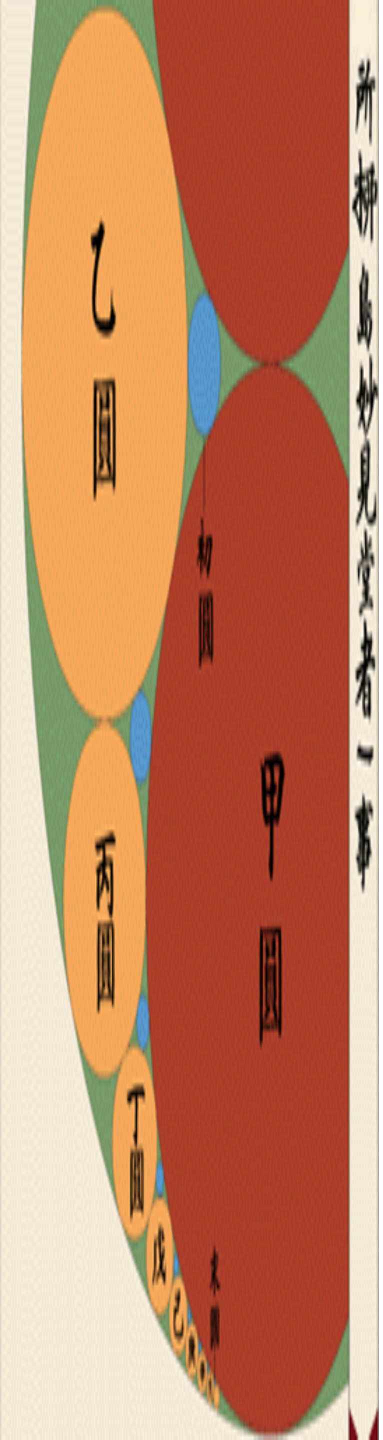

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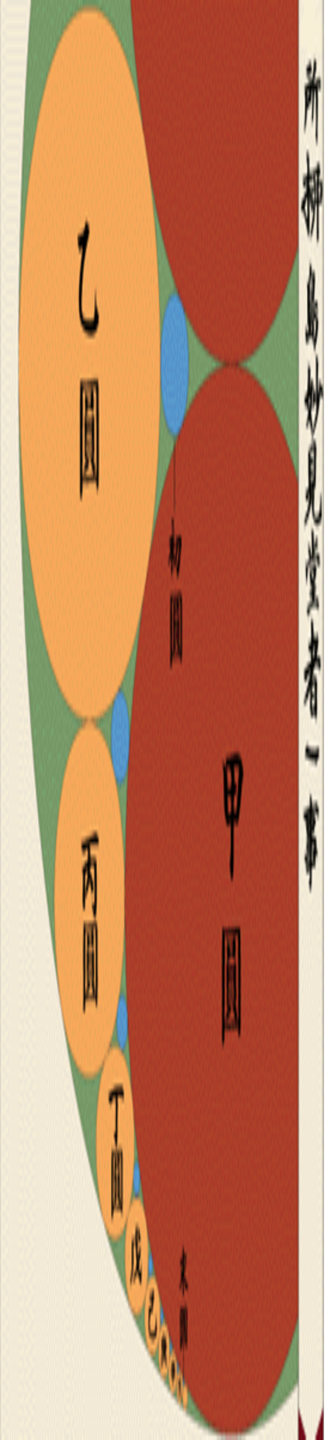

A priori estimates and reduction principles for quasilinear elliptic problems and applications (*)

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The organization of the seminar

- Introduction: Problems and motivations
- Some references
- Quasilinear weakly elliptic operators:
  Kato's inequality
- Reduction principles: the role of positive solutions
- Some classical applications of the reduction principles

The organization of the seminar

- The role of the behavior at infinity of the nonlinearity
- Positivity and Liouville Theorems: Applications to Schröedinger equations

A priori estimates of solutions of quasilinear elliptic equations has been a subject of fundamental and remarkable interest in recent years. For quasilinear elliptic problems, significant and interesting results are dealing with nonnegative solutions associated to nonlinearities that grow faster than the differential part.

Recently, Serrin [41] considered quasilinear coercive equations and inequalities with source term changing sign and proved some interesting Liouville theorems. These results (see also [14, 15] for related contributions) are consequence of appropriate *a priori* estimates on the possible solutions or on suitable functionals of them.

It is well known that when looking for Liouville theorems of *non coercive* nonlinear equations or inequalities, the fact that the nonlinearity has definite sign is of fundamental importance. This is because, in general, canonical examples of this type show that when the nonlinearity changes sign, the problem may posses infinitely many solutions with no *a priori* bound. A canonical example in this direction is the following,

$$-\Delta u = |u|^{q-1}u \quad on\ \mathbb{R}^N. \tag{1}$$

Indeed, it is well known that if $1 < q < \frac{N+2}{N-2}$, $N > 2$, then (1) admits infinitely many radial solutions with increasing number of zeroes.

On the other hand, when the problem is *coercive*, then the situation may be completely different as the following striking result due to Brezis [6] shows.

Theorem (Brezis) *Let $q > 1$. If $u \in L^q_{loc}(\mathbb{R}^N)$ is a distributional solution of*

$$\Delta u \geq |u|^{q-1} u \quad on\ \mathbb{R}^N, \tag{2}$$

then $u \leq 0$ a.e. on $\mathbb{R}^N$. In particular if equality holds in (2), then $u \equiv 0$ a.e. on $\mathbb{R}^N$.

It is worth pointing out that, besides the quite general functional framework, there are no assumptions on the behavior of the possible solutions of (2) at infinity.

Brezis's technique is based on a form of Kato's inequality [24, 6, 2] and on a construction of a suitable Loewner-Nirenberg barrier function. See [27] and [26, 36].

Some generalizations of Brezis's result for quasilinear elliptic inequalities of second order have been obtained in [14, 15, 16] and more recently in a series of papers by Farina and Serrin [17, 18] and Pucci and Serrin [38].

One common aspect in these recent contributions is that from the technical point of view, none of them use a form of Kato's inequality.

Thus one natural question is the extent to which Kato's inequality might be satisfied in the quasilinear case. A positive answer to this problem will allow to develop a general strategy for proving positivity type results as well as Liouville theorems for wide classes of quasilinear inequalities. This will bring together some aspects of qualitatively different problems, namely, coercive and non coercive quasilinear elliptic inequalities of second order. To get an idea of some preliminary results contained in this paper we mention the following special cases of Theorem 3.1 proved in the next section.

Example 1. The p-Laplacian type operator.
Let $\Omega \subset \mathbb{R}^N$ be an open set. Let $f \in L^1_{loc}(\Omega)$ and let $u \in W^{1,p}_{loc}(\Omega)$ be a solution of the inequality,

$$\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) \geq f \qquad on\ \Omega.$$

Then,

$$\operatorname{div}\left(\left|\nabla u^+\right|^{p-2}\nabla u^+\right) \geq \operatorname{sign}^+(u) f \qquad on\ \Omega.$$

Example 2. The 1-Laplacian type operator.

Let $\Omega \subset \mathbb{R}^N$ be an open set. Let $f \in L^1_{loc}(\Omega)$ and let $u \in W^{1,1}_{loc}(\Omega)$ be a solution of the inequality,

$$\operatorname{div}\left(|\nabla u|^{-1}\nabla u\right) \geq f \qquad on\ \Omega.$$

Then,

$$\operatorname{div}\left(\left|\nabla u^+\right|^{-1}\nabla u^+\right) \geq \operatorname{sign}^+(u) f \qquad on\ \Omega.$$

Example 3. The mean curvature operator in non parametric form.
Let $\Omega \subset \mathbb{R}^N$ be an open set. Let $f \in L^1_{loc}(\Omega)$ and let $u \in W^{1,2}_{loc}(\Omega)$ be a solution of the inequality

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) \geq f \qquad on\ \Omega.$$

Then,

$$\operatorname{div}\left(\frac{\nabla u^+}{\sqrt{1+|\nabla u^+|^2}}\right) \geq \operatorname{sign}^+(u) f \qquad on\ \Omega.$$

The main goal of this seminar is to discuss some positivity results and Liouville Theorems for (3).

$$\operatorname{div}_L\left(\mathscr{A}(x,u,\nabla_L u)\right) \geq f(x,u,\nabla_L u) \quad \text{on } \Omega \subset \mathbb{R}^N. \tag{3}$$

Here $\Omega \subset \mathbb{R}^N$ is an open set, $\mathscr{A} : \Omega \times \mathbb{R} \times \mathbb{R}^l \to \mathbb{R}^l$ is a Caratheodory vector field, $f : \Omega \times \mathbb{R} \times \mathbb{R}^l \to \mathbb{R}$ is a Caratheodory function and ∇_L is a quite general vector field.

Our study of (3) can be shortly described as follows.

i) *Reduction of the problem (3) to an inequality that may posses only nonnegative solutions.*

ii) *Good a priori bounds of the possible nonnegative solutions of the reduced problem.*

iii) *Nonexistence of nonnegative solutions of the reduced problem.*

iv) *Nonexistence of nonnegative and changing sign solutions of (3)*

In the above scheme, we shall see that point i) depends on the weak ellipticity of the differential operators. On the other hand, roughly speaking, ii) depends on the behavior of the nonlinearity at infinity. Notice that when dealing with non coercive problems, step ii) depends only on the behavior of the nonlinearity near zero. See [16].

Altogether the above considerations suggest the following natural problem for elliptic equations and inequalities.

Problem A: *What kind of second order elliptic inequalities of type (3) on $\mathbb{R}^N$, admits only solutions of definite sign?*

The possibility to exclude solutions changing sign is of fundamental importance when looking for Liouville theorems. We point out that an interesting consequence of the validity of Kato's inequality is that for a large class of differential inequalities associated to coercive operators, the non existence of positive solutions implies that all possible solutions of the given problem must be of definite (negative) sign. In other words, the problem cannot have oscillatory solutions. This fact is obviously false if the problem is *non coercive*, see (1).

In this paper we will give an answer to the Problem A for inequalities of type (3) and illustrate some general implications. We shall call these consequences *reduction principles*. As we shall see during the course, these consequences imply some *maximum and comparison principles*, which are new in our general framework, and some of them are new even in the Euclidean setting (see Theorems 5.12 and 5.13).

Another point of interest is that our contribution shows that, when looking for Liouville theorems for coercive inequalities of type (3) with $f(x, t, \xi)\, t \geq 0$, the assumption that the possible solutions are nonnegative involves no loss of generality.

Consequently, to our knowledge, most of the Liouville theorems concerning positive solutions proved in the literature for coercive problems, are indeed results on the non oscillatory character of the possible solutions of (3).

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1 Notations and definitions

In this paper ∇ and $|\cdot|$ stand respectively for the usual gradient in $\mathbb{R}^N$ and the Euclidean norm. $\Omega \subset \mathbb{R}^N$ open. Let $\mu \in \mathscr{C}(\mathbb{R}^N; \mathbb{R}^l)$ be a matrix $\mu := (\mu_{ij})$, $i = 1, \dots, l$, $j = 1, \dots, N$ and assume that for any $i = 1, \dots, l$, $j = 1, \dots, N$ the derivative $\frac{\partial}{\partial x_j}\mu_{ij} \in \mathscr{C}(\Omega)$. For $i = 1, \dots, l$, let X_i and its formal adjoint X_i^* be defined as

$$X_i := \sum_{j=1}^{N} \mu_{ij}(\xi) \frac{\partial}{\partial \xi_j}, \qquad X_i^* := -\sum_{j=1}^{N} \frac{\partial}{\partial \xi_j} \left(\mu_{ij}(\xi) \cdot\right), \tag{4}$$

and let ∇_L be the vector field defined by

$$\nabla_L := (X_1, \dots, X_l)^T = \mu \nabla,$$

and

$$\nabla_L^* := (X_1^*, \dots, X_l^*)^T.$$

For any vector field $h = (h_1, \dots, h_l)^T \in \mathscr{C}^1(\Omega, \mathbb{R}^l)$, we shall use the following notation $\mathrm{div}_L(h) := \mathrm{div}\left(\mu^T h\right)$, that is

$$\mathrm{div}_L(h) = -\sum_{i=1}^{l} X_i^* h_i = -\nabla_L^* \cdot h.$$

Examples of vector fields, which we are interested in, are the usual gradient acting on $l(\leq N)$ variables (see Example B.1), vector fields related to Bouendi-Grushin operator (see Example B.2), Heisenberg-Kohn sub-Laplacian (see Example B.3), Heisenberg-Greiner operator (see Example B.4), sub-Laplacian on Carnot Groups (see Appendix A). Another motivation for considering these kind of operators is the following. Let $A = (a_{ij}(x))_{1 \leq i,j \leq N}$ be a matrix with continuous entries. Consider the linear operator $Lu := \text{div}\,(A(x)\nabla u)$. Assume that A is symmetric and positive semidefinite (that is $a_{ij} = a_{ji}$ and $A(x)\xi \cdot \xi \geq 0$ for any $\xi \in \mathbb{R}^N$.) With this assumption the operator L is weakly elliptic see Definition 1.1 below. Since A is symmetric and positive semidefinite, there exists a matrix μ such that $A = \mu^T \mu$. Let l be the rank of μ. Since A may be singular, in general we shall have $l \leq N$. Therefore, setting $\nabla_L := \mu\nabla$ and $\text{div}_L(\cdot) := \text{div}\left(\mu^T \cdot\right)$, the operator L can be rewritten as $Lu = \text{div}_L(\nabla_L u)$ (formally as the Laplace operator). Finally, even if the entries of the matrix A are smooth, in general then nothing can be said on the regularity of the entries of μ.

Since we are interested in weak solutions of the problems under consideration, we shall allow that the entries of the matrix μ are singular. However, for simplicity we shall assume that μ_{ij} are continuous.

Let $\delta := (\delta_1, \ldots, \delta_N)$ be an N-uple of positive real numbers. Let $R > 0$, we shall denote by δ_R the anisotropic dilation $\delta_R : \mathbb{R}^N \to \mathbb{R}^N$ defined by

$$\delta_R(x) = \delta_R(x_1, \ldots, x_N) := (R^{\delta_1} x_1, \ldots, R^{\delta_N} x_N). \tag{5}$$

The Jacobian of the transformation δ_R is given by $J(\delta_R) = R^Q$, where $Q := \delta_1 + \delta_2 + \cdots + \delta_N$.

The Jacobian of the transformation δ_R is given by $J(\delta_R) = R^Q$, where $Q := \delta_1 + \delta_2 + \cdots + \delta_N$.

In Chapter III we shall require that ∇_L is *pseudo homogeneous of degree 1 with respect to dilation* δ_R, that there exist $\delta_i > 0$ $(i = 1..N)$ such that

$$\text{for each } \phi \in \mathscr{C}^1(\mathbb{R}^N) \text{ and } R > 0: \quad \nabla_L(\phi(\delta_R(\cdot))) = R(\nabla_L \phi)(\delta_R)(\cdot). \tag{6}$$

A nonnegative continuous function $S : \mathbb{R}^N \to \mathbb{R}_+$ is called a *homogeneous norm*, if
i) $S(x) = 0$ if and only if $x = 0$, and
ii) it is homogeneous of degree 1 with respect to δ_R (i.e. $S(\delta_R(x)) = RS(x)$).
An example of homogeneous norm which is differentiable for $x \neq 0$ is given by

$$S_\delta(x) := \left(\sum_{i=1}^{N} (x_i^r)^{\frac{d}{\delta_i}} \right)^{\frac{1}{rd}}, \tag{7}$$

where $d := \delta_1 \delta_2 \cdots \delta_N$ and r is the lowest even integer such that $r \geq \max\{\delta_1/d, \ldots, \delta_N/d\}$.

Notice that if S is a homogeneous norm differentiable a.e. and ∇_L is pseudo homogeneous of degree 1 with respect to δ_R, then $|\nabla_L S|$ is homogeneous of degree 0 with respect to δ_R. Hence the function $|\nabla_L S|$ is bounded.

In Chapter III we shall fix a homogeneous norm S differentiable away from 0 and we shall set

$$\psi := |\nabla_L S(\cdot)| \tag{8}$$

We define B_R the ball of radius $R > 0$ generated by the norm S, i.e. $B_R := \{x : S(x) < R\}$ and A_R stands for the annulus $B_{2R} \setminus \overline{B_R}$. Therefore we have

$$|B_R| = \int_{B_R} dx = R^Q \int_{S(x)<1} dx = c_S R^Q \quad \text{and} \quad |A_R| = c_S(2^Q - 1)R^Q.$$

A canonical framework for which our results apply, see next chapters, is the Euclidean space $(\mathbb{R}^N, |\cdot|)$ with $|\cdot|$ the Euclidean norm. In this case $\mu = I_N$ the identity matrix in N dimension, $\nabla_L = \nabla$ is the isotropic gradient and div_L is the divergence operator. The dilation δ_R defined by

$$\delta_R(x) = \delta_R(x_1, \ldots, x_N) := (Rx_1, \ldots, Rx_N),$$

is isotropic. Here, $Q = N$ is the dimension of the space. In this case, $\psi \equiv 1$ and B_R is the Euclidean open ball of radius R centered at the origin.

Another setting in which our results apply is the framework of Carnot Groups. For more details see Appendix A. Further examples will be discussed in Appendix B below.

In what follows $\mathscr{A}: \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^l \to \mathbb{R}^l$ shall be assumed to be a Caratheodory function, that is for each $t \in \mathbb{R}$ and $\xi \in \mathbb{R}^l$ the function $\mathscr{A}(\cdot, t, \xi)$ is measurable; and for a.e. $x \in \mathbb{R}^N$, $\mathscr{A}(x, \cdot, \cdot)$ is continuous.

We consider operators L "generated" by $\mathscr{A}$, that is

$$L(u)(x) = \mathrm{div}_L \left(\mathscr{A}(x, u(x), \nabla_L u(x)) \right).$$

Our model cases are the p-Laplacian operator, the mean curvature operator and some related generalizations. See Examples 1.3 below.

Definition 1.1 *Let* $\mathscr{A}: \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^l \to \mathbb{R}^l$ *be a Caratheodory function. The function* $\mathscr{A}$ *is called* weakly elliptic *if it generates a weakly elliptic operator* L *i.e.*

$$\begin{aligned} &\mathscr{A}(x, t, \xi) \cdot \xi \geq 0 \quad \textit{for each } x \in \mathbb{R}^N, t \in \mathbb{R}, \xi \in \mathbb{R}^l, \\ &\mathscr{A}(x, 0, \xi) = 0 \quad \textit{or} \quad \mathscr{A}(x, t, 0) = 0 \end{aligned} \qquad (WE)$$

Definition 1.2 *Let* $\Omega \subset \mathbb{R}^N$ *be an open set and let* $f : \Omega \times \mathbb{R} \times \mathbb{R}^l \to \mathbb{R}$ *be a Caratheodory function. Let* $p \geq 1$. *We say that* $u \in W^{1,p}_{loc}(\Omega)$ *is a* weak solution *of*

$$\mathrm{div}_L \left(\mathscr{A}(x, u, \nabla_L u)\right) \geq f(x, u, \nabla_L u) \qquad on \quad \Omega,$$

if $\mathscr{A}(\cdot, u, \nabla u) \in L^{p'}_{loc}(\Omega)$, $f(\cdot, u, \nabla_L u) \in L^1_{loc}(\Omega)$, *and for any nonnegative* $\phi \in \mathscr{C}^1_0(\Omega)$ *we have*

$$-\int_\Omega \mathscr{A}(x, u, \nabla_L u) \cdot \nabla_L \phi \geq \int_\Omega f(x, u, \nabla_L u)\phi.$$

Definition 2.1 *Let $\mathscr{A} : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^l \to \mathbb{R}^l$ be a Caratheodory function. The function $\mathscr{A}$ is called* weakly elliptic *if it generates a weakly elliptic operator L i.e.*

$$\begin{aligned} \mathscr{A}(x,t,w) \cdot w \geq 0 \quad \textit{for each } x \in \mathbb{R}^N, t \in \mathbb{R}, w \in \mathbb{R}^l, \\ \mathscr{A}(x,0,w) = 0 \quad \textit{or} \quad \mathscr{A}(x,t,0) = 0 \end{aligned} \tag{WE}$$

*Let $p > 1$, the function $\mathscr{A}$ is called (**W**-p-**C**) (weakly-p-coercive) if it generates a weakly-p-coercive operator L i.e. if there exists a constant $a > 0$ such that*

$$(\mathscr{A}(x,t,w) \cdot w) \geq a\,|\mathscr{A}(x,t,w)|^{p'} \text{ for each } x \in \mathbb{R}^N, t \in \mathbb{R}, w \in \mathbb{R}^l. \tag{$\mathbf{W}$-p-$\mathbf{C}$}$$

Example 2.2

1. *Let $p > 1$. The p-Laplacian operator defined on suitable functions u by,*

   $$\Delta_{p,L} u = \operatorname{div}_L(|\nabla_L u|^{p-2} \nabla_L u)$$

   *is an operator generated by $\mathscr{A}(x,t,w) := |w|^{p-2} w$ which is **W**-p-**C** and **S**-p-**C**.*

2. *If $\mathscr{A}$ is **S**-p-**C**, then $\mathscr{A}$ is **W**-p-**C**.*

3. *The mean curvature operator in non parametric form*

   $$T(u) := \operatorname{div}_L\left(\frac{\nabla_L u}{\sqrt{1 + |\nabla_L u|^2}}\right),$$

   *is generated by $\mathscr{A}(x,t,w) := \frac{w}{\sqrt{1+|w|^2}}$. In this case $\mathscr{A}$ is **W**-2-**C** and of mean curvature type but it is not **S**-2-**C**.*

Let $p \geq 1$, the function $\mathscr{A}$ is called **W**-p-**C** *(weakly-p-coercive) (see [4]), if $\mathscr{A}$ is (WE) and it generates a weakly-p-coercive operator L, i.e. if there exists a constant $k_2 > 0$ such that*

$$(\mathscr{A}(x,t,\xi)\cdot\xi)^{p-1} \geq k_2\,|\mathscr{A}(x,t,\xi)|^p \text{ for each } x \in \mathbb{R}^N,\, t \in \mathbb{R},\, \xi \in \mathbb{R}^l. \qquad (\mathbf{W}\text{-}p\text{-}\mathbf{C})$$

Let $p > 1$, the function $\mathscr{A}$ is called **S-**p**-C** *(strongly-p-coercive) (see [40, 4, 32]), if there exist $k_1, k_2 > 0$ constants such that*

$$(\mathscr{A}(x,t,\xi)\cdot\xi) \geq k_1\,|\xi|^p \geq k_2\,|\mathscr{A}(x,t,\xi)|^{p'} \text{ for each } x \in \mathbb{R}^N,\, t \in \mathbb{R},\, \xi \in \mathbb{R}^l. \qquad (\mathbf{S}\text{-}p\text{-}\mathbf{C})$$

Examples 1.3 *1. Let $p > 1$. The p-Laplacian operator defined on suitable functions u by,*

$$\Delta_{p,L} u = \operatorname{div}_L \left(|\nabla_L u|^{p-2} \nabla_L u \right)$$

is an operator generated by $\mathscr{A}(x,t,\xi) := |\xi|^{p-2}\,\xi$ *which is* $\mathbf{S}$-p-$\mathbf{C}$.

2. *If $\mathscr{A}$ is of mean curvature type, that is $\mathscr{A}$ can be written as $\mathscr{A}(x,t,\xi) := A(|\xi|)\xi$ with $A : \mathbb{R} \to \mathbb{R}$ a positive bounded continuous function (see [31, 4]), then $\mathscr{A}$ is* **W-2-C**.

3. *The mean curvature operator in non parametric form*

$$Tu := \operatorname{div}_L \left(\frac{\nabla_L u}{\sqrt{1 + |\nabla_L u|^2}} \right),$$

is generated by $\mathscr{A}(x,t,\xi) := \frac{\xi}{\sqrt{1+|\xi|^2}}$. *In this case $\mathscr{A}$ is* $\mathbf{W}$-p-$\mathbf{C}$ *with* $1 \le p \le 2$ *and of mean curvature type but it is not* **S-2-C**.

4. *Let $m > 1$. The operator*

$$T_m u := \operatorname{div} \left(\frac{|\nabla u|^{m-2} \nabla u}{\sqrt{1 + |\nabla u|^m}} \right)$$

is $\mathbf{W}$-p-$\mathbf{C}$ *for* $m \ge p \ge m/2$.

5. *Let $p > 1$ and define*

$$Lu := \sum_{i=1}^{N} \partial_i \left(|\partial_i u|^{p-2} \partial_i u \right).$$

The operator L is **S**-p-**C**.

6. *The operator defined by*

$$\operatorname{div} \left(\frac{|u| \nabla u}{|u| + |\nabla u|} \right)$$

is **W-**2**-C**.

7. *Let $\nu > 0$ and define*

$$B_\nu u := \nu \operatorname{div} \left(\frac{|u| \nabla u}{\sqrt{u^2 + \frac{\nu^2}{c^2} |\nabla u|^2}} \right).$$

The operator B_ν is related to the so called "tempered diffusion equation" *or* "relativistic heat equation" *(here ν is a constant representing a kinematic viscosity and c the speed of light). See [5] and [39]. This operator is* **W-**2**-C**.

8. *Letting $\nu \to +\infty$ in B_ν above, we obtain the operator that appears in the so called* "diffusion equation in transparent media",

$$B_\infty u := c \operatorname{div} \left(\frac{|u| \nabla u}{|\nabla u|} \right)$$

See [5]. This operator is obviously (WE).

3 Quasilinear weakly elliptic operators

In this section we consider a class of quasilinear elliptic operators for which we can prove a suitable version of inequality (9). We point out that the following results hold for a wide class of differential operators for which no group invariance is required. Of course the price to pay for this generality is that we need to consider solutions that belong to the space $W^{1,p}_{loc}(\Omega)$. Under additional assumption on the underline group structure and suitable invariance, it is possible to handle solutions that belong to the more natural space $W^{1,p}_{L,loc}(\Omega)$. See Remark 3.5 for the exact meaning.

Let Ω be an open set contained in $\mathbb{R}^N$, $p \geq 1$ and $u \in W^{1,p}_{loc}(\Omega)$.

Theorem 3.1 (Kato's inequality: The quasilinear case) *Let $\mathscr{A}$ be such that*

$$\mathscr{A}(x,t,\xi)\cdot\xi \geq 0 \qquad for\ any\ x \in \Omega,\ t \in \mathbb{R},\ \xi \in \mathbb{R}^l. \tag{14}$$

Let $f \in L^1_{loc}(\Omega)$ and let $u \in W^{1,p}_{loc}(\Omega)$ be a weak solution of

$$\mathrm{div}_L\left(\mathscr{A}(x,u,\nabla_L u)\right) \geq f \qquad on\ \ \Omega. \tag{15}$$

Then

$$\mathrm{div}_L\left(\mathrm{sign}^+ u\ \mathscr{A}(x,u,\nabla_L u)\right) \geq \mathrm{sign}^+ u\ f \qquad on\ \ \Omega. \tag{16}$$

Moreover if

$$\mathrm{div}_L\left(\mathscr{A}(x,u,\nabla_L u)\right) = f \qquad \textit{on} \quad \Omega, \tag{17}$$

then

$$\mathrm{div}_L\left(\mathrm{sign}\, u\ \mathscr{A}(x,u,\nabla_L u)\right) \geq \mathrm{sign}\, u\ f \qquad \textit{on} \quad \Omega. \tag{18}$$

In particular, if $\mathscr{A}$ *is (WE), then* u^+ *is a weak solution of*

$$\mathrm{div}_L\left(\mathscr{A}(x,u^+,\nabla_L u^+)\right) \geq \mathrm{sign}^+ u\ f \qquad \textit{on} \quad \Omega. \tag{19}$$

If in addition $\mathscr{A}$ *is odd i.e.*

$$\mathscr{A}(x,-t,-\xi) = -\mathscr{A}(x,t,\xi), \tag{20}$$

and u *is a solution of (17), then* $|u|$ *satisfies,*

$$\mathrm{div}_L\left(\mathscr{A}(x,|u|,\nabla_L |u|)\right) \geq \mathrm{sign}\, u\ f \qquad \textit{on} \quad \Omega. \tag{21}$$

The proof of Kato's inequality is based on the following:

Lemma 3.2 *Let $\mathscr{A}$ satisfy (14). Let $f \in L^1_{loc}(\Omega)$ and let $u \in W^{1,p}_{loc}(\Omega)$ be a weak solution of*

$$\mathrm{div}_L\left(\mathscr{A}(x, u, \nabla_L u)\right) \geq f \qquad \textit{on} \quad \Omega.$$

Let $\gamma \in \mathscr{C}^1(\mathbb{R})$ be nonnegative and such that γ, γ' are bounded. Then,

$$\int_\Omega f\gamma(u)\phi + \int_\Omega \mathscr{A}(x, u, \nabla_L u) \cdot \nabla_L u\ \gamma'(u)\phi \leq -\int_\Omega \mathscr{A}(x, u, \nabla_L u) \cdot \nabla_L \phi\ \gamma(u). \qquad (22)$$

In particular if $\gamma' \geq 0$, we have

$$\mathrm{div}_L\left(\gamma(u)\mathscr{A}(x, u, \nabla_L u)\right) \geq \gamma(u) f \qquad \textit{on} \quad \Omega. \qquad (23)$$

Moreover if

$$\mathrm{div}_L\left(\mathscr{A}(x, u, \nabla_L u)\right) = f \qquad \textit{on} \quad \Omega, \qquad (24)$$

then (23) holds provided $\gamma' \geq 0$ regardless the nonnegativity assumption on γ.

Proof of Theorem 3.1. In order to prove (16) it suffices to approximate sign^+ with a family of nonnegative smooth bounded functions which are nondecreasing and with bounded derivative.

To this end we introduce,

$$\gamma_\epsilon(t) := \begin{cases} \left(\frac{2}{\pi}\arctan(t/\epsilon)\right)^2, & if\ t \geq 0; \\ 0 & if\ t < 0. \end{cases}$$

Then $0 \leq \gamma_\epsilon < 1$ and $\gamma_\epsilon(t) \to sign^+(t)$. Applying Lemma 3.2, from (23) with γ replaced by γ_ϵ we obtain,

$$\int_\Omega f\gamma_\epsilon(u)\phi \leq -\int_\Omega \gamma_\epsilon(u)\mathscr{A}(x,u,\nabla_L u)\cdot\nabla_L\phi. \tag{27}$$

Passing to the limit $\epsilon \to 0$ in (27), by Lebesgue dominated convergence theorem we finally obtain (16), i.e.

$$\int_\Omega \text{sign}^+(u)f\phi \leq -\int_\Omega \text{sign}^+(u)\mathscr{A}(x,u,\nabla_L u)\cdot\nabla_L\phi. \tag{28}$$

In addition, if $\mathscr{A}$ is (WE) from the identity

$$\text{sign}^+(u)\ \mathscr{A}(x,u,\nabla_L u) = \mathscr{A}(x,u^+,\nabla_L u^+)\quad on\quad \Omega,$$

inequality (19) follows.

The proof of (18) follows once again by applying the above argument to the family of functions

$$\gamma_\epsilon(t) := \frac{2}{\pi}\arctan(t/\epsilon).$$

□

4 Examples

Inequality (19) holds for all (WE) operators, in particular for all operators listed in Examples 1.3. In this section we illustrate in detail some classes of operators for which Kato's inequality holds.

4.1 p-Laplacian type operators

Let $f \in L^1_{loc}(\Omega)$ and let $u \in W^{1,p}_{loc}(\Omega)$ be a solution of the inequality,

$$L_p u := \mathrm{div}_L \left(|\nabla_L u|^{p-2} \nabla_L u \right) \geq f \qquad on\ \Omega.$$

Then,

$$L_p u^+ \geq \mathrm{sign}^+(u) f \qquad on\ \Omega.$$

In particular if ∇_L is the Euclidean gradient ∇ and $u \in W^{1,p}_{loc}(\Omega)$ is a weak solution of

$$\Delta_p u \geq f \qquad on\ \Omega,$$

then $u^+ \in W^{1,p}_{loc}(\Omega)$ is a weak solution of

$$\Delta_p u^+ \geq \mathrm{sign}^+(u) f \qquad on\ \Omega. \tag{34}$$

As a consequence of (34), we have the following. See [41, 17] for a different proof under stronger assumption on the solutions.

Proposition 4.1 *Let $q > p-1 > 0$. If $u \in W^{1,p}_{loc}(\mathbb{R}^N) \cap L^q_{loc}(\mathbb{R}^N)$ is a weak solution of*

$$\Delta_p u \geq |u|^{q-1} u \quad on\ \mathbb{R}^N, \tag{35}$$

then $u \leq 0$ a.e. on $\mathbb{R}^N$. In particular, if in (35) the equality sign holds, then $u \equiv 0$ a.e. on $\mathbb{R}^N$.

Proof. Let $q > p-1$ and set

$$u_R(x) := \frac{cR^\beta}{\left(R^{p/(p-1)} - |x|^{p/(p-1)}\right)^\alpha} \qquad x \in B_R,$$

with

$$\alpha := \frac{p}{q-p+1}, \qquad \beta := \begin{cases} 0 & if \quad q \leq 1, \\ \frac{\alpha p}{p-1} - \frac{p}{q-1} & if \quad q > 1, \end{cases}$$

and the positive constant c satisfies $c^{q-p+1} = (\dfrac{\alpha p}{p-1})^{p-1} \max\{N, p(\alpha+1)\}$.

The function u_R is a slight modification of the Loewner-Nirenberg [27] function used by Brezis in his original argument [6] for $p = 2$. It is easy to check that for $R > 0$, u_R is a solution of the inequality

$$-\Delta_p u_R + u_R^q \geq 0 \qquad on\ B_R.$$

Indeed

$$\frac{\Delta_p u_1}{u_1^q} = \left(\frac{\alpha p}{p-1}\right)^{p-1} c^{-q+p-1}[N + (p\alpha + p - N)r^{p/(p-1)}] \leq 1.$$

Now, since

$$u_R = \frac{cR^{\beta-\alpha p/(p-1)}}{\Big(1-(\frac{|x|}{R})^{p/(p-1)}\Big)^{\alpha}} = R^{\beta-\alpha p/(p-1)}u_1(\frac{|x|}{R}),$$

for $R \geq 1$ we have

$$\frac{\Delta_p u_R}{u_R^q} = \frac{R^{\beta-\alpha p/(p-1)}R^{-p}(\Delta_p u_1)(|x|/R)}{R^{q(\beta-\alpha p/(p-1))}u_1^q(|x|/R)} \leq R^{(1-q)(\beta-\alpha p/(p-1))-p} \leq 1.$$

Let $u \in W^{1,p}_{loc}(\mathbb{R}^N) \cap L^q_{loc}(\mathbb{R}^N)$ be a weak solution of (35).
Applying inequality (34) it follows that, in the weak sense we have,

$$\Delta_p u^+ \geq (u^+)^q \qquad on \ \ \mathbb{R}^N.$$

Since u^+ is p-subharmonic, from [29] we deduce $u^+ \in L^\infty_{loc}(\mathbb{R}^N)$. By the weak comparison principle we deduce that, for any $R > 1$ we have $u^+ \leq u_R$ a.e. on B_R. Since $u_R \to 0$ for $R \to +\infty$, it follows that $u^+ \leq 0$ a.e. on $\mathbb{R}^N$. This completes the proof. □

The reduction principles

Throughout the following sections, unless otherwise stated, Ω stands for an open subset contained in $\mathbb{R}^N$ and $\mathscr{A}$ is (WE).

5 The role of positive solutions

In this section we are going to develop the main ideas that we shall use throughout this paper when studying quasilinear elliptic inequalities of coercive type. It is known [16], that dealing with non coercive problems of the form,

$$-\operatorname{div}_L(\mathscr{A}(x, v, \nabla_L v)) \geq f(x, v), \qquad v \geq 0, \quad \textit{on} \;\; \mathbb{R}^N, \tag{36}$$

where $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a nonnegative function, the existence or nonexistence of positive solutions in a suitable functional space is determined only by the behavior of the non linearity f near zero. On the other hand in the coercive case, that is

$$\operatorname{div}_L(\mathscr{A}(x, v, \nabla_L v)) \geq g(x, v) \qquad \textit{on} \;\; \mathbb{R}^N, \tag{37}$$

and $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a given function, a first step for the understanding the solutions set, is to reduce our problem to an inequality with solutions having a definite sign. A remarkable fact is that this reduction is always possible for weakly elliptic quasilinear inequalities. Even though, as we shall see during the course, this reduction leads to nontrivial problems in finding *good a priori* estimates on the possible nonnegative solutions of (37).

In keeping with the notation and terminology introduced above, we have.

Theorem 5.1 *Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^l \to \mathbb{R}$ be a Caratheodory function satisfying*

$$f(x, 0, \xi) = 0 \quad or \quad f(x, t, 0) = 0. \tag{38}$$

Let $p \geq 1$ and let $X \subset W^{1,p}_{loc}(\Omega)$ be a set such that if $u \in X$ then $u^+ \in X$. Assume that the problem

$$\operatorname{div}_L (\mathscr{A}(x, v, \nabla_L v)) \geq f(x, v, \nabla_L v) \qquad v \geq 0 \quad on \quad \Omega, \tag{39}$$

has no nontrivial weak solutions in X.

Then any weak solution of the problem

$$\operatorname{div}_L (\mathscr{A}(x, u, \nabla_L u)) \geq f(x, u, \nabla_L u) \qquad u \in X, \tag{40}$$

is nonpositive, i.e.

$$u(x) \leq 0 \quad a.e. \quad x \in \Omega.$$

Proof. Let $u \in X$ be a solution of (40). By inequality (19) and by hypothesis (38) it follows that

$$\operatorname{div}_L \left(\mathscr{A}(x, u^+, \nabla_L u^+)\right) \geq \operatorname{sign}^+ u\, f(x, u, \nabla_L u) = f(x, u^+, \nabla_L u^+) \qquad on \quad \Omega.$$

Hence $u^+ \in X$ is a nonnegative solution of (39). Thus $u^+ \equiv 0$ a.e. on Ω. This completes the proof. □

In what follows for a given function $\mathscr{A} : \Omega \times \mathbb{R} \times \mathbb{R}^l \to \mathbb{R}^l$, we shall denote with $\overline{\mathscr{A}}$ the function $\overline{\mathscr{A}} : \Omega \times \mathbb{R} \times \mathbb{R}^l \to \mathbb{R}^l$ defined by

$$\overline{\mathscr{A}}(x, t, \xi) := -\mathscr{A}(x, -t, -\xi). \tag{41}$$

Notice that if $\mathscr{A}$ is weakly elliptic or **W**-p-**C** or **S**-p-**C** then $\overline{\mathscr{A}}$ has the same properties. Moreover if $\mathscr{A}$ is odd (see (20)), then $\overline{\mathscr{A}} = \mathscr{A}$.

An immediate implication of the above theorems is the following obvious consequence for non coercive problems.

Theorem 5.3 *Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^l \to \mathbb{R}$ be a Caratheodory function satisfying (38). Let $p \geq 1$ and let $X \subset W^{1,p}_{loc}(\Omega)$ be a set such that if $u \in X$ then $-u, u^+ \in X$. Assume that the problem*

$$\operatorname{div}_L\left(\overline{\mathscr{A}}(x, v, \nabla_L v)\right) \geq f(x, -v, -\nabla_L v) \qquad v \geq 0 \quad on \quad \Omega, \tag{42}$$

has no nontrivial weak solutions in X.

Then any weak solution of the problem

$$-\operatorname{div}_L\left(\mathscr{A}(x, u, \nabla_L u)\right) \geq f(x, u, \nabla_L u) \qquad on \quad \Omega, \quad u \in X, \tag{43}$$

is nonnegative, i.e.

$$u(x) \geq 0 \quad a.e. \quad x \in \Omega.$$

Proof. Let $u \in X$ be a solution of (43). The function $w := -u \in X$ is a solution of

$$\mathrm{div}_L\left(\overline{\mathscr{A}}(x, v, \nabla_L v)\right) \geq f(x, -v, -\nabla_L v) \quad \textit{on} \quad \Omega, \quad v \in X.$$

Since $\overline{\mathscr{A}}$ is weakly elliptic and f satisfies (38) we are in the position to apply Theorem 5.1, which yields $w \leq 0$ on Ω. This completes the proof. □

Theorem 5.4 *Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Caratheodory function satisfying (38) and set*

$$\overline{f}(x, t, \xi) = -f(x, -t, -\xi).$$

Let $p \geq 1$ and let $X \subset W^{1,p}_{loc}(\Omega)$ be a set such that if $u \in X$ then $-u, u^+ \in X$. Assume that the problems,

$$\mathrm{div}_L\left(\mathscr{A}(x, v, \nabla_L v)\right) \geq f(x, v, \nabla_L v), \qquad v \geq 0, \quad \textit{on} \quad \Omega, \tag{44}$$

$$\mathrm{div}_L\left(\overline{\mathscr{A}}(x, v, \nabla_L v)\right) \geq \overline{f}(x, v, \nabla_L v), \qquad v \geq 0, \quad \textit{on} \quad \Omega, \tag{45}$$

have no nontrivial weak solutions in X.

Then the problem

$$\mathrm{div}_L\left(\mathscr{A}(x, u, \nabla_L u)\right) = f(x, u, \nabla_L u) \qquad \textit{on} \quad \Omega \quad u \in X, \tag{46}$$

has no nontrivial weak solutions.

Proof. Let $u \in X$ be a solution of (43). The function $w := -u \in X$ is a solution of

$$\operatorname{div}_L\left(\overline{\mathscr{A}}(x, v, \nabla_L v)\right) \geq f(x, -v, -\nabla_L v) \quad on \quad \Omega, \quad v \in X.$$

Since $\overline{\mathscr{A}}$ is weakly elliptic and f satisfies (38) we are in the position to apply Theorem 5.1, which yields $w \leq 0$ on Ω. This completes the proof. □

Theorem 5.4 *Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Caratheodory function satisfying (38) and set*

$$\overline{f}(x, t, \xi) = -f(x, -t, -\xi).$$

Let $p \geq 1$ and let $X \subset W^{1,p}_{loc}(\Omega)$ be a set such that if $u \in X$ then $-u, u^+ \in X$. Assume that the problems,

$$\operatorname{div}_L(\mathscr{A}(x, v, \nabla_L v)) \geq f(x, v, \nabla_L v), \qquad v \geq 0, \quad on \quad \Omega, \tag{44}$$

$$\operatorname{div}_L\left(\overline{\mathscr{A}}(x, v, \nabla_L v)\right) \geq \overline{f}(x, v, \nabla_L v), \qquad v \geq 0, \quad on \quad \Omega, \tag{45}$$

have no nontrivial weak solutions in X.

Then the problem

$$\operatorname{div}_L(\mathscr{A}(x, u, \nabla_L u)) = f(x, u, \nabla_L u) \qquad on \quad \Omega \quad u \in X, \tag{46}$$

has no nontrivial weak solutions.

5.1 Applications: maximum and comparison principles

Although it is not exactly the direction in which we have been going, it seems appropriate to include here some interesting examples and applications of the reduction ideas.

Let $\Omega \subset \mathbb{R}^N$ be an open set and let $u, v \in L^1_{loc}(\Omega)$. In what follows the inequality $u \leq v$ in $\partial\Omega$ should be understood in the sense that for every $\epsilon > 0$ there exists a neighborhood V of $\partial\Omega$ such that for a.e. $x \in V$ we have $u(x) \leq v(x) + \epsilon$.

Moreover we shall need of the following hypothesis on ∇_L.

$$\text{If } O \subset \mathbb{R}^N \text{ is an open connected set and } \nabla_L u \equiv 0 \ \Rightarrow u \equiv \text{const} \ \text{ on } O. \tag{47}$$

This assumption obviously holds if $\nabla_L = \nabla$, the standard Euclidean gradient. It also holds in Carnot group setting as well as in all the examples of Appendix B except for the gradient of l variables, see Example B.1. A general condition assuring the validity of (47) is related to the Hörmander condition and to Caratheodory-Chow-Rashevsky theorem, see [8].

Theorem 5.7 (The weak maximum principle) *Let $\mathscr{A}$ be weakly elliptic such that for a.e.$x \in \Omega$,*

$$if \quad \mathscr{A}(x,t,\xi) = 0 \quad then \quad t = 0 \ or \ \xi = 0. \tag{48}$$

Assume that (47) holds.

Let $p \geq 1$ and let $u \in W^{1,p}_{loc}(\Omega)$ be a weak solution of

$$\mathrm{div}_L \left(\mathscr{A}(x,u,\nabla_L u)\right) \geq 0 \qquad on \quad \Omega.$$

Suppose that $\Omega' \subset\subset \Omega$ and $u \leq 0$ on $\partial\Omega'$. Then $u \leq 0$ a.e. on Ω'.

Theorem 5.8 (The weak comparison principle) *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set. Let $\mathscr{A}$ be a monotone function. Let $f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ be Caratheodory functions such that*

$$f(x,t) \geq g(x,t), \qquad a.e.\ x \in \Omega, \quad t \in \mathbb{R}, \tag{50}$$

and at least one of them is nondecreasing with respect to t variable. Assume that (47) holds and one of the following conditions

1. *$\mathscr{A}$ is strictly monotone;*

2. *$f(x,t)$ or $g(x,t)$ is increasing with respect to t variable;*

is satisfied.

Let $u, v \in W^{1,p}_{loc}(\Omega)$ be such that

$$-\,\mathrm{div}_L \left(\mathscr{A}(x,v,\nabla_L v)\right) + g(x,v) \geq -\mathrm{div}_L \left(\mathscr{A}(x,u,\nabla_L u)\right) + f(x,u). \tag{51}$$

If $u \leq v$ on $\partial\Omega$, then $u \leq v$ a.e. in Ω.

Theorem 5.9 (A generalized weak maximum principle) *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and (47) holds. Suppose that there exists a Caratheodory function $G : \Omega \times \mathbb{R} \to \mathbb{R}$ and a constant $\overline{\lambda} > 0$ such that for any nonnegative $v \in W_0^{1,p}(\Omega)$ we have $G(\cdot, v(\cdot)) \in L^1(\Omega)$ and*

$$\int_\Omega \mathscr{A}(x, v, \nabla_L v) \cdot \nabla_L v \geq \overline{\lambda} \int_\Omega G(x, v) \quad \textit{for any } v \geq 0,\ v \in W_0^{1,p}(\Omega). \tag{54}$$

Assume that either (48) holds or

$$\textit{if} \quad v \in W_0^{1,p}(\Omega), v \geq 0, v \not\equiv 0 \quad \Rightarrow \quad \int_\Omega G(x, v) > 0. \tag{55}$$

Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Caratheodory function such that $g(x, 0) = 0$, and there exists $c_g > 0$ such that $0 \leq g(x,t)t \leq c_g G(x,t)$ for $t > 0$. Let $u \in W^{1,p}(\Omega)$ be a weak solution of

$$\mathrm{div}_L \left(\mathscr{A}(x, u, \nabla_L u)\right) + \lambda g(x, u) \geq 0, \quad \textit{on } \Omega, \qquad u \leq 0 \ \textit{ on } \partial\Omega.$$

i) If $\lambda < c_g/\overline{\lambda}$, then $u \leq 0$ a.e. on Ω.

ii) If $\lambda = c_g/\overline{\lambda}$ and the constant $\overline{\lambda}$ in (54) is not achieved in $W_0^{1,p}(\Omega)$, then $u \leq 0$ a.e. on Ω.

The following Hardy inequality will play an important role in what follows (see [12] for the proof and several other results).

Theorem 5.11 *Let $p > 1$. Let $d : \Omega \to \mathbb{R}$ be a nonnegative non constant measurable function and $\alpha \in \mathbb{R}$, $\alpha \neq 0$ such that*

$$d^{-p} |\nabla_L d|^p \, , \, d^{(\alpha-1)(p-1)} |\nabla_L d|^{p-1} \in L^1_{loc}(\Omega).$$

If $-L_p(d^\alpha) \geq 0$ in the weak sense, then for every $u \in \mathscr{C}^1_0(\Omega)$ we have

$$\left(\frac{|\alpha|\,(p-1)}{p}\right)^p \int_\Omega \frac{|u|^p}{d^p} |\nabla_L d|^p \, dx \leq \int_\Omega |\nabla_L u|^p \, dx. \tag{57}$$

In particular:

1. *If ∇_L is the horizontal gradient on a Carnot group $\mathbb{G}$ and S is a homogeneous norm such that $L_p S^{\frac{p-Q}{p-1}} = c\,\delta_0$*[3] *on $\mathbb{G}$ with $Q > p > 1$, then*

$$\left(\frac{Q-p}{p}\right)^p \int_{\mathbb{G}} \frac{|u|^p}{S^p} |\nabla_L S|^p \, dx \leq \int_{\mathbb{G}} |\nabla_L u|^p \, dx, \quad u \in D^{1,p}_L(\mathbb{G})^4, \tag{58}$$

   where the constant $\left(\frac{Q-p}{p}\right)^p$ is sharp and it is not achieved.

2. *If the first column of the matrix μ is such that $\mu_{11} = 1$ and $\mu_{k1} = 0$ for $k = 2..l$*[5] *and Ω is bounded in the x_1 direction, then there exists $c > 0$ such that*

$$c^p \int_\Omega |u|^p \leq \int_\Omega |\nabla_L u|^p \, , \quad u \in \mathscr{C}^1_0(\Omega). \tag{59}$$

Some direct consequences of Theorem (5.9) are the following.

Theorem 5.12 *Let ∇_L be the horizontal gradient on a Carnot group $\mathbb{G}$. Let $Q > p > 1$ and let S be a homogeneous norm such that $L_p S^{\frac{p-Q}{p-1}} = c\,\delta_0$ on $\mathbb{G}$.*
Let $\Omega \subset \mathbb{G}$ be a bounded open set. Let $u \in W_L^{1,p}(\Omega)$ be a weak solution of

$$L_p u + \lambda \frac{|\nabla_L S|^p}{S^p} |u|^{p-2} u \geq 0 \quad on\ \Omega, \qquad u \leq 0 \ \ on\ \partial\Omega,$$

with $\lambda \leq \left(\frac{Q-p}{p}\right)^p$. Then $u \leq 0$ a.e. on Ω.

Another simple application of Theorem (5.9) is the following.

Theorem 5.14 *Let ∇ be the Euclidean gradient on $\mathbb{R}^N$. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz boundary and $p > 1$. Set*

$$\delta(x) := dist(x, \partial\Omega) \qquad x \in \Omega.$$

Then there exists $\lambda(\Omega, p) > 0$ such that if $\lambda < \lambda(\Omega, p)$ and $u \in W^{1,p}(\Omega)$ is a weak solution of

$$\Delta_p u + \lambda \frac{|u|^{p-2}\, u}{\delta^p} \geq 0 \quad on\ \Omega, \qquad u \leq 0 \ \ on\ \partial\Omega,$$

then $u \leq 0$ a.e. on Ω. Moreover, $0 < \lambda(\Omega, p) \leq (\frac{p-1}{p})^p$,

The proof is based on the Hardy inequality

$$\int_\Omega |\nabla u|^p \geq \lambda(\Omega, p) \int_\Omega \frac{|u|^p}{\delta^p}, \tag{60}$$

It is known that the best constant $\lambda(\Omega, p)$ in (60) is such that $\lambda(\Omega, p) \leq (\frac{p-1}{p})^p$ and if Ω is convex then $\lambda(\Omega, p) = (\frac{p-1}{p})^p$. See [30]. Notice that if $\lambda = \lambda(\Omega, p)$, then the above theorem holds provided $\lambda(\Omega, p)$ is not achieved. For further information on (60) we refer to [1, 30, 12, 25] and the references therein.

A priori estimates, positivity results and Liouville theorems

In what follows we shall assume that $\mathscr{A}$ is **W**-p-**C** with $p > 1$. Throughout all sections, except Section 10, we shall assume that the vector field ∇_L satisfies (6), that is it homogeneous of degree one with respect to a dilation δ_R as specified in Section 1. However for convenience of the reader we state our assumptions at the beginning of each sections.

10 General a priori estimates

Let $\Omega \subset \mathbb{R}^N$ be an open set. Let $V \in L^\infty_{loc}(\Omega)$ be nonnegative and let $\mathscr{A}$ be **W**-p-**C** with $p > 1$. The following preliminary lemmata will play an important role in the proof of our main result (see Theorem 10.5 below).

Lemma 10.1 *Let $g \in L^1_{loc}(\Omega)$ be nonnegative and let $u \in W^{1,p}_{loc}(\Omega)$ be a weak solution of*

$$\mathrm{div}_L \left(\mathscr{A}(x, u, \nabla_L u)\right) + V u^{p-1} \geq g, \qquad u \geq 0, \quad \textit{on} \;\; \Omega. \tag{94}$$

Let $s \geq 1$*. If* $u^{s+p-1} \in L^1_{loc}(\Omega)$*, then*

$$gu^s, \quad \mathscr{A}(x, u, \nabla_L u) \cdot \nabla_L u \; u^{s-1} \in L^1_{loc}(\Omega) \tag{95}$$

and for any nonnegative $\phi \in \mathscr{C}^1_0(\Omega)$ *we have,*

$$\int_\Omega g u^s \phi + c_1 s \int_\Omega \mathscr{A}(x, u, \nabla_L u) \cdot \nabla_L u \; u^{s-1}\phi \leq c_2 s^{1-p} \int_\Omega u^{s+p-1} \frac{|\nabla_L \phi|^p}{\phi^{p-1}} + \int_\Omega V u^{s+p-1}\phi, \tag{96}$$

where $c_1 = 1 - \frac{\epsilon^{p'}}{p' k_2} > 0$*,* $c_2 = \frac{p^p}{p\epsilon^p}$ *and* $\epsilon > 0$ *is sufficiently small.*

Remark 10.2 *i) Notice that form the above result it follows that if* $u \in W^{1,p}_{loc}(\Omega)$ *is a weak solution of (94), then* $g\,u \in L^1_{loc}(\Omega)$*.*

ii) The above lemma still holds if we replace the function $g \in L^1_{loc}(\Omega)$ *with a regular Borel measure on* Ω*.*

Remark 10.3 *i) The above lemma holds for* $s > 0$*. Indeed if* $0 < s < 1$ *the proof follows the same arguments as above. To this end in (97) it is enough to choose* $\gamma := \gamma_n(u + \delta)$ *where* γ_n *is defined by (98).*

ii) If $V \leq 0$*, then the assumption* $u^{s+p-1} \in L^1_{loc}(\Omega)$*, is not needed for the validity of (95). Indeed what that really matters is* $u^{s+p-1} \in L^1_{loc}(S)$ *where* S *is the support of* $\nabla_L \phi$*. This remark will be useful when dealing with inequalities on unbounded set.*

Let $a : \Omega \to \mathbb{R}$ be a nonnegative measurable function. Let u be weak solution of

$$\operatorname{div}_L\left(\mathscr{A}(x,u,\nabla_L u)\right) + V u^{p-1} \geq a(x)\, u^q, \quad u \geq 0, \quad on\ \Omega. \tag{99}$$

The main strategy to obtain *a priori* estimates is to use the family of test functions $u^\alpha \phi$ where $\alpha > 0$ is a suitable constant that will be chosen according to our needs. See [31]. However, *a priori* it is not clear why, after multiplying the inequality by $u^\alpha\phi$, this family is admissible, i.e. why $u^{q+\alpha} \in L^1_{loc}(\Omega)$. A sufficient condition for the admissibility of the family $u^\alpha\phi$ is contained in the following.

Lemma 10.4 *Let u be a weak solution of (99) with $q > p-1$. Assume that there exists $\overline{\alpha} > 1$ such that $a^{-\frac{\overline{\alpha}+p-1}{q-p+1}} \in L^1_{loc}(\Omega)$.*

If $1 \leq \alpha < \overline{\alpha}$, then

$$a\, u^{q+\alpha}, \ u^{\alpha+p-1} \in L^1_{loc}(\Omega), \tag{100}$$

and for any nonnegative $\phi \in \mathscr{C}^1_0(\Omega)$, the following inequalities hold

$$\int_\Omega a\, u^{q+\alpha}\phi + c_1\alpha \int_\Omega \mathscr{A}(x,u,\nabla_L u)\cdot\nabla_L u\ u^{\alpha-1}\phi \leq c_2\alpha^{1-p}\int_\Omega u^{\alpha+p-1}\frac{|\nabla_L\phi|^p}{\phi^{p-1}} + \int_\Omega V u^{\alpha+p-1}\phi, \tag{101}$$

$$\int_\Omega a\, u^{q+\alpha}\phi + c_1\alpha \int_\Omega \mathscr{A}(x,u,\nabla_L u)\cdot\nabla_L u\ u^{\alpha-1}\phi \leq$$
$$c_2\alpha^{1-p}\left(\int_S a\, u^{q+\alpha}\phi\right)^{1/\chi}\left(\int_S \frac{|\nabla_L\phi|^{p\chi'}}{\phi^{p\chi'-1}} a^{-\frac{\alpha+p-1}{q-p+1}}\right)^{1/\chi'} + \int_\Omega V u^{\alpha+p-1}\phi, \tag{102}$$

where $\chi := \frac{q+\alpha}{\alpha+p-1}$, $\chi' := \frac{q+\alpha}{q-p+1}$ and S is the support of $\nabla_L\phi$.

In particular if

$$\text{for any } C \subset\subset \Omega : \quad \operatorname*{ess\,inf}_C a > 0,$$

then for any $\alpha > 0$, we have $a\, u^{q+\alpha} \in L^1_{loc}(\Omega)$.

Theorem 10.5 *Let u be a weak solution of (99) with $q > p-1$ and $V \leq 0$. Assume that there exists $\overline{\alpha} > 1$ such that $a^{-\frac{\overline{\alpha}+p-1}{q-p+1}} \in L^1_{loc}(\Omega)$.*

Then

$$a\, u^{q+\overline{\alpha}},\ u^{\overline{\alpha}+p-1} \in L^1_{loc}(\Omega),$$

and for any $1 \leq \alpha \leq \overline{\alpha}$ and for any nonnegative $\phi \in \mathscr{C}^1_0(\Omega)$ the inequalities (101), (102) hold and

$$\int_\Omega a\, u^{q+\alpha}\phi \leq (c_2\alpha^{1-p})^{\chi'} \int_\Omega \frac{|\nabla_L \phi|^{p\chi'}}{\phi^{p\chi'-1}} a^{-\frac{\alpha+p-1}{q-p+1}}, \qquad (106)$$

where $\chi' := \frac{q+\alpha}{q-p+1}$.

11 Universal a priori estimates

In this section $\Omega \subset \mathbb{R}^N$ is an open set, ∇_L is the horizontal gradient on a Carnot group $\mathbb{G}$ and $\mathscr{A}$ is **S**-p-**C** (see Definition 1.1). We shall denote with $|\cdot|_L$ a homogeneous norm on $\mathbb{G}$.

Theorem 11.1 *Let $q > p-1$ and $c > 0$. Assume that $f \in \mathscr{C}(\mathbb{R})$ satisfies $f(t) \geq ct^q$ for $t > 0$. Then there exists a constant $C = C(f, \mathbb{G}, \mathscr{A}) > 0$ such that if u is a weak solution of*

$$\operatorname{div}_L(\mathscr{A}(x,u,\nabla_L u)) + \mathscr{B}(x,u,\nabla_L u) \geq f(u) \quad \text{on } \Omega, \tag{108}$$

with $\mathscr{B}(x,t,\xi) \leq 0$ for $t \geq 0$, then

$$u(x) \leq C dist(x, \partial\Omega)^{-\frac{p}{q-p+1}} \qquad a.e.\ x \in \Omega. \tag{109}$$

In particular if u is a weak solution of

$$\operatorname{div}_L(\mathscr{A}(x,u,\nabla_L u)) + \mathscr{B}(x,u,\nabla_L u) = f(u), \quad \text{on } \Omega. \tag{110}$$

with $\mathscr{B}(x,t,\xi)t \leq 0$ and $f(t)t \geq ct^{q+1}$ for $t \in \mathbb{R}$, then

$$|u(x)| \leq C dist(x, \partial\Omega)^{-\frac{p}{q-p+1}} \qquad a.e.\ x \in \Omega. \tag{111}$$

Remark 11.2 *In general, inequality (109) is sharp as the following examples show.*

For $q > \frac{N}{N-2}$ *the function* $u(x) := c\,|x|^{-\frac{2}{q-1}}$, *for a suitable* $c > 0$, *is a solution of*

$$\Delta u = u^q \quad on \quad \mathbb{R}^N \setminus \{0\}.$$

For $q > 1$ *the function* $u(x) := c{x_1}^{-\frac{2}{q-1}}$, *for a suitable* $c > 0$ *is a solution of*

$$\Delta u = u^q \quad on \quad]0, +\infty[\times \mathbb{R}^{N-1}.$$

12 Some Liouville theorems for coercive inequalities

In this section we study Liouville theorems for a class of quasilinear elliptic inequalities on $\mathbb{R}^N$.

Recently, a wide class of weakly elliptic quasilinear problems were also considered by Farina and Serrin [18]and Pucci and Serrin [38], where sharp interesting cases were handled. The main technique we use throughout this section, is a combination of three ingredients: the Kato inequalities (16) and (18), a slight modification of the test functions method together with an idea introduced in [31].

More precisely, we shall consider problems of the type

$$\mathrm{div}_L\left(\mathscr{A}(x, u, \nabla_L u)\right) + V(x)\,|u|^{p-2}\,u = a(x)f(u) \quad on\ \mathbb{R}^N, \tag{114}$$

where $V \leq 0$ and $a : \mathbb{R}^N \to \mathbb{R}$ is a nonnegative measurable function. The proof or our main results will be organized in two steps. The first is to apply Kato's inequality (16) and (18) to (114) reducing the problem to the study of the nonnegative solutions of

$$\mathrm{div}_L\left(\mathscr{A}(x, u, \nabla_L u)\right) \geq a(x)\,u^q, \quad u \geq 0, \quad on\ \mathbb{R}^N. \tag{115}$$

A second one will be the application of *a priori* estimates proved in Section 10 to (115). These estimates depend on two parameters α and R. By using an idea first introduced in [31, see proof of Theorem 4.1], we can choose α large enough and then by letting $R \to +\infty$ we conclude.

We point out that when dealing with equations or inequalities other fine techniques based on Keller and Osserman ideas ([26] and [36] respectively) are available. However, the application of these later ideas need special stronger assumptions on the differential operator and on the nonlinearity. For recent contribution see [33, 20, 28].

Throughout this section we shall assume that $\mathscr{A}$ is **W**-p-**C** with $p > 1$, the vector field ∇_L satisfies (6) (that is ∇_L is homogeneous of degree one with respect to a dilation δ_R as specified in Section 1) and $|\cdot|_L$ stands for a homogeneous norm.

Theorem 12.1 *Let* $V \in L^{\infty}_{loc}(\mathbb{R}^N)$ *be such that* $V \leq 0$. *Suppose that* $f \in \mathscr{C}(\mathbb{R})$ *satisfies*

$$f(t) \geq c\, t^q, \qquad for \quad t > 0,$$

where $q > p-1$ *and* $c > 0$. *Assume that there exists* $\overline{\alpha} > 1$ *such that* $a^{-\frac{\overline{\alpha}+p-1}{q-p+1}} \in L^1_{loc}(\mathbb{R}^N)$ *and*

$$\liminf_{R\to+\infty} R^{-p\frac{q+\overline{\alpha}}{q-p+1}} \int_{A_R} a^{-\frac{\overline{\alpha}+p-1}{q-p+1}} < +\infty. \tag{116}$$

Let u *be a weak solution of*

$$\mathrm{div}_L\left(\mathscr{A}(x,u,\nabla_L u)\right) + V(x)\,|u|^{p-2}\,u \geq a(x)\, f(u) \qquad on\ \mathbb{R}^N. \tag{117}$$

Then $u \leq 0$ *a.e. on* $\mathbb{R}^N$.

Moreover if

$$f(t)\, t \geq c\,|t|^{q+1} \qquad t \in \mathbb{R},$$

and u *is a weak solution of the equation*

$$\mathrm{div}_L\left(\mathscr{A}(x,u,\nabla_L u)\right) + V(x)\,|u|^{p-2}\,u = a(x)\, f(u) \qquad on\ \mathbb{R}^N, \tag{118}$$

then $u \equiv 0$ *a.e. on* $\mathbb{R}^N$.

Theorem 12.4 *Let* $\mathscr{A} = \mathscr{A}(x, \nabla_L u)$ *and let* $V \in L^\infty_{loc}(\mathbb{R}^N)$ *be such that* $V \leq 0$. *Assume that there exist* $q > p-1$ *and* $\overline{\alpha} > 1$ *such that* $a^{-\frac{\overline{\alpha}+p-1}{q-p+1}} \in L^1_{loc}(\mathbb{R}^N)$ *and (116) holds. Suppose that* $f \in \mathscr{C}(\mathbb{R})$ *satisfies*

$$\liminf_{t\to+\infty} \frac{f(t)}{t^q} > 0. \tag{120}$$

Then,

1. *If* u *is a weak solution of inequality (117), then* $u \leq \max(Z(f) \cup \{0\})^7$ *a.e. on* $\mathbb{R}^N$.
2. *If* $V \equiv 0$ *and* u *is a weak solution of inequality (117), then* $u \leq \max Z(f)$ *a.e. on* $\mathbb{R}^N$.
3. *If* $V \equiv 0$ *and* f *is positive, then inequality (117) has no weak solutions.*
4. *If* $V \equiv 0$ *and* u *is a weak solution of equation (118) with* f *satisfying*

   $$\liminf_{t\to+\infty} \frac{f(t)}{|t|^q} > 0, \qquad \limsup_{t\to-\infty} \frac{f(t)}{|t|^q} < 0, \tag{121}$$

   then,

   $$\min Z(f) \leq u \leq \max Z(f) \quad a.e.\ on\ \mathbb{R}^N.$$

Corollary 12.6 *Let* $\mathscr{A} = \mathscr{A}(x, \nabla_L u)$ *and* $V \in L^\infty_{loc}(\mathbb{R}^N)$ *be such that* $V \leq 0$. *Let* a *be a continuous positive function satisfying*

$$a(x) \geq c\,|x|_L^{-\theta} \quad for \;\; |x|_L \;\; large,$$

with $\theta < p$. *Let* $f \in \mathscr{C}(\mathbb{R})$ *be such that (120) holds for some* $q > p - 1$. *Then,*

1. *If* u *is a weak solution of inequality (117), then* $u \leq \max(Z(f) \cup \{0\})$ *a.e. on* $\mathbb{R}^N$.
2. *If* $V \equiv 0$ *and* u *is a weak solution of inequality (117), then* $u \leq \max Z(f)$ *a.e. on* $\mathbb{R}^N$.
3. *If* $V \equiv 0$ *and* f *is positive, then inequality (117) has no weak solutions.*
4. *If* $V \equiv 0$ *and* u *is a weak solution of equation (118) with* f *satisfying (121), then,*

$$\min Z(f) \leq u \leq \max Z(f) \quad a.e. \; on \; \mathbb{R}^N.$$

Theorem 12.10 *Assume that* ∇_L *is the usual gradient* ∇ *on* $\mathbb{R}^N$ *or the horizontal gradient on the Heisenberg group* $\mathbb{H}^n (= \mathbb{R}^{2n+1} = \mathbb{R}^N)$. *Let* $f \in \mathscr{C}(\mathbb{R})$ *be such that:*

$$for\ some\ c > 0\ f\ is\ nondecreasing,\ positive\ on\ [c, +\infty[\ and\ \int_c^{+\infty} \left(\int_c^t f(s)\ ds \right)^{-\frac{1}{p}} dt < +\infty. \tag{124}$$

Assume that $V \in L^\infty_{loc}(\mathbb{R}^N)$ *satisfies* $V \leq 0$.

Then,

1. *If* u *is a weak solution of*

$$\mathrm{div}_L \left(|\nabla_L u|^{p-2} \nabla_L u \right) + V(x) |u|^{p-2} u \geq f(u) \qquad on \quad \mathbb{R}^N, \tag{125}$$

   then $u \leq \max(Z(f) \cup \{0\})$ *a.e. on* $\mathbb{R}^N$.

2. *If* $V \equiv 0$ *and* u *is a weak solution of (125), then* $u \leq \max Z(f)$ *a.e. on* $\mathbb{R}^N$.

3. *If* $V \equiv 0$ *and* f *is positive, then (125) has no weak solutions.*

4. *If* u *is a weak solution of*

$$\mathrm{div}_L \left(|\nabla_L u|^{p-2} \nabla_L u \right) = f(u) \qquad on \quad \mathbb{R}^N, \tag{126}$$

   with f *and* $\bar{f}(t) := -f(-t)$ *satisfying (124), then*

$$\min Z(f) \leq u \leq \max Z(f) \quad a.e.\ on\ \mathbb{R}^N.$$

15 Schrödinger's equations and inequalities

In this section we shall study nonexistence of solutions of Schrödinger's type equations of the form

$$\operatorname{div}_L(\mathscr{A}(x,u,\nabla_L u)) + \lambda V(x)|u|^{p-2}u = a(x)|u|^{q-1}u \qquad on\ \mathbb{R}^N, \tag{152}$$

where $\mathscr{A}$ is **W**-p-**C** and a, V and λ will be specified during the course.

Similar problems have been studied in the semilinear case in [7], where nonexistence of solutions of the equation

$$\Delta u + \lambda V(x)u = f(x,u) \qquad on \quad \mathbb{R}^N \setminus \{0\}, \tag{153}$$

was proved by reducing to an o.d.e. inequality by applying the spherical mean operator to (153) and using some convexity argument. For our problem (152), a radial reduction in general is not applicable even if the differential operator is linear. So we need to proceed differently.

Being interested in nonexistence theorem for (152), by reduction principles it is enough to consider possible nonnegative solutions. Our results allow us to consider, as special case in the Euclidean setting,

$$V(x) = \frac{1}{|x|^p}, \qquad a(x) = \frac{c}{|x|^\theta} \qquad for\ \ |x|\ \ large.$$

Considering a more general operator we shall require that

$$a(x) \geq c\frac{\psi^k}{|x|_L^\theta} \qquad for\ \ |x|_L\ \ large, \tag{154}$$

$$C_1\frac{\psi^h}{|x|_L^\nu} \geq V(x) \geq C_2\frac{\psi^p}{|x|_L^p} \qquad for\ \ |x|_L\ \ large, \tag{155}$$

and a Hardy's inequality holds for the weight V, that is there exists $\lambda_H > 0$ such that

$$\int_{\mathbb{R}^N} |\nabla_L \phi|^p \geq \lambda_H \int_{\mathbb{R}^N} V\, |\phi|^p \quad for\ any\ \phi \in \mathscr{C}_0^1(\mathbb{R}^N). \tag{156}$$

In what follows, for simplicity, we deal with locally bounded solutions in the setting of Carnot groups. We note that if the function $\mathscr{A}$ is **S**-p-**C** and V belongs to $L^{Q/p}_{loc}(\mathbb{R}^N)$ or to the Morrey space $M^{Q/(p-\epsilon)}(\mathbb{R}^N)$ then the positive solutions of (152) belongs $L^\infty_{loc}(\mathbb{R}^N)$. This is due to the fact that for **S**-p-**C** operator a weak Harnack inequality holds. See [29] for the Euclidean case and [9] for the Carnot group setting.

In this section we endow $\mathbb{R}^N$ with a group law such that it becomes a Carnot group. As usual, ∇_L stands for the horizontal gradient as described in Appendix A.

The validity of (156) with $V = \psi^p / |\cdot|_L^p$ is established among other Hardy inequalities in [12].

We notice that in view of the reduction principles stated in Chapter II, it suffices to study nonnegative solutions of the inequality related to (152). Notice that the case $\lambda \leq 0$ has been considered in Section 12. Hence, in what follows we shall focus our attention to the case $\lambda > 0$.

Theorem 15.1 *Let $Q > p > 1$. Let $\mathscr{A}$ be* **S**-p-**C** *and let $a, V \in L^1_{loc}(\mathbb{R}^N)$ be nonnegative functions satisfying (154) and (155) with $p \geq \nu > \theta$ and $p \geq h \geq k \geq 0$. Assume that (156) holds and let λ be such that $0 < \lambda \leq \lambda_H k_1$ where λ_H is the best constant in (156) and k_1 is the constant structure appearing in the definition of* **S**-p-**C** *(see Definition 1.1).*

Let $u \in W^{1,p}_{L,loc}(\mathbb{R}^N) \cap L^\infty_{loc}(\mathbb{R}^N)$ be a weak solution of

$$\mathrm{div}_L\left(\mathscr{A}(x, u, \nabla_L u)\right) + \lambda V(x) u^{p-1} \geq a(x) u^q, \qquad u \geq 0, \qquad \text{on} \quad \mathbb{R}^N. \tag{157}$$

If

$$p - 1 < q \leq \frac{(Q-\theta)(p-1) + x_0(p-\theta)}{Q-p}, \tag{158}$$

where $x_0 \geq 1$ is the unique solution of the equation

$$(x - 1 + p)^p \lambda = x \lambda_H k_1 p^p, \qquad x \geq 1,$$

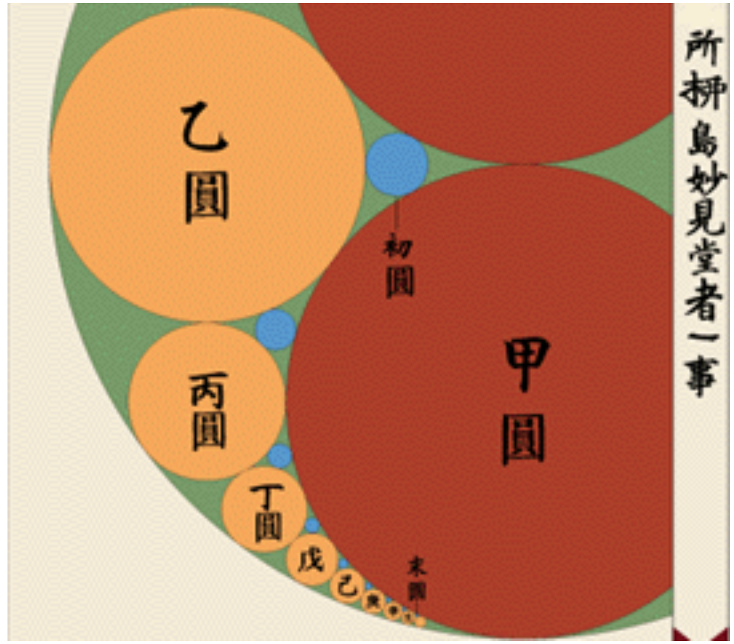
then $au \equiv 0$ a.e. on $\mathbb{R}^N$. Moreover, if $a > 0$ or if $\lambda < \lambda_H k_1$, then $u \equiv 0$ a.e. on $\mathbb{R}^N$.

An interesting consequence of Theorem 5.1 is the following.

Corollary 15.2 *Assume that $\mathscr{A}$, a, V, λ and q satisfy the hypotheses of Theorem 15.1. Let $u \in W^{1,p}_{L,loc}(\mathbb{R}^N) \cap L^\infty_{loc}(\mathbb{R}^N)$ be a weak solution of*

$$\mathrm{div}_L\left(\mathscr{A}(x, u, \nabla_L u)\right) + \lambda V(x) |u|^{p-2} u \geq a(x) |u|^{q-1} u, \qquad \text{on} \quad \mathbb{R}^N. \tag{159}$$

Then $au \leq 0$ a.e. on $\mathbb{R}^N$. Moreover, if $\lambda < \lambda_H k_1$, then $u \leq 0$ a.e. on $\mathbb{R}^N$.



すべてに感謝　！

THANK YOU FOR YOUR ATTENTION!

GRAZIE A TUTTI！