Apriori estimates and comparison principles for some classes of nonlinear elliptic equations

A. Mercaldo

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$$|H(x,\xi)| \leq h\,|\xi|^q\,, \qquad p-1 \leq q \leq p$$

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Ω bounded open subset of $\mathbb{R}^N$

$f \in W^{-1,p'}(\Omega)$ or $f \in L^m(\Omega)$

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- Comparison principles $\Rightarrow$ UNIQUENESS results

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Apriori estimates for weak solution u in $W_0^{1,p}(\Omega)$
and existence result

[Leray-Lions, 1965]

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GILBARG -TRUDINGER

CARILLO - CHIPOT, 1985

ARTOLA, 1986

CHIPOT - MICHAILLE, 1989

BOCCARDO-GALLOÜET-MURAT, 1992

ARTOLA - BOCCARDO, 1996

ANDRÈ - CHIPOT, 1996

CASADO-DIAZ, MURAT, PORRETTA, 2007 ....

Comparison principle : Structural assumptions

$$Qu = -\mathrm{div}\,(\mathbf{a}(x,u,\nabla u))\,, \qquad\qquad u \in W^{1,p}(\Omega)$$

Strong monotonicity

$$(\mathbf{a}(x,s,\xi) - \mathbf{a}(x,s,\xi')) \cdot (\xi - \xi') \geq \begin{cases} \alpha \dfrac{|\xi - \xi'|^2}{(|\xi| + |\xi'|)^{2-p}} & 1 \leq p \leq 2, \\ \\ \alpha |\xi - \xi'|^2 (1 + |\xi| + |\xi'|)^{p-2} & p \geq 2, \end{cases}$$

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$$Qu \equiv -\mathrm{div}(|\nabla u|^{p-2}\nabla u) \equiv -\Delta_p u \qquad p \leq 2$$

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Locally Lipschitz continuity with respect to s

$$|\mathbf{a}(x,s,\xi) - \mathbf{a}(x,s',\xi)| \leq \beta(|\xi|^{p-1} + c)|s - s'|\,,$$

[Alvino-Betta-M., 2010]

Models

$$Qu \equiv -\text{div}(A(x,u)|\nabla u|^{p-2}\nabla u) \qquad p \leq 2$$

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with $A(x,s)$ is positive, bounded and Lipschitz continuous with respect to s.

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[Alvino-Betta-M., 2010]
[Casado-Diaz, Murat, Porretta 2007]

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A counterexample shows that strong monotonicity is sharp when $p > 2$

[Alvino-Betta-M., 2010]
[Boccardo-Galloüet-Murat, 1992]

Euristic argument

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$$|\{x\,:\,(u-v)^+ > 0\}| = 0$$

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  Function test $\varphi = \frac{T_t(u-v)}{t}$ $t > 0$

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- Isoperimetric inequality

$$|(u-v)^+ > t|^{1-1/N} \leq const \left(\int_{0<u-v<t} (1 + |\nabla u| + |\nabla v|)^{p-2} |\nabla \varphi|^2 \, dx \right)^{\frac{1}{2}}$$

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- Conclusion : $t \to 0$

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Generalizations

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$$|\mathbf{a}(x, s, \xi) - \mathbf{a}(x, s', \xi)| \le \beta \left[|\xi|^{p-1} + (|s| + |s'|)^{\gamma} + c\right] |s - s'|\,,$$

with

$$0 < \gamma \le \frac{Np}{2(N-p)}, \qquad p > 2\,,$$

when $p < N$, $\gamma > 0$ when $p \ge N$.

Model

$$Qu \equiv -\text{div}(A(x, u)(1 + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u) - \text{div}(\,|u|^{\gamma - 1} u)$$

Operators with first order terms

$$\begin{cases} Qu \equiv -\operatorname{div}\left(\mathbf{a}(x,u,\nabla u)\right) = H(x,\nabla u) + f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$$\boxed{H(x,\nabla u) \leq h\,|\nabla u|^q}, \qquad \boxed{p-1 \leq q \leq p}$$

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$\boxed{H(x,\nabla u) \le h|\nabla u|^q}$, $\boxed{p-1 \le q \le p}$

$q = p - 1$ | $p - 1 < q < p$ | $q = p$

Simplest Model

$$\begin{cases} Qu \equiv -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = h|\nabla u|^q + f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Operators with first order terms: Apriori estimates

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$$\boxed{H(x,\nabla u) \le h\,|\nabla u|^q}, \qquad \boxed{f \in W^{-1,p'}}$$

$q < p - 1$

Apriori estimates for $u \in W_0^{1,p}$ and existence results

[Leray-Lions, 1965]

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Apriori estimates for $u \in W_0^{1,p}$ and existence result

[Maz'ya,1969], [Bottaro-Marina, 1973], [Talenti, 1976], [Del Vecchio-Porzio, 1995], [Betta-M.-Murat-Porzio, 2005]

Apriori estimates for $u \in W_0^{1,q}$, $q < \frac{N(p-1)}{N-1}$ and existence result

[Betta-M.-Murat-Porzio, 2003], [Alvino-M., 2008]

Operators with first order terms: Comparison principle

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[Gilbarg-Trudinger],
[Serrin],
[Betta-M.-Murat-Porzio, 2005],

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Locally Lipschitz continuity of H

$$|H(x,\xi)-H(x,\xi')| \le \begin{cases} \beta\dfrac{|\xi-\eta|}{(1+|\xi|+|\xi'|)^{2-p}} & 1<p\le 2,\\ \beta(|\xi|+|\xi'|)^{p-2}|\xi-\xi'| & p>2,\end{cases}$$

Models

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$$Qu \equiv -\mathrm{div}(|\nabla u|^{p-2}\nabla u) + (1 + |\nabla u|^2)^{\frac{p-1}{2}} \qquad p \leq 2$$

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[Alvino-Betta-M., 2010]
[Betta-M.-Murat-Porzio, 2004]
[Porretta, 2009]

Operators with first order terms: Apriori estimates

$$\begin{cases} Qu \equiv -\text{div}(|\nabla u|^{p-2}\nabla u) = h|\nabla u|^q + f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$p - 1 < q < p$

Necessary conditions on the data in Lebesgue spaces:

$$f \in L^m,\ m \geq \frac{N(q-p+1)}{q},\ \text{when } q > \frac{N(p-1)}{N-1}$$

$\|f\|_{L^m}$ small enough

[Alaar-Pierre, 1993], [Hansson-Maz'ya-Verbitsky, 1999]

Operators with first order terms: Apriori estimates

$$\begin{cases} Qu \equiv -\text{div}\,(\mathbf{a}(x,u,\nabla u)) = h\,|\nabla u|^q + f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$p - 1 < q < p$

Necessary conditions on the data:

$f \in L^m$, $\boxed{m \geq \frac{N(q-p+1)}{q}}$, when $\boxed{q > \frac{N(p-1)}{N-1}}$

$\|f\|_{L^m}$ small enough

[Alaar-Pierre, 1993], [Hansson-Maz'ya-Verbitsky, 1999]

Radial case

$1 < q < 2$

Linear Case ($p = 2$)

$$(1) \qquad \begin{cases} Qu \equiv -\Delta u = |\nabla u|^q & \text{in } B_1 \\ u = 0 & \text{on } \partial B_1 \end{cases}$$

- $q > \frac{N}{N-1}$

$$u(r) = \int_r^1 \frac{1}{t^{\frac{1}{q-1}} \left[\frac{1-q}{1+(N-1)(1-q)} + c t^{(q-1)(N-1)-1} \right]^{\frac{1}{q-1}}} \, dt$$

solves (1)

Radial case

$1 < q < 2$

Linear Case ($p = 2$)

$q > \frac{N}{N-1}$

$$(2) \qquad \begin{cases} Qu \equiv -\Delta u = |\nabla u|^q & \text{in } B_1 \\ u = 0 & \text{on } \partial B_1 \end{cases}$$

- $q > 1 + \frac{2}{N} \Rightarrow u \in H_0^1(\Omega)$ (comparison principle does not hold)

Radial case

$1 < q < 2$

Linear Case ($p = 2$)

$q > \frac{N}{N-1}$

$$(2) \qquad \begin{cases} Qu \equiv -\Delta u = |\nabla u|^q & \text{in } B_1 \\ u = 0 & \text{on } \partial B_1 \end{cases}$$

- $q > 1 + \frac{2}{N} \Rightarrow u \in H_0^1(\Omega)$ (comparison principle does not hold)
- $u(x) \in W_0^{1,q}(\Omega)$

$1 < q < 2$

Linear Case ($p = 2$)

$q > \frac{N}{N-1}$

$$(2) \qquad \begin{cases} Qu \equiv -\Delta u = |\nabla u|^q & \text{in } B_1 \\ u = 0 & \text{on } \partial B_1 \end{cases}$$

- $q > 1 + \frac{2}{N} \Rightarrow u \in H_0^1(\Omega)$ (comparison principle does not hold)
- $u(x) \in W_0^{1,q}(\Omega)$
- u is not regular enough (apriori estimates does not hold)

Radial case

$1 < q < 2$

Linear Case ($p = 2$)

$q > \frac{N}{N-1}$

$$(3) \qquad \begin{cases} Qu \equiv -\Delta u = |\nabla u|^q & \text{in } B_1 \\ u = 0 & \text{on } \partial B_1 \end{cases}$$

- $q > 1 + \frac{2}{N}$ $\Rightarrow u \in H_0^1(\Omega)$ (comparison principle does not hold)
- $u(x) \in W_0^{1,q}(\Omega)$
- u is not regular enough (apriori estimates does not hold)

Radial case

$1 < q < 2$

Linear Case ($p = 2$)

$q > \frac{N}{N-1}$ $\quad$ $q > 1 + \frac{2}{N}$

$$(4) \qquad \begin{cases} Qu \equiv -\Delta u = |\nabla u|^q & \text{in } B_1 \\ u = 0 & \text{on } \partial B_1 \end{cases}$$

$1 < q < \frac{N}{N-1}$ $\quad$ $\frac{N}{N-1} < q < 1 + \frac{2}{N}$ $\quad$ $1 + \frac{2}{N} < q < 2$

Operators with first order terms: Apriori estimates

$$\begin{cases} Qu \equiv -\mathrm{div}\,(\mathbf{a}(x,u,\nabla u)) = h\,|\nabla u|^q + f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$p - 1 < q < p$

$p - 1 < q < \frac{N(p-1)}{N-1}$ with $f \in L^1$

$\frac{N(p-1)}{N-1} < q < p - 1 + \frac{p}{N}$ with $f \in L^m,\ m \geq \frac{N(q-p+1)}{q}$,

$p - 1 + \frac{p}{N} \leq q < p$ with $f \in L^m,\ m \geq \frac{N(q-p+1)}{q}$.

[Grenon-Murat-Porretta, preprint], [Ferone-Messano, 2007]

Operators with first order terms: Apriori estimates

$$\begin{cases} Qu \equiv -\mathrm{div}\,(\mathbf{a}(x,u,\nabla u)) = h|\nabla u|^q + f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$p-1 < q < p$

$p-1 < q < \frac{N(p-1)}{N-1}$ with $f \in L^1$

$\frac{N(p-1)}{N-1} < q < p-1+\frac{p}{N}$ with $f \in L(\frac{N(q-p+1)}{q}, \frac{p}{q})$,

$p-1+\frac{p}{N} \leq q < p$ with $f \in L(\frac{N(q-p+1)}{q}, \frac{p}{q})$.

[Alvino-M., in preparation]

Operators with first order terms: Apriori estimates

$$\begin{cases} Qu \equiv -\text{div}\,(\mathbf{a}(x,u,\nabla u)) = h\,|\nabla u|^q + f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$p - 1 < q < p$

$p - 1 + \frac{p}{N} \leq q < p$

If $\boxed{f \in L(\frac{N(q-p+1)}{q}, \frac{p}{q})}$ with norm small enough, then at least a *weak solution* exists and

$$u \in L((N(q-p+1))^*, p)$$

[Alvino-M., in preparation]

Operators with first order terms: Apriori estimates

$$\begin{cases} Qu \equiv -\text{div}\,(\mathbf{a}(x,u,\nabla u)) = h\,|\nabla u|^q + f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$p-1 < q < p$

$\frac{N(p-1)}{N-1} < q < p - 1 + \frac{p}{N}$

If $f \in L(\frac{N(q-p+1)}{q}, \frac{p}{q})$ with norm small enough, then at least a *solution obtained as a limit of approximations* exists and

$$u \in L((N(q-p+1))^*, p) \qquad |\nabla u| \in L(N(q-p+1), p)$$

[Alvino-M., in preparation]

Operators with first order terms: Apriori estimates

$$\begin{cases} Qu \equiv -\operatorname{div}(\mathbf{a}(x,u,\nabla u)) = h|\nabla u|^q + f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$p-1 < q < p$

Limit case $q = \frac{N(p-1)}{N-1}$

If $f \in L^m\ ,\ m > 1$ with norm small enough, then at least a *solution obtained as a limit of approximations* exists and

$$|\nabla u| \in L(m^*(p-1), p)$$

[Alvino-M., in preparation]

Radial case

$1 < q < 2$

Linear Case ($p = 2$)

$$(5)\qquad \begin{cases} Qu \equiv -\Delta u = |\nabla u|^q + f & \text{in } B_1 \\ u = 0 & \text{on } \partial B_1 \end{cases}$$

$q = \frac{N}{N-1}$,

$$\int_{B_1} f(x)|\log|x||^{N-1} dx < +\infty .$$

Operators with first order terms: Apriori estimates

$$\begin{cases} Qu \equiv -\mathrm{div}\,(\mathbf{a}(x,u,\nabla u)) = h\,|\nabla u|^q + f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$p-1 < q < p$

$p - 1 < q < \frac{N(p-1)}{N-1}$

If $\boxed{f \in L^1(\Omega)}$ with norm small enough, then at least a *solution obtained as a limit of approximations* exists and

$$u \in L\left(\left(\frac{N(p-1)}{N-1}\right)^*, \infty\right) \qquad |\nabla u| \in L\left(\frac{N(p-1)}{N-1}, \infty\right)$$

[Alvino-M., in preparation]

- Approximated problems

$$\begin{cases} Qu \equiv -\mathrm{div}\,(\mathbf{a}(x, u_n, \nabla u_n)) = H_n(x, \nabla u_n) + f_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases}$$

$$f_n \in C_0^\infty(\Omega), \qquad f_n \to f \qquad L^m\text{-strongly}$$

- Apriori estimates for $|\nabla u_n|$ which imply $u_n \to u$
- Passage to the limit

u is a Solution Obtained as Limit of Approximations

Operators with first order terms: Apriori estimates

$$\begin{cases} Qu \equiv -\operatorname{div}(\mathbf{a}(x,u,\nabla u)) = H(x,\nabla u) + f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$$\boxed{H(x,\nabla u) \le h\,|\nabla u|^q}$$

$q = p$

If $\boxed{f \in L(\frac{\beta}{p},1)}$, $\beta > N$ with norm small enough, then at least a *weak solution* exists and

$$u \in L^\infty(\Omega)$$

[Alvino-M., in preparation]

Apriori estimates : main tools

$p-1 < q < p$

- Test Functions :

$$\varphi(x) = \text{sign}(u(x)) \int_0^{|u(x)|} \frac{1}{[\mu(t)]^\alpha}\, dt,$$

or

$$\psi(x) = \text{sign}(u(x)) \int_0^{|u(x)|} [\mu(t)]^\alpha\, dt,$$

for a suitable $\alpha > 0$.

Apriori estimates : main tools

$p-1 < q < p$

- Test Functions :

$$\varphi(x) = \mathrm{sign}(u(x)) \int_0^{|u(x)|} \frac{1}{[\mu(t)]^\alpha}\, dt,$$

or

$$\psi(x) = \mathrm{sign}(u(x)) \int_0^{|u(x)|} [\mu(t)]^\alpha\, dt,$$

for a suitable $\alpha > 0$.

- Sobolev-Hardy inequalities

$$\int_{\Omega^\#} |x|^{N\alpha - p} \left[u^\#(x)\right]^p dx \le \omega_N^{-\alpha} \left(\frac{p}{N-p+N\alpha}\right)^p \int_\Omega [\mu(|u(x)|)]^\alpha\, |\nabla u|^p\, dx$$

$$\int_{\Omega^\#} |x|^{-N\alpha - p} \left[u^\#(x)\right]^p dx \le \omega_N^{-\alpha} \left(\frac{p}{N-p-N\alpha}\right)^p \int_\Omega \frac{|\nabla u|^p}{[\mu(|u(x)|)]^\alpha}\, dx\,,$$

Estimates for the gradient

$$\begin{cases} Qu \equiv -\mathrm{div}\,(\mathbf{a}(x,u,\nabla u)) = h\,|\nabla u|^q + f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$p-1<q<p$

$p-1<q<\frac{N(p-1)}{N-1}$

$\frac{N(p-1)}{N-1}<q<p-1+\frac{p}{N}$

$p-1+\frac{p}{N}\le q<p$

[Betta-Di Nardo - M.- Perrotta, in preparation]

Estimates for the gradient

$$\begin{cases} Qu \equiv -\text{div}\,(\mathbf{a}(x,u,\nabla u)) = h\,|\nabla u|^q + f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$p-1 < q < p$

$\frac{N(p-1)}{N-1} < q < p - 1 + \frac{p}{N}$

$$\boxed{f \in L(m,k) \Rightarrow |\nabla u| \in L(m^*(p-1), k(p-1)).}$$

Here

$$m > \frac{N(q-p+1)}{q} \qquad 0 < k \leq +\infty$$

[Betta-Di Nardo - M.- Perrotta, in preparation]

Estimates for the gradient: main tool

$$\begin{cases} Qu \equiv -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = h\,|\nabla u|^{q} + f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$p - 1 < q < p$

Pointwise estimates of the gradient

[Alvino, V. Ferone, Trombetti, 2000]

Comparison principle

$$\begin{cases} Qu \equiv -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = h|\nabla u|^q + f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$p - 1 < q < p$

$p - 1 < q < \frac{N(p-1)}{N-1}$

$\frac{N(p-1)}{N-1} < q < p - 1 + \frac{p}{N}$

$p - 1 + \frac{p}{N} < q < p$

NO uniqueness of weak solution .

[Betta-Di Nardo - M.- Perrotta, in preparation]
[Porretta, 2009]