptotically constant, then u is differentiable at $x = 0$ and

$$\partial_j u(0) = \lim_{r \to 0} \frac{n}{r} \fint_{S^{n-1}} u(r\theta)\, \theta_j \, ds_\theta.$$

Remark. If the coefficients a_{ij} are radial functions, then $R(r) \equiv 0$ and we only require

$$\sup_{|x|=r} |a_{ij}(x) - \delta_{ij}| \leq \omega(r) \quad \text{as } r \to 0,$$

and the square-Dini condition to conclude that weak solutions are differentiable at $x = 0$.

Moreover, if

$$a_{ij}(x) = a^0_{ij}(|x|) + a^1_{ij}(x),$$

then the R in (6) is completely determined by a^1_{ij}.

For example, if the a^1_{ij} are Dini continuous then weak solutions are differentiable even though a_{ij} need only be square-Dini continuous.

Corollaries of Main Theorem

We investigate specific analytic conditions on the coefficients a_{ij} that imply the desired asymptotic properties of the dynamical system (6).
We introduce the symmetric matrix

$$\mathcal{S} = -\frac{1}{2}(R + R^t)$$

and

$$\mu(\mathcal{S}) = \text{largest eigenvalue of } \mathcal{S}.$$

We use the theory of dynamical systems to show that if there exist positive constants ε and K so that

$$\int_{r_1}^{r_2} \mu(\mathcal{S}(\rho)) \, \frac{d\rho}{\rho} < K \quad \text{for all } \varepsilon > r_2 > r_1 > 0, \tag{7}$$

then the dynamical system (6) is uniformly stable.

As a consequence, Main Theorem implies:
Corollary 1 Suppose that

$$\sup_{|x|=r} |a_{ij}(x) - \delta_{ij}| \leq \omega(r) \quad \text{as } r \to 0,$$

the square-Dini condition, and (7) are satisfied. Then every weak solution u of (1) is Lipschitz continuous at $x = 0$.

Conditions for differentiability at $x=0$

As already observed, $r^{-1}R(r) \in L^1(0,\varepsilon)$ is sufficient, but is there a weaker condition? Let us suppose that for $r \in (0,\varepsilon)$ the improper integral

$$\int_0^r R(\rho)\,\frac{d\rho}{\rho} \quad \text{converges (perhaps not absolutely).} \tag{8a}$$

Our examples show that this condition is not sufficient to ensure that the dynamical system (6) is uniformly stable.

We require an additional condition such as

$$\frac{R(r)}{r}\int_0^r R(\rho)\,\frac{d\rho}{\rho} \in L^1(0,\varepsilon), \tag{8b}$$

which is also weaker than assuming $R(r)\,r^{-1} \in L^1(0,\varepsilon)$.

We show that (8a) and (8b) together imply not only that the dynamical system (6) is uniformly stable but its solutions are asymptotically constant.

Consequently, Main Theorem yields:

Corollary 2. Suppose that

$$\sup_{|x|=r} |a_{ij}(x) - \delta_{ij}| \leq \omega(r) \quad \text{as } r \to 0,$$

the square-Dini condition, as well as both (8a) and (8b) are satisfied. Then every weak solution u of

$$\partial_i(a_{ij}(x)\partial_j u) = 0 \quad \text{in } U$$

is differentiable at $x = 0$.

Remark

The condition (8a) can be expressed as a volume integral (in the sense of Cauchy principal value):

$$\int_{|x|<r} \left(A(x) - n\,\frac{A(x)x}{|x|} \cdot \frac{x}{|x|} \right) \frac{dx}{|x|^n} \quad \textit{converges for } r \in (0, \varepsilon).$$

This form of the condition is better suited for changes of coordinates, so can be expressed without assumption $a_{ij}(0) = \delta_{ij}$. However, (8b) is not easily handled in this way. Similarly, the condition

$$\int_{|x|<\varepsilon} |A(x) - I| \frac{dx}{|x|^n} < \infty$$

is sufficient for Corollary 2 and easily generalizes to the case $a_{ij}(0) \neq \delta_{ij}$. However, it implies $r^{-1}R(r) \in L^1(0, \varepsilon)$, so is less general than (8a) and (8b).

One more consequence of Main Theorem. We show that

$$\int_r^{\varepsilon} \mu(\mathcal{S}(\rho))\frac{d\rho}{\rho} \to -\infty \quad \text{as } r \to 0 \tag{9}$$

implies that the null solution of the dynamical system (6) is asymptotically stable. Thus Main Theorem yields:
Corollary 3. Suppose that

$$\sup_{|x|=r} |a_{ij}(x) - \delta_{ij}| \leq \omega(r) \quad \text{as } r \to 0,$$

the square-Dini condition is satisfied, and

$$\int_{r_1}^{r_2} \mu\left(\mathcal{S}(\rho)\right) \frac{d\rho}{\rho} < K \quad \text{for all } \varepsilon > r_2 > r_1 > 0.$$

Moreover, assume (9). Then every weak solution u of $\partial_i(a_{ij}(x)\partial_j u) = 0$ is differentiable at $x = 0$ and all derivatives are zero: $\partial_j u(0) = 0$ for $j = 1, \ldots, n$.

Main ideas of the proof

We write the solution u in the form

$$u(x) = u_0(|x|) + \vec{v}(|x|) \cdot \vec{x} + w(x), \tag{10}$$

where

$$u_0(r) := \fint_{S^{n-1}} u(r\theta)\, ds_\theta, \qquad v_k(r) := \frac{n}{r} \fint_{S^{n-1}} u(r\theta)\theta_k\, ds_\theta, \tag{11}$$

and w has zero spherical mean and first spherical moments:

$$\fint_{S^{n-1}} w(r\theta)\, ds_\theta = 0 = \fint_{S^{n-1}} w(r\theta)\theta_i\, ds_\theta \quad \text{for } i = 1, \ldots, n. \tag{12}$$

Ingradient I

We find that $\vec{v}$ satisfies a second-order differential system depending upon u_0 and w, but it is equivalent to a first-order system that only depends on w.
Let $r = e^{-t}$. Consider the $2n \times 2n$ system on $(0, \infty)$

$$\frac{d}{dt}\begin{pmatrix}\phi \\ \psi\end{pmatrix} + \begin{pmatrix}0 & 0 \\ 0 & -nI\end{pmatrix}\begin{pmatrix}\phi \\ \psi\end{pmatrix} + \mathcal{R}(t)\begin{pmatrix}\phi \\ \psi\end{pmatrix} = g(t), \tag{13}$$

where i) $\mathcal{R}$ is a $2n \times 2n$ matrix of the form

$$\mathcal{R}(t) = \begin{pmatrix}R_1(t) & R_2(t) \\ R_3(t) & R_4(t)\end{pmatrix} \quad \text{with } \left|R_j(t)\right| \le \varepsilon(t) \text{ on } 0 < t < \infty,$$

and ii) $g = (g_1, g_2)$ with $g_1 \in L^1(0, \infty)$. We have $R_1 \sim R$ as $t \to \infty$.

Proposition. Suppose that

$$\frac{d\phi}{dt} + R_1\phi = 0 \quad \text{for } t > 0 \tag{14}$$

is uniformly stable. Then all solutions (ϕ, ψ) of (13) that satisfy the "finite-energy condition"

$$\int_0^\infty \left(|\psi|^2 + |\psi_t|^2\right) e^{-nt}\, dt < \infty$$

remain bounded as $t \to \infty$, and $\psi(t) \to 0$.
In addition, if all solutions of (14) are asymptotically constant as $t \to \infty$, then the solution (ϕ, ψ) of the system (13) also has a limit:

$$(\phi(t), \psi(t)) \to (\phi_\infty, 0) \quad \text{as } t \to \infty.$$

Ingradient II

The function w in (10) satisfies the PDE

$$\Delta w + [\mathrm{div}(\Omega\nabla w)]^{\perp} + [\mathrm{div}(\Omega\nabla(\vec{v}\cdot\vec{x}))]^{\perp} + [\mathrm{div}(\Omega\nabla u_0)]^{\perp} = 0,$$

where

$$f(r\theta)^{\perp} = f(r\theta) - \Big(\fint_{S^{n-1}} f(r\phi)\, ds_{\phi} + n\theta_k \fint_{S^{n-1}} \phi_k f(r\phi)\, ds_{\phi}\Big).$$

and the matrix $\Omega = (\Omega_{ij})$ has the entries

$$\Omega_{ij} = a_{ij} - \delta_{ij}$$

Proposition. If

$$-\Delta w = [\mathrm{div}(f)]^{\perp}$$

then $\|\nabla w\|_p$ can be estimated in terms of $\|f\|_p$ and

$$⨍_{S^{n-1}} w(r\phi)\, ds_\phi = 0 = ⨍_{S^{n-1}} \phi_k w(r\phi)\, ds_\phi$$

Combine I and II

Recall that we want

$$u_0(r) \sim u_0 + o(r)$$

,

$$\vec{v}(r) = O(1) \quad \text{or} \quad \vec{v}(r) = \vec{v}(0) + o(1)$$

and

$$w(x) = o(|x|).$$

We introduce the L^p-means

$$M_p(w,r) = \left(\fint_{|x|<r} |w(x)|^p \, dx \right)^{1/p}$$

and use Morrey's inequality

$$\sup_{|x|<r} |\nabla w(x)| \leq c\, r\, M_p(\nabla w, r), \quad p > n,$$

to show that

$$M_p(\nabla w, r) \leq c\, w(r). \tag{15}$$

Then we define a Banach space in which we seek w and use (15) to show that $u_0(r)$ and $\vec{v}(r)$ behave as desired.

Examples of Gilbarg-Serrin type

Gilbarg and Serrin (1956) considered examples of the form

$$a_{ij}(x) = \delta_{ij} + g(r)\theta_i\theta_j, \tag{16}$$

where $g(0) = 0$ but vanishes slowly as $r \to 0$. They used such examples to show that Dini continuity is essential for their "extended maximum principle" to hold, but we shall use them to explore the conditions in Main Theorem and its corollaries.

We assume that $|g(r)| \leq \omega(r)$ for r near 0 with ω satisfying the square-Dini condition.

The dynamical system (6) reduces to the scalar equation

$$\frac{d\phi}{dt} = \frac{n-1}{n}\,\tilde{g}(t)\,\phi, \tag{17}$$

where $\tilde{g}(t) = g(e^{-t})$. The solution is

$$\phi(t) = \phi(0)\exp\Big(\int_0^t \tilde{g}(\tau)d\tau.$$

According to Main Theorem, u is Lipschitz continuous at $x = 0$ provided (17) is uniformly stable for $t > T$ with T sufficiently large and this is the case if and only if

$$\int_s^t \tilde{g}(\tau)\, d\tau < K \quad \text{for } t > s > T. \tag{18}$$

Moreover, $\mu(\mathcal{R}(r)) = (1 - n^{-1})g(r)$, so (7) agrees with (18) and we see that Corollary 1 is sharp for this class of examples.

On the other hand, solutions of (17) are asymptotically constant if and only if the improper integral

$$\int_T^\infty \tilde{g}(\tau)\,d\tau \text{ converges to an extended real number} < \infty. \quad (19)$$

Thus Main Theorem implies that u is differentiable at $x = 0$ if both (18) and (19) hold. The case

$$\int_T^\infty \tilde{g}(t)\,dt = -\infty \quad (20)$$

in (19) pertains to Corollary 3, which is sharp for this class of examples.

On the other hand, the case that $\tilde{g}(t)$ is integrable pertains to Corollary 2 and coincides with the hypothesis (8a). However, in Corollary 2 we also require (8b), since the condition (8a) alone does not imply the uniform stability of (6).

In fact, this class of examples may be used to show not only this last statement, but in general that uniform stability is not implied by every solution being asymptotically constant: we only need to construct $\tilde{g}(t)$ for which (19) holds but (18) fails. If we construct $\tilde{g}$ for which (20) holds and yet (18) fails, then we see that (7) is not implied by (9), so both conditions are necessary in Corollary 3. In this regard, let us observe that the book by W.A.Coppel *Stability and Asymptotic Behavior of Differential Equations*, 1965, gives an explicit example of a function $\tilde{g}(t)$ satisfying (20) and yet (18) fails: there exist $t_j \to \infty$ for which

$$\int_{t_{2j}}^{t_{2j+1}} \tilde{g}(\tau)\, d\tau \to \infty$$

and yet

$$\int_{t_{2j+1}}^{t_{2j+2}} \tilde{g}(\tau)\, d\tau \to -\infty$$

more rapidly so that (20) still holds.

The example by Coppel does not have

$$\tilde{g}(t) \to 0 \quad \text{as} \quad t \to \infty,$$

but it can be modified to achieve this; in fact, we can even arrange $\tilde{g}(t) = O(t^{-2/3})$, which implies that $\tilde{g} \in L^2(T, \infty)$ and so the a_{ij} are square-Dini continuous at $x = 0$. Moreover, the example can be modified so that (20) is replaced by the condition that $\tilde{g}$ is integrable on $(0, \infty)$. Thus (18) and (19) are completely independent conditions, even under the assumption that the coefficients a_{ij} are square-Dini continuous at $x = 0$.

Pathological solutions to equations with continuous coefficients

Consider the equation

$$-\mathrm{div}\,(A\nabla u) = 0 \qquad \text{in } \Omega$$

for $\Omega \subset \mathbb{R}^n$. If $A : \Omega \to \mathbb{R}^{n\times n}$ is bounded, measurable and uniformly elliptic, then one can define a weak solution $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ by requiring that for every $\varphi \in C^1_0(\Omega)$

$$\int_\Omega (A\nabla u) \cdot \nabla\varphi = 0.$$

J. Serrin showed that the assumption $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ is essential in E. De Giorgi's result by constructing for every $p \in (1, 2)$ a function $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ that solves such an elliptic equation but which is not locally bounded. In these counterexamples A is not continuous.

J. Serrin conjectured that if A is Hölder continuous, then any weak solution $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ is in $W^{1,2}_{\mathrm{loc}}(\Omega)$ and one can apply E. De Giorgi's theory.

This conjecture was confirmed for $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ by R.A. Hager and R. Ross and for $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ by H. Brezis. The proof by Brezis extends to the case where the modulus of continuity of A

$$\omega_A(t) = \sup_{x,y\in\Omega, |x-y|\le t} |A(x) - A(y)|,$$

satisfies the Dini condition

$$\int_0^1 \frac{\omega_A(s)}{s} ds < \infty.$$

In the case where A is merely continuous, H. Brezis obtained the following result.

Theorem. (H. Brezis) *Assume that* $A \in C(\Omega; \mathbb{R}^{n\times n})$ *is elliptic. If* $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ *solves the equation*

$$-\mathrm{div}\,(A\nabla u) = 0 \qquad \text{in } \Omega$$

then $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$ *for every* $q \in [p, \infty)$.

H. Brezis asked two questions about the cases $p = 1$ and $q = \infty$ in the previous theorem. We answer both questions, with a negative answer.

Main assertions

Proposition 1. *There exists* $u \in W^{1,1}_{\mathrm{loc}}(B(0,1))$ *and an elliptic* $A \in C(B(0,1); \mathbb{R}^{n\times n})$ *such that* u *solves* $-\mathrm{div}\,(A\nabla u) = 0$ *but* $u \notin W^{1,p}_{\mathrm{loc}}(B(0,1))$ *for every* $p > 1$.

As a byproduct we obtain

Proposition 2. *There exists* $A \in C(B(0,1); \mathbb{R}^{n\times n})$ *such that the problem*

$$\begin{array}{rcll} -\mathrm{div}(A\nabla u) & = & 0 & \textit{in } B(0,1) \\ u & = & 0 & \textit{on } \partial B(0,1) \end{array}$$

has a nontrivial solution.

Our construction in Proposition 1 allows to show that the counterexamples can be esed to obtain the following stronger assertions.

Proposition 3. *There exists* $u \in W^{1,1}_{\mathrm{loc}}(B(0,1))$ *and an elliptic* $A \in C(B(0,1); \mathbb{R}^{n\times n})$ *such that* u *solves*

$$-\mathrm{div}\,(A\nabla u) = 0$$

and $\nabla u \in (L\log L)_{\mathrm{loc}}(B(0,1))$ *but* $u \notin W^{1,p}_{\mathrm{loc}}(B(0,1))$ *for every* $p > 1$.

Concerning the possibility of Lipschitz estimates, we have

Proposition 4. *There exists* $u \in W^{1,1}_{\mathrm{loc}}(B(0,1))$ *and an elliptic* $A \in C(B(0,1); \mathbb{R}^{n\times n})$ *such that* u *solves*

$$-\mathrm{div}\,(A\nabla u) = 0$$

and $\nabla u \in W^{1,p}_{\mathrm{loc}}(B(0,1))$ *for every* $p > 1$, $\nabla u \in \mathrm{BMO}_{\mathrm{loc}}(B(0,1))$ *but* $u \notin W^{1,\infty}_{\mathrm{loc}}(B(0,1))$.

This shows that $\nabla u \in L^p(B(0,1))$ does not imply $\nabla u \in L^\infty(B(0,1/2))$ and one can wonder whether it implies $\nabla u \in \mathrm{BMO}_{\mathrm{loc}}(B(0,1/2))$. The answer is still negative.

Proposition 5. *There exists $u \in W^{1,1}_{\mathrm{loc}}(B(0,1))$ and an elliptic $A \in C(B(0,1);\mathbb{R}^{n\times n})$ such that u solves*

$$-\mathrm{div}\,(A\nabla u) = 0$$

and $u \in W^{1,p}_{\mathrm{loc}}(B(0,1))$ for every $p \in (1,\infty)$, but $\nabla u \notin \mathrm{BMO}_{\mathrm{loc}}(B(0,1))$.

The construction of counterexamples is made by explicit formulas, inspired by the construction of J. Serrin. They can be also obtained from asymptotic formulas of V. Kozlov and V. Maz'ya.