Nonradial maximizers for a Henon type problem and symmetry breaking bifurcation for a Liouville Gel'fand equation with a vanishing coefficient

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Problem

Maximize the functional

$$\sup_{u \in H_0^1(D), \|u\| = \beta} \int_D r^\alpha (e^u - 1)\, dx\,.$$

$$D := \{(x, y) \in \mathbf{R}^2; x^2 + y^2 < 1\}$$

$$\alpha > 0, \beta > 0$$

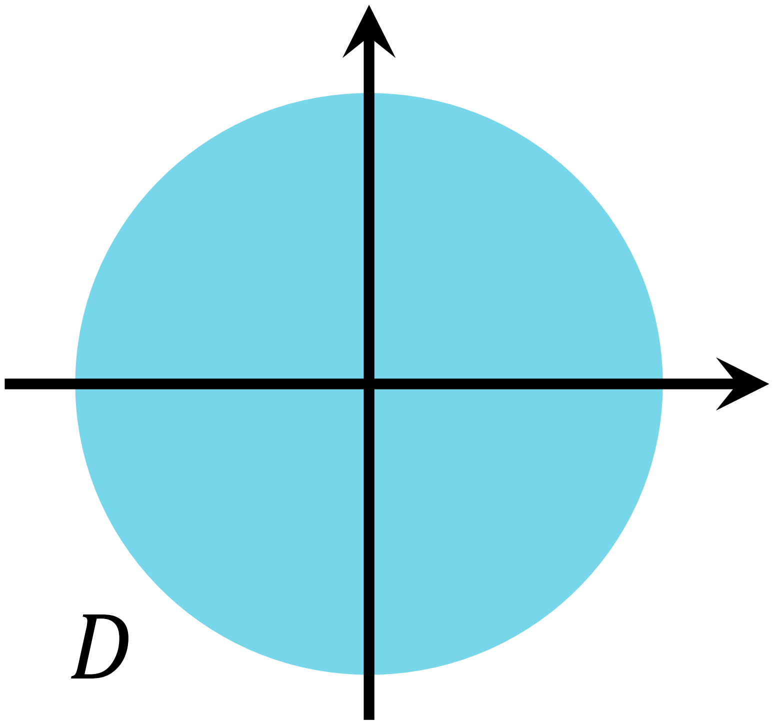
$$\|u\|^2 := \int_D |\nabla u|^2 dx$$

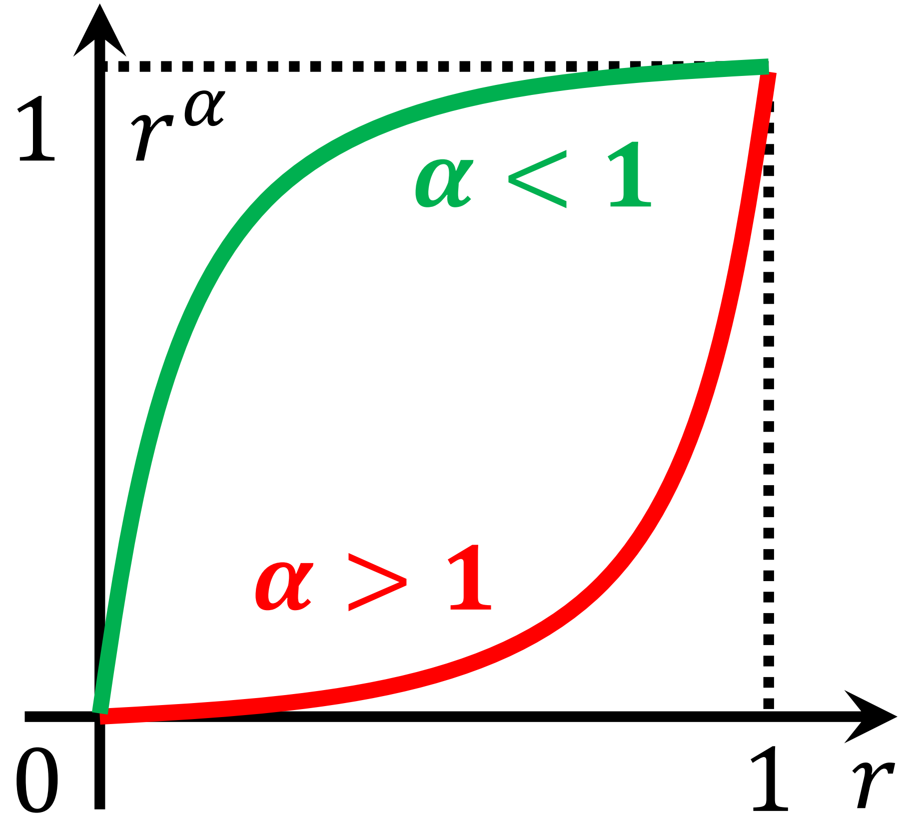
We consider **the shape** of the maximizer.

$$\sup_{u \in H_0^1(D), \|u\| = \boldsymbol{\beta}} \int_D r^{\boldsymbol{\alpha}} (e^u - 1) dx$$

There are two positive parameter $\boldsymbol{\alpha}, \boldsymbol{\beta}$.

$$\sup_{u\in H_0^1(D),\|u\|=\beta} \int_D r^{\alpha}(e^u - 1)dx \qquad \beta\text{: fixed}$$



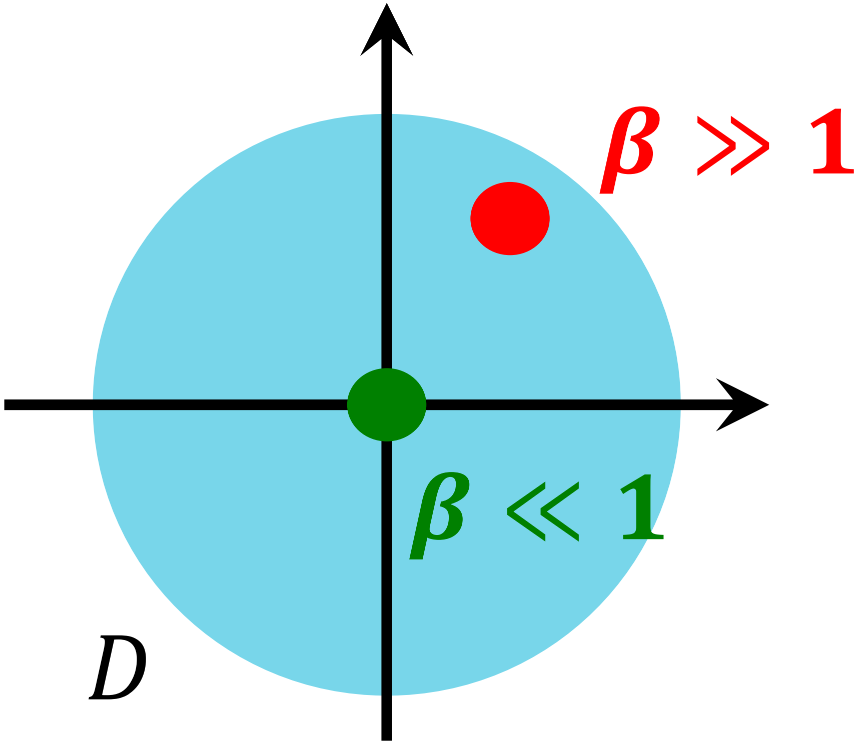


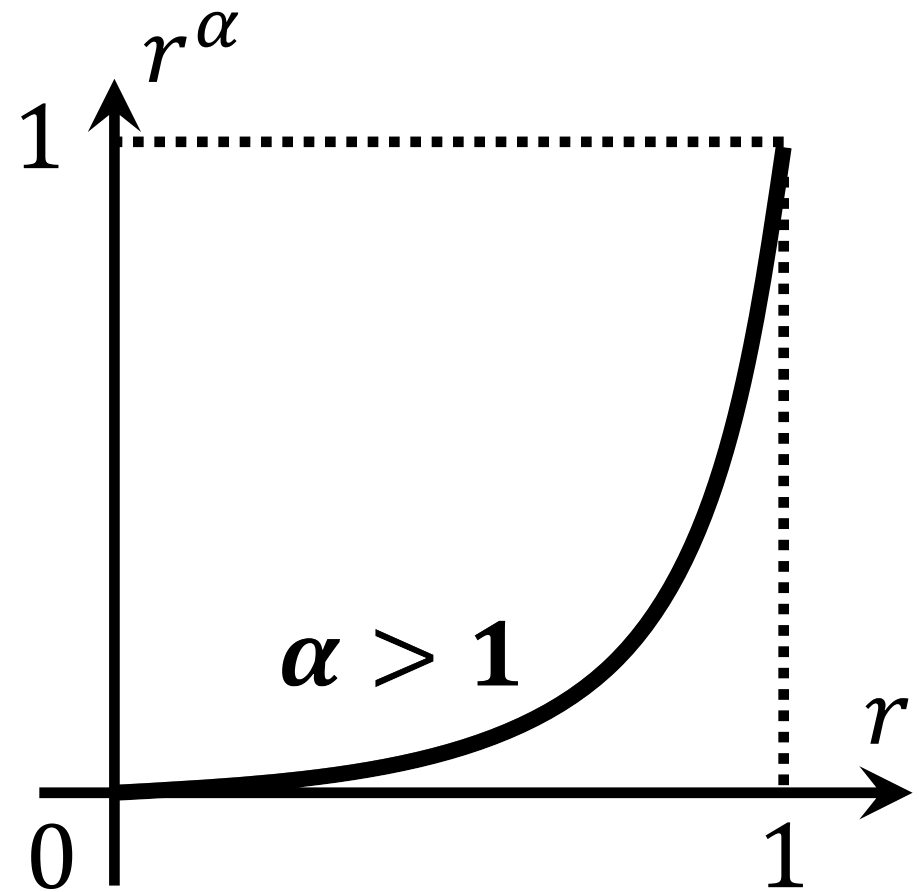
Expectation:

If α is small, then the maximizer is radial.

If α is large, then the maximizer is nonradial.

$$\sup_{u \in H_0^1(D), \|u\| = \beta} \int_D r^\alpha (e^u - 1) dx \qquad \alpha\text{: fixed}$$





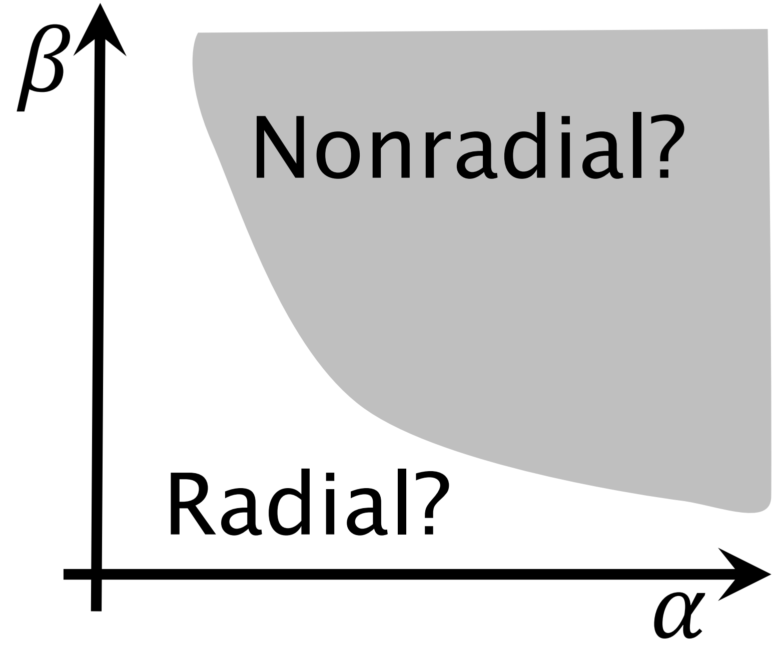
Expectation:

If β is small, then the maximizer is radial.

If β is large, then the maximizer is nonradial.

Goal

Draw the phase diagram in the $\alpha\beta$-plane.



Related results

$$2 < p < \frac{n+2}{n-2}, \qquad n \geq 3$$

$$S_{\alpha,p} := \inf_{u \in H_0^1, u \neq 0} \frac{\int_B |\nabla u|^2 dx}{\left(\int_B r^\alpha |u|^p \, dx\right)^{2/p}}$$

$$S_{\alpha,p}^R := \inf_{u \in H_0^1, u \neq 0, \textbf{radial}} \frac{\int_B |\nabla u|^2 dx}{\left(\int_B r^\alpha |u|^p dx\right)^{2/p}}$$

If $\alpha > \alpha^*$ is large, then $S_{\alpha,p} < S_{\alpha,p}^R$, thus the ground state is nonradial.

D. Smets, M. Willem and J. Su,
Non-radial ground states for the Henon equation,
Commun. Contemp. Math **4** (2002), 467—480

Euler–Lagrange equation of

$$S_{\alpha,p} := \inf_{u \in H_0^1, u \neq 0} \frac{\int_B |\nabla u|^2 dx}{\left(\int_B r^\alpha |u|^p dx\right)^{2/p}}$$ is

$$\begin{cases} \Delta u + \lambda r^\alpha u^p = 0 & \text{in } B \\ u = 0 & \text{on } \partial B \end{cases}$$

$$2 < p < \frac{n+2}{n-2}, \; n \geq 3,$$

which is called the Henon equation.

$$\begin{cases} \Delta u + r^1 u^3 = 0 & \text{in } D \\ u = 0 & \text{on } \partial D \end{cases} \qquad D \in \mathbf{R}^2$$

The ground state

Radially symmetric solution

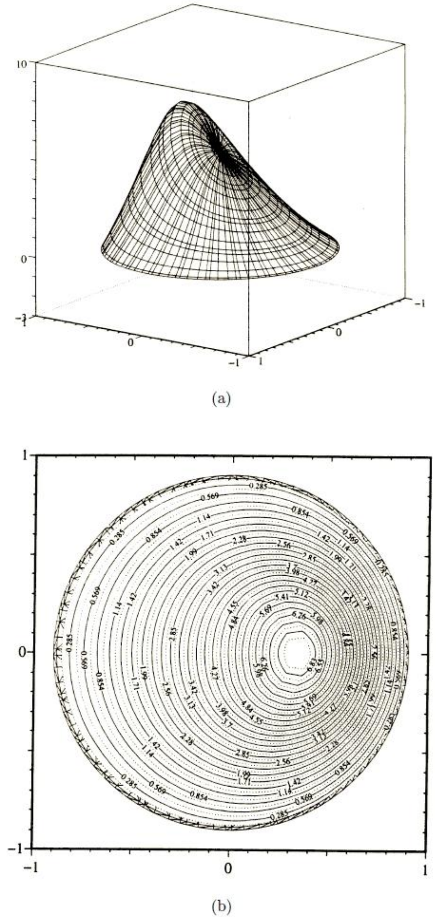


Fig. 41. A ground state (a) and its contours (b) of Henon's equation (100) on the unit disk Ω_1, with $\ell = 1$, $p = 3$, and $k = 0.81$.

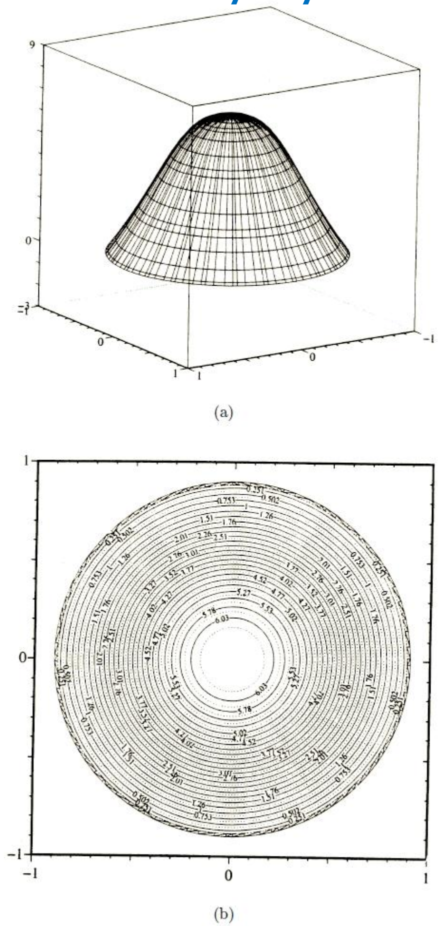


Fig. 42. A radially symmetric positive solution (a) and its contours (b) of Henon's equation (100) on the unit disk, with $\ell = 1$, and $p = 3$. This solution has an energy level slightly higher than that of the ground state displayed in Fig. 41.

G. Chen, J. Zhou and W. Ni,
Algorithms and visualization for solutions of nonlinear elliptic equations,
Internat. J . Bifur. Chaos Appl. Sci. Engrg 10 (2000), 1656—1612.

The problem

$$\sup_{u \in H_0^1(D), \|u\|=\beta} \int_D r^\alpha (e^u - 1) dx$$

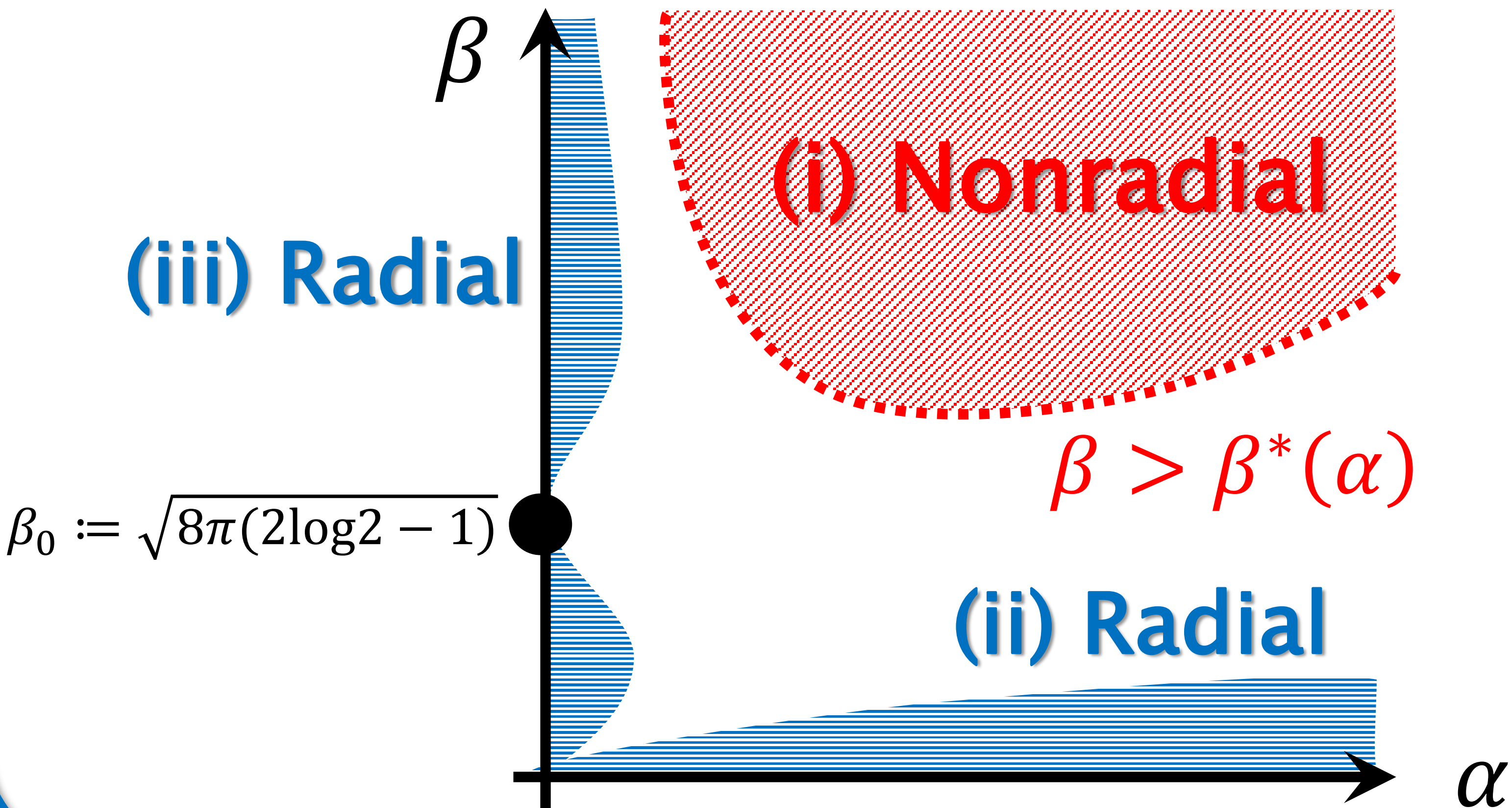
can be seen as a two dimensional version of

$$S_{\alpha,p} := \inf_{u \in H_0^1, u \neq 0} \frac{\int_B |\nabla u|^2 dx}{\left(\int_B r^\alpha |u|^p dx\right)^{2/p}} .$$

Theorem A

$$\sup_{u \in H_0^1(D), \|u\| = \beta} \int_D r^\alpha (e^u - 1) dx$$

$$\beta^*(\alpha) := \sqrt{8\pi \left\{ (\alpha + 2) \log \frac{2\alpha + 4}{\alpha} - \frac{\alpha + 4}{2} \right\}}$$



Euler–Lagrange equation of

$$\sup_{u\in H_0^1(D),\|u\|=\beta} \int_D r^\alpha(e^u-1)dx \text{ is}$$

$$(\mathrm{LG}) \quad \begin{cases} \Delta u + \lambda r^\alpha e^u = 0 & \text{in } D \\ u = 0 & \text{on } \partial D \end{cases}$$

$$\beta^2 = \|u\|^2 = \lambda \int_D r^\alpha u e^u dx$$

$$(\mathrm{EL}) \quad \begin{cases} \Delta u + \beta^2 \dfrac{r^\alpha e^u}{\int_D r^\alpha u e^u dx} = 0 & \text{in } D \\ u = 0 & \text{on } \partial D \end{cases}$$

(LG) $$\begin{cases} \Delta u + \lambda r^{\alpha} e^{u} = 0 & \text{in } D \\ u = 0 & \text{on } \partial D \end{cases}$$

$$(\lambda, u)$$

$$\lambda = \frac{\beta^2}{\int_D r^{\alpha} u e^{u} dx}$$

$$\beta = \sqrt{\lambda \int_D r^{\alpha} u e^{u} dx}$$

$$(\beta, u)$$

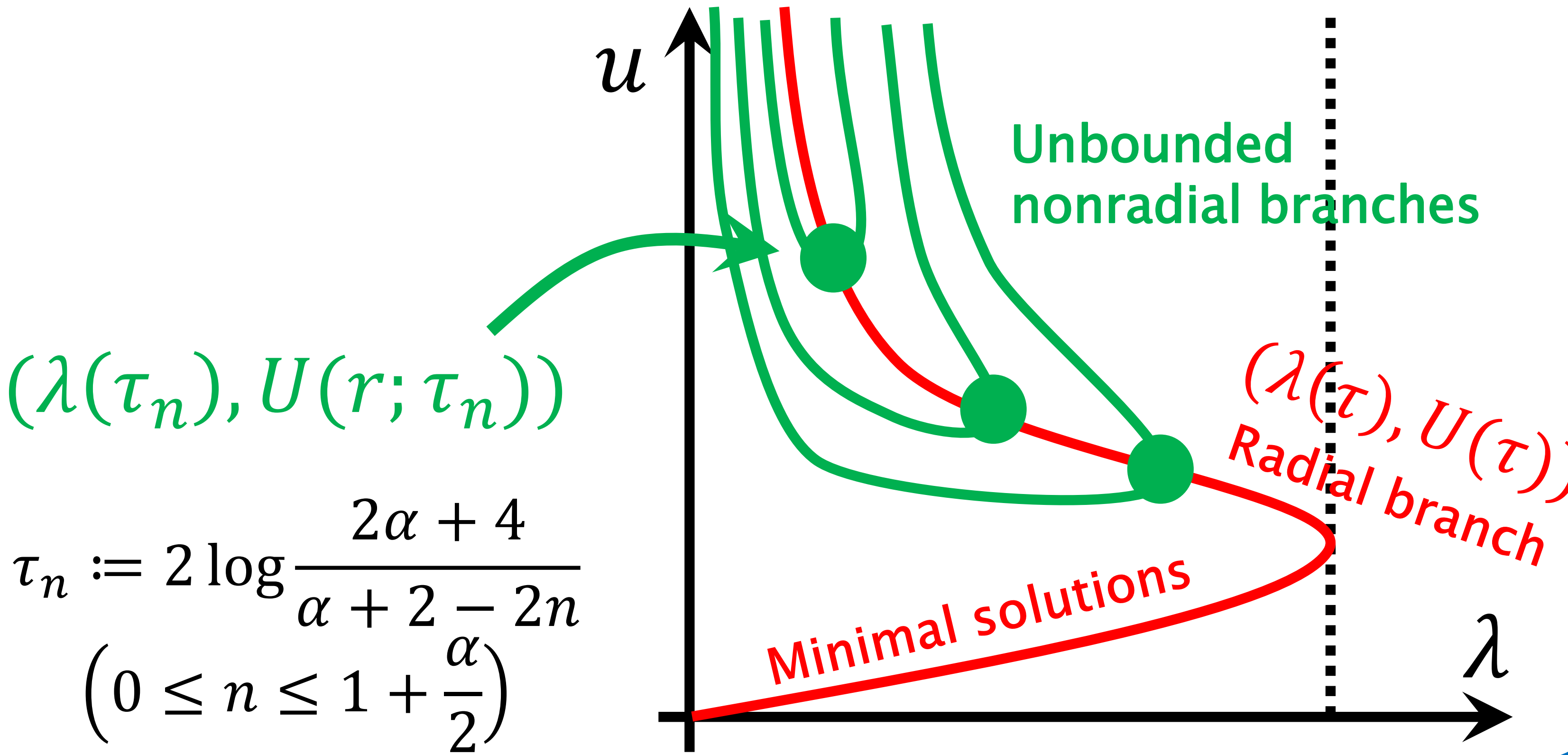
(EL) $$\begin{cases} \Delta u + \beta^2 \dfrac{r^{\alpha} e^{u}}{\int_D r^{\alpha} u e^{u} dx} = 0 & \text{in } D \\ u = 0 & \text{on } \partial D \end{cases}$$

Theorem B

(LG) $\begin{cases} \Delta u + \lambda r^{\alpha} e^{u} = 0 & \text{in } D \\ u = 0 & \text{on } \partial D \end{cases}$

$$\lambda(\tau) := 2(\alpha+2)^2\left(e^{-\tau/2} - e^{-\tau}\right)$$

$$U(r;\tau) := \log \frac{e^{\tau}}{\left(1 + (e^{\tau/2} - 1) r^{\alpha+2}\right)^2}$$

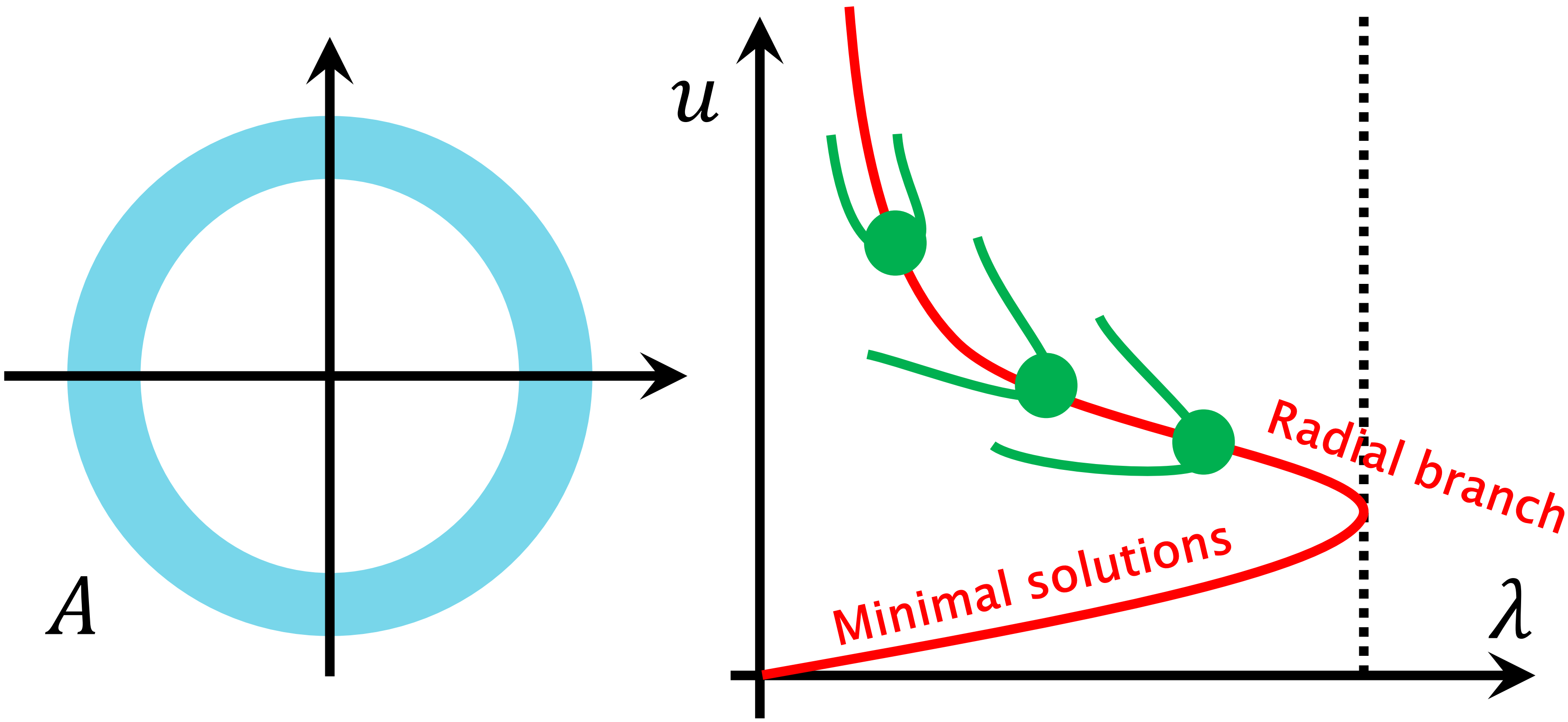


$$\tau_n := 2\log\frac{2\alpha+4}{\alpha+2-2n}$$

$$\left(0 \le n \le 1 + \frac{\alpha}{2}\right)$$

Remark

$$\begin{cases} \Delta u + \lambda e^u = 0 & \text{in} \quad A \\ u = 0 & \text{on} \quad \partial A \end{cases}$$



S.-S. Lin,
On non-radially symmetric bifurcation in the annulus,
J. Differential Equations **80** (1989), 251—279.

(LG) $$\begin{cases} \Delta u + \lambda r^{\alpha} e^{u} = 0 & \text{in } D \\ u = 0 & \text{on } \partial D \end{cases}$$

$$(\lambda, u)$$

$$\lambda = \frac{\beta^2}{\int_D r^{\alpha} u e^{u} dx} \qquad \beta = \sqrt{\lambda \int_D r^{\alpha} u e^{u} dx}$$

$$(\beta, u)$$

(EL) $$\begin{cases} \Delta u + \beta^2 \dfrac{r^{\alpha} e^{u}}{\int_D r^{\alpha} u e^{u} dx} = 0 & \text{in } D \\ u = 0 & \text{on } \partial D \end{cases}$$

Corollary C

(EL) $$\begin{cases} \Delta u + \beta^2 \dfrac{r^\alpha e^u}{\int_D r^\alpha u e^u dx} = 0 & \text{in } D \\ u = 0 & \text{on } \partial D \end{cases}$$

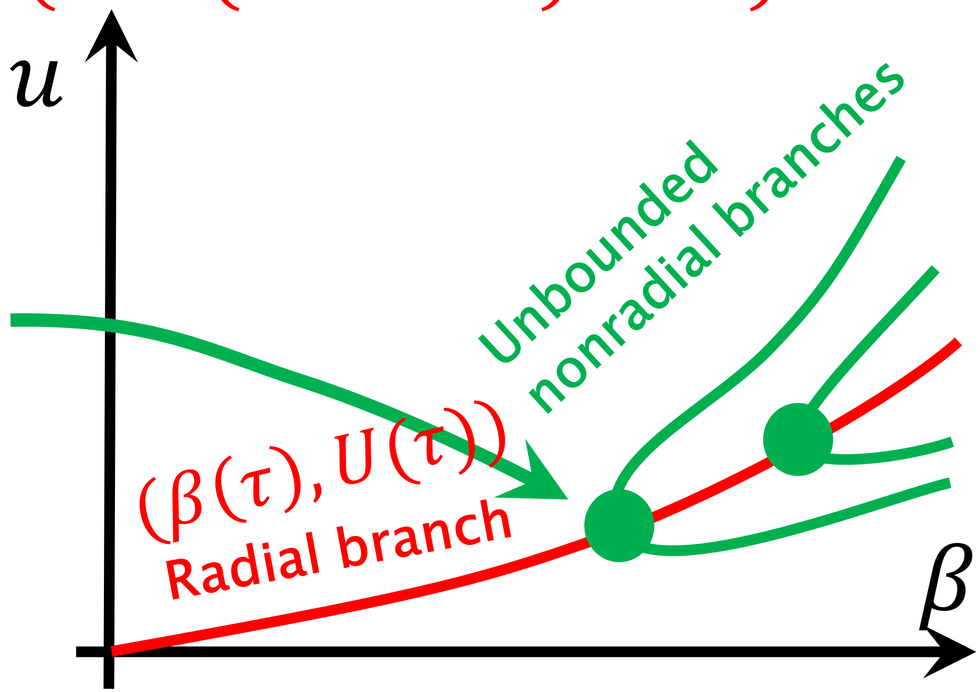
$$\beta(\tau) := \sqrt{8\pi(\alpha+2)(e^{-\tau/2} - 1 + \tau/2)}$$

$$U(r;\tau) := \log \frac{e^\tau}{(1+(e^{\tau/2}-1)r^{\alpha+2})^2}$$

$(\beta(\tau_n), U(r;\tau_n))$

$$\tau_n := 2\log\frac{2\alpha+4}{\alpha+2-2n}$$

$$\left(0 \le n \le 1 + \frac{\alpha}{2}\right)$$



$$
\text{(EL)} \quad \begin{cases} \Delta u + \beta^2 \dfrac{r^\alpha e^u}{\int_D r^\alpha \boldsymbol{u} e^u dx} = 0 & \text{in } D \\ u = 0 & \text{on } \partial D \end{cases}
$$

$$
\text{(MF)} \quad \begin{cases} \Delta u + \rho \dfrac{r^\alpha e^u}{\int_D r^\alpha e^u dx} = 0 & \text{in } D \\ u = 0 & \text{on } \partial D \end{cases}
$$

$$\text{(LG)} \quad \begin{cases} \Delta u + \lambda r^{\alpha} e^{u} = 0 & \text{in } D \\ u = 0 & \text{on } \partial D \end{cases}$$

$$(\lambda, u)$$

$$\lambda = \frac{\rho}{\int_D r^{\alpha} e^{u} dx}$$

$$\rho = \lambda \int_D r^{\alpha} e^{u} dx$$

$$(\rho, u)$$

$$\text{(MF)} \quad \begin{cases} \Delta u + \rho \dfrac{r^{\alpha} e^{u}}{\int_D r^{\alpha} e^{u} dx} = 0 & \text{in } D \\ u = 0 & \text{on } \partial D \end{cases}$$

Corollary D

$$\text{(MF)}\quad \begin{cases} \Delta u + \rho \dfrac{r^\alpha e^u}{\int_D r^\alpha e^u dx} = 0 & \text{in } D \\ u = 0 & \text{on } \partial D \end{cases}$$

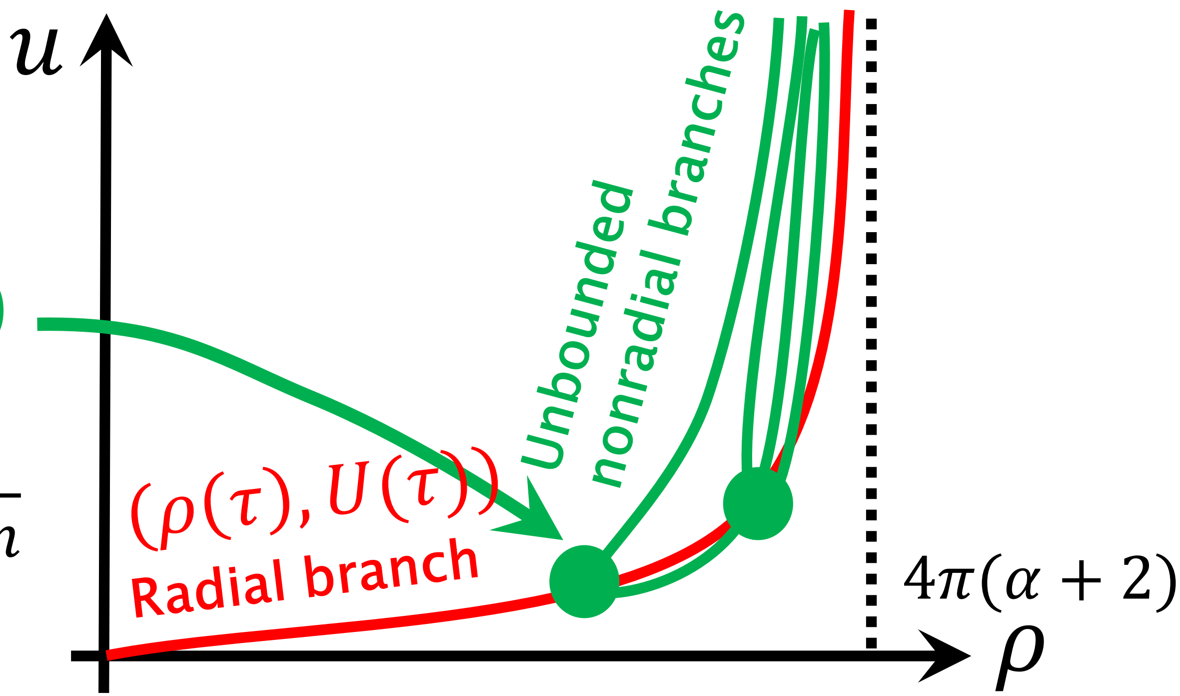
$$\rho(\tau) := 4\pi(\alpha+2)(1 - e^{-\tau/2})$$

$$U(r;\tau) := \log \frac{e^\tau}{(1 + (e^{\tau/2} - 1) r^{\alpha+2})^2}$$

$$(\rho(\tau_n), U(r;\tau_n))$$

$$\tau_n := 2\log\frac{2\alpha+4}{\alpha+2-2n}$$

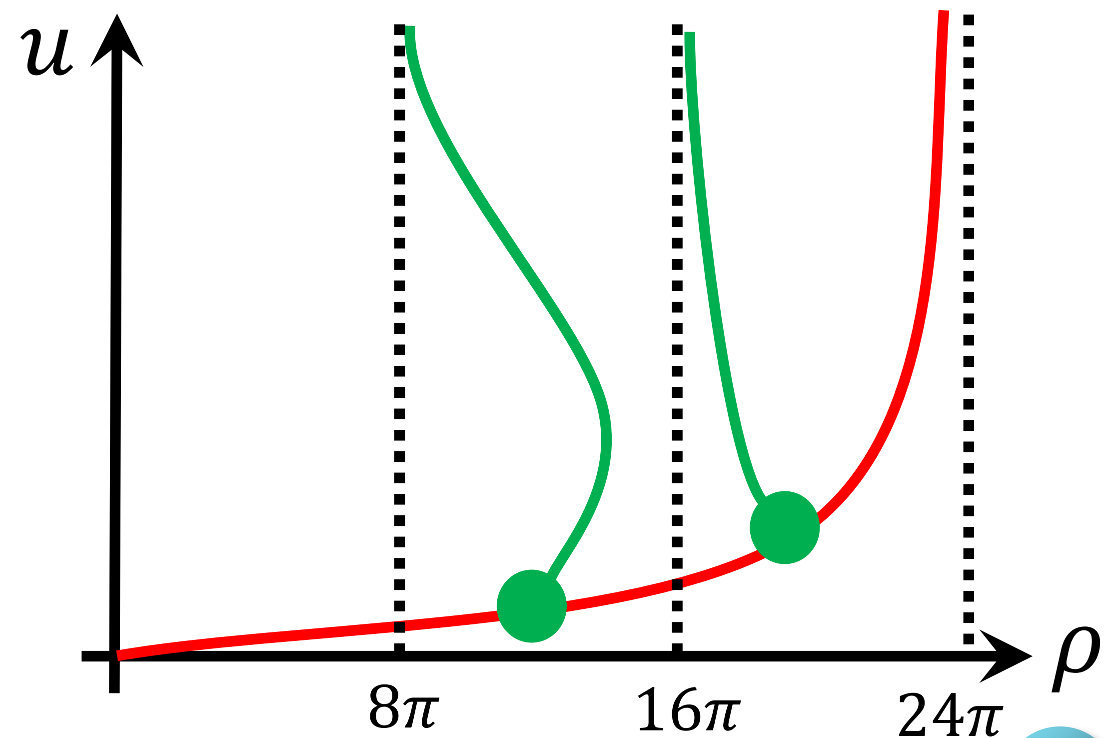
$$\left(0 \le n \le 1 + \frac{\alpha}{2}\right)$$



Remark

$$\text{(MF)} \quad \begin{cases} \Delta u + \rho \dfrac{h(x)e^u}{\int_\Omega h(x)e^u dx} = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

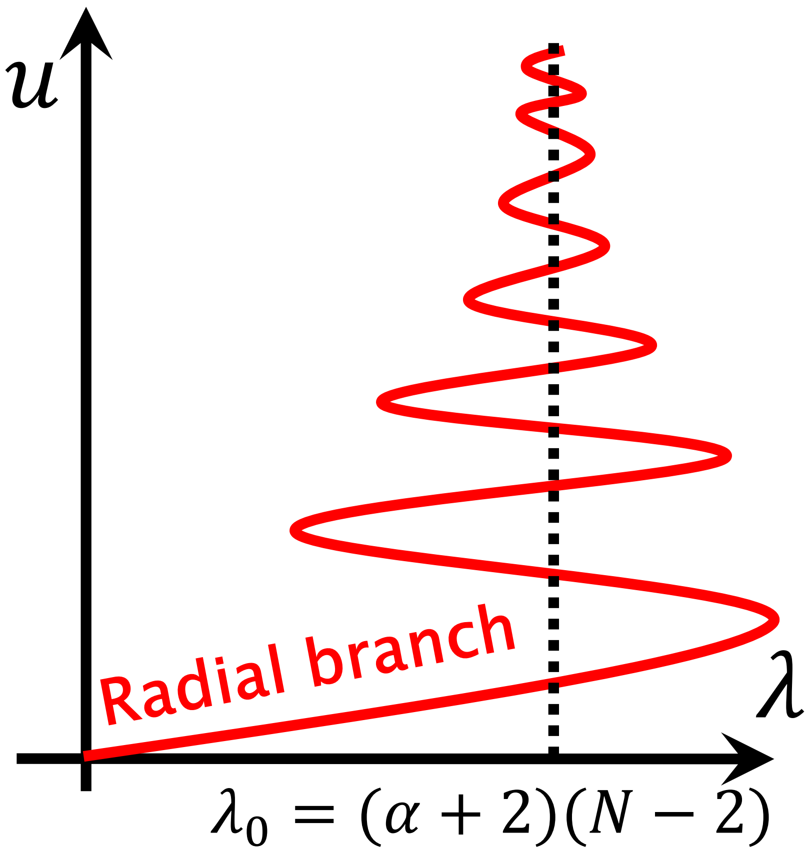
If $0 < c_0 < h(x) < c_1$, then a blow-up occurs only at $\rho = 8\pi n$.



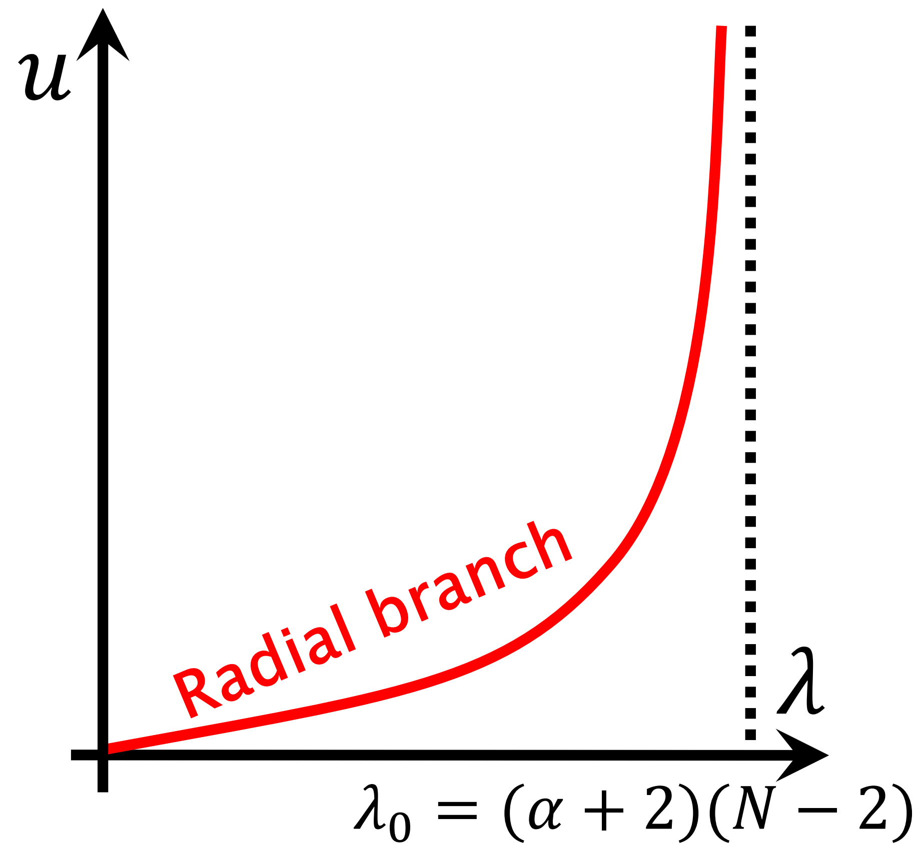
Corollary E

$$\text{(LG)} \quad \begin{cases} \Delta u + \lambda r^\alpha e^u = 0 & \text{in} \quad B \\ u = 0 & \text{on} \quad \partial B \end{cases}$$

$3 \leq N < 10 + 4\alpha$



$10 + 4\alpha \leq N$



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We will find the **radial** solutions of

$$(\mathrm{LG})\quad \begin{cases} \Delta u + \lambda r^{\alpha} e^{u} = 0 & \text{in } D \\ u = 0 & \text{on } \partial D \end{cases}, \text{ i.e.,}$$

$$\begin{cases} u_{rr} + \dfrac{u_r}{r} + \lambda r^{\alpha} e^{u} = 0 \quad \text{in } D \\ u_r(0) = u(1) = 0 \end{cases}$$

Emden's transformation works well.

$$u_{rr} + \frac{u_r}{r} + \lambda r^{\alpha} e^{u} = 0$$

$r := e^{t}$

$$e^{-2t} u_{tt} + e^{u + \log \lambda + \alpha \log r} = 0$$

$t = \log r$

$$u_{tt} + e^{u + \log \lambda + (\alpha + 2)t} = 0$$

$v := u + \log \lambda + (2 + \alpha)t$

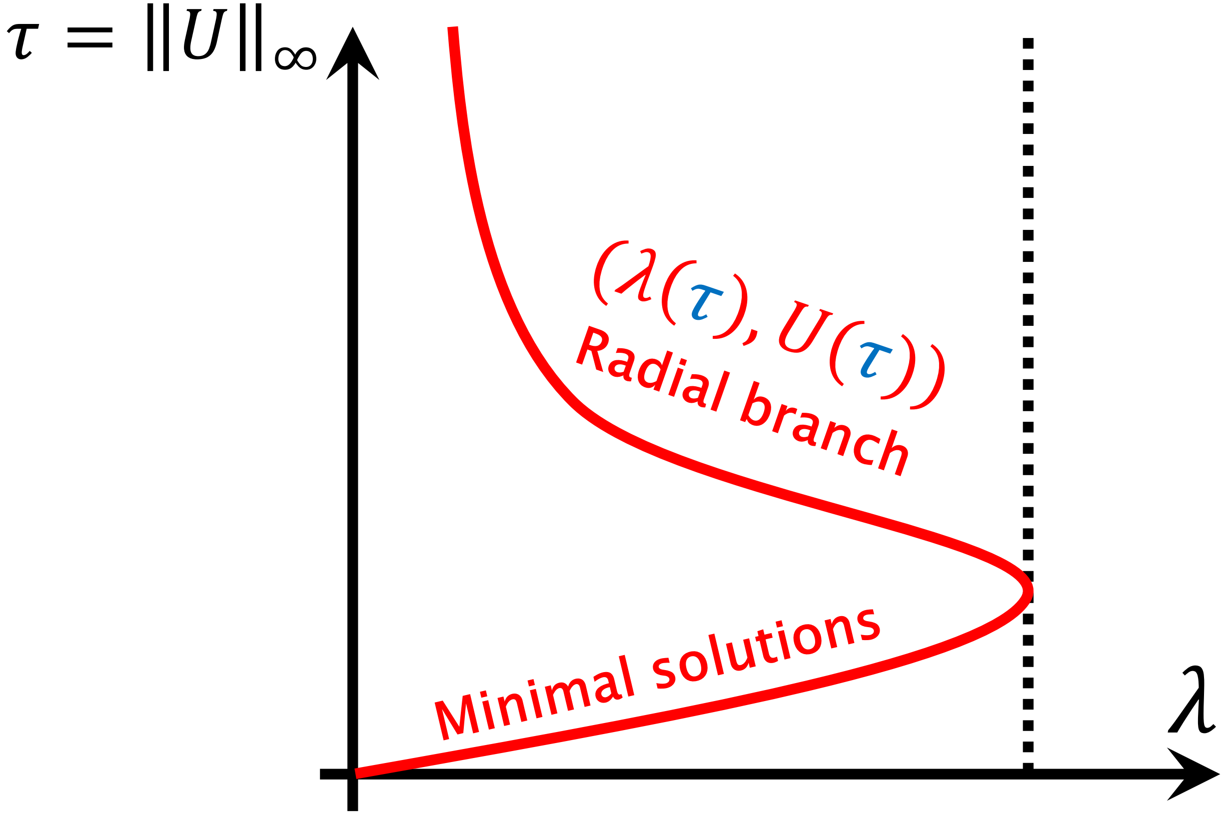
$$v_{tt} + e^{v} = 0$$

All the solutions can be written explicitly!

All radial solutions can be parameterized by τ.

$$\lambda(\tau) := 2(\alpha+2)^2\left(e^{-\tau/2} - e^{-\tau}\right)$$

$$U(r;\tau) := \log\frac{e^{\tau}}{\left(1+\left(e^{\tau/2}-1\right)r^{\alpha+2}\right)^2}$$



Next, we find symmetry breaking bifurcation points on the radial branch of

$$(\mathrm{LG})\quad \begin{cases} \Delta u + \lambda r^{\alpha} e^{u} = 0 & \text{in } \ D \\ u = 0 & \text{on } \ \partial D \end{cases}.$$

We will find the **degenerate** solutions.

$$\text{(LG)} \quad \begin{cases} \Delta u + \lambda r^{\alpha} e^{u} = 0 & \text{in } D \\ u = 0 & \text{on } \partial D \end{cases}.$$

linearize at $U(r;\tau)$

$$\Delta \Phi + \frac{2(\alpha+2)^2\left(e^{\tau/2}-1\right)r^{\alpha}}{\left(1+\left(e^{\tau/2}-1\right)r^{\alpha+2}\right)^2}\Phi = 0$$

$\Phi(r,\theta) \coloneqq \phi(r)\cos n\theta$

$$\phi_{rr} + \frac{\phi_r}{r} - \frac{n^2\phi}{r^2} + \frac{2(\alpha+2)^2\left(e^{\tau/2}-1\right)r^{\alpha}}{\left(1+\left(e^{\tau/2}-1\right)r^{\alpha+2}\right)^2}\phi = 0$$

$$\phi_{rr} + \frac{\phi_r}{r} - \frac{n^2\phi}{r^2} + \frac{2(\alpha+2)^2\left(e^{\tau/2}-1\right)r^{\alpha}}{(1+(e^{\tau/2}-1)r^{\alpha+2})^2}\phi = 0$$

$$\lim_{r\downarrow 0} r^{-n}\phi(r) < \infty, \qquad \phi(1) = 0$$

Two fundamental solutions can be written explicitly.

$$\phi^{(1)}(r) \coloneqq \frac{q-\left(e^{\tau/2}-1\right)r^{\alpha+2}}{1+(e^{\tau/2}-1)r^{\alpha+2}} r^{n}$$

$$\phi^{(2)}(r) \coloneqq \frac{q-\left(e^{\tau/2}-1\right)r^{\alpha+2}}{1+(e^{\tau/2}-1)r^{\alpha+2}} r^{-n}$$

Here, $q \coloneqq (\alpha+2+2n)/(\alpha+2-2n)$.

$$\phi^{(1)}(1) = 0$$

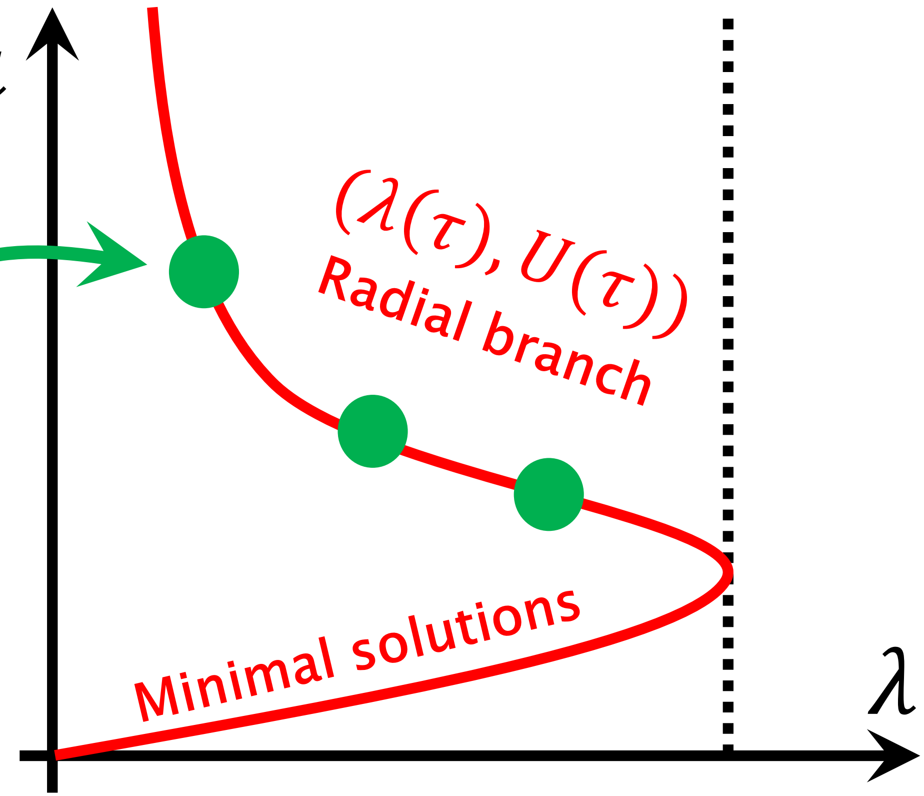
⇕

$$\phi^{(1)}(1) \coloneqq \frac{q - \left(e^{\tau/2} - 1\right)}{1 + \left(e^{\tau/2} - 1\right)} = 0$$

Here, $q \coloneqq (\alpha + 2 + 2n)/(\alpha + 2 - 2n)$.

⇕

The equation has a unique solution when $\tau_n = 2\log(2\alpha + 4)/(\alpha + 2 - 2n)$.

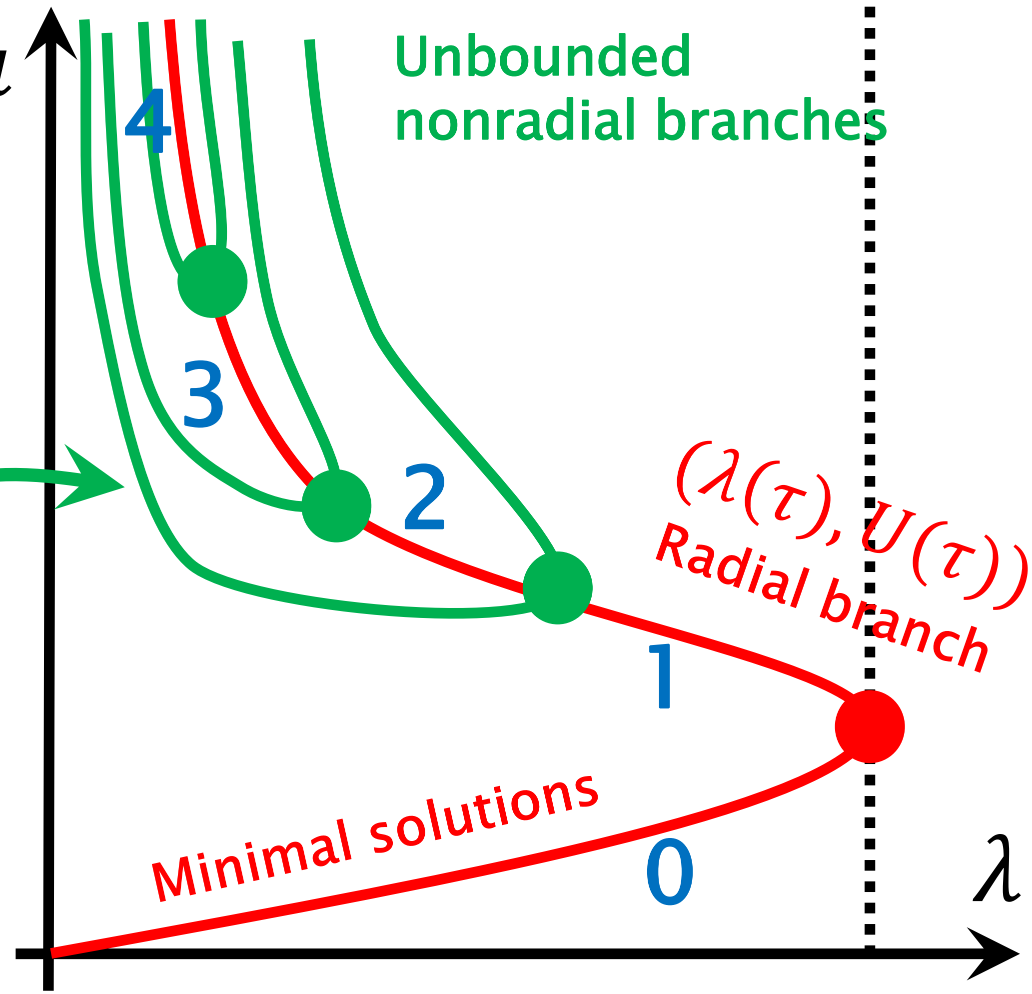


$$\tau_n \coloneqq 2\log\frac{2\alpha+4}{\alpha+2-2n}$$
$$\left(0 \le n \le 1+\frac{\alpha}{2}\right)$$

$\Phi(r,\theta) \coloneqq \phi(r)\cos n\theta$: eigenfunction

$$\phi = \frac{(\alpha+2+2n)(1-r^{\alpha+2})}{\alpha+2-2n+(\alpha+2+2n)r^{\alpha+2}}r^n$$

Applying the Rabinowitz global bifurcation theorem, we obtain unbounded nonradial branches.



$(\lambda(\tau_n), U(r;\tau_n))$

$$\tau_n := 2\log\frac{2\alpha+4}{\alpha+2-2n}$$

$$\left(0 \le n \le 1 + \frac{\alpha}{2}\right)$$

$$\text{(LG)} \quad \begin{cases} \Delta u + \lambda r^{\alpha} e^{u} = 0 & \text{in } D \\ u = 0 & \text{on } \partial D \end{cases}$$

$$(\lambda, u)$$

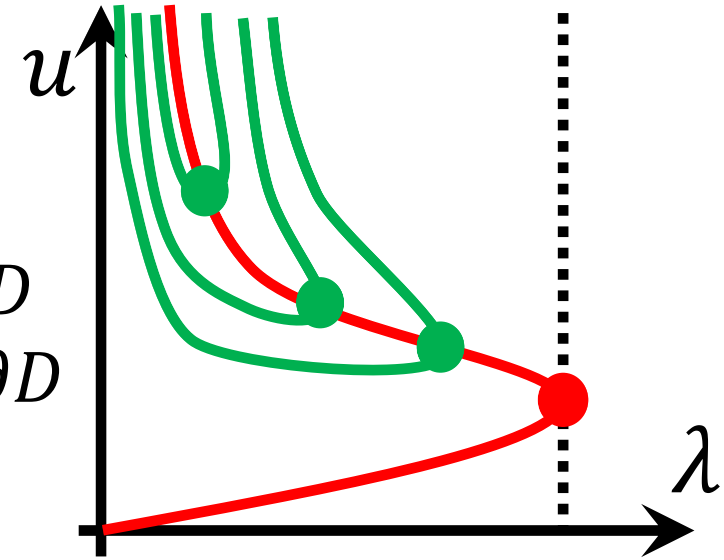
$$\lambda = \frac{\beta^2}{\int_D r^{\alpha} u e^{u} dx}$$

$$\beta = \sqrt{\lambda \int_D r^{\alpha} u e^{u} dx}$$

$$(\beta, u)$$

$$\text{(EL)} \quad \begin{cases} \Delta u + \beta^2 \dfrac{r^{\alpha} e^{u}}{\int_D r^{\alpha} u e^{u} dx} = 0 & \text{in } D \\ u = 0 & \text{on } \partial D \end{cases}$$

(LG) $\begin{cases} \Delta u + \lambda r^{\alpha} e^{u} = 0 & \text{in } D \\ u = 0 & \text{on } \partial D \end{cases}$

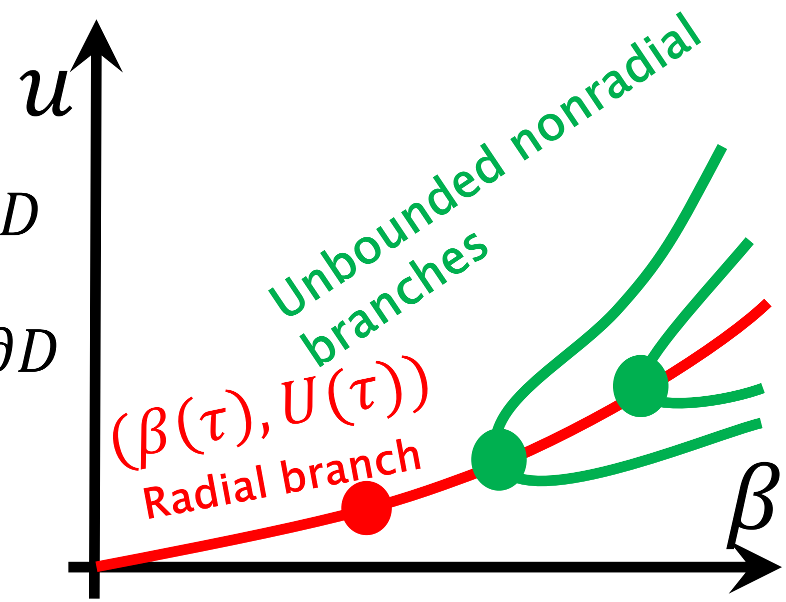


$$\lambda = \frac{\beta^2}{\int_D r^{\alpha} u e^{u} dx}$$

$$\beta = \sqrt{\lambda \int_D r^{\alpha} u e^{u} dx}$$

(EL) $\begin{cases} \Delta u + \beta^2 \dfrac{r^{\alpha} e^{u}}{\int_D r^{\alpha} u e^{u} dx} = 0 & \text{in } D \\ u = 0 & \text{on } \partial D \end{cases}$

$$\beta(\tau) := \sqrt{8\pi(\alpha + 2)(e^{-\tau/2} - 1 + \tau/2)}$$



$$\sup_{u\in H_0^1(D),\,\|\boldsymbol{u}\|=\boldsymbol{\beta}} \int_D r^{\alpha}(e^{u}-1)\,dx$$

$$\Updownarrow$$

$$\sup_{u\in H_0^1(D),\,u\not\equiv 0} \int_D r^{\alpha}\left(e^{\boldsymbol{\beta u}/\|\boldsymbol{u}\|}-1\right)dx$$

$$F(u) := \int_D r^{\alpha}\left(e^{\beta u/\|u\|}-1\right)dx$$

Lemma

Let $(\beta(\tau), U(\tau))$ be a radial solution of (EL).
Let (μ, ξ) be an eigenpair of

$$\Delta\xi + \beta^2 \frac{r^\alpha u}{\int r^\alpha U e^U dx} \xi = -\mu\xi$$

with the Dirichlet boundary condition.
If $\langle U, \xi \rangle = 0$, then

$$D_u^2 F(U)[\xi, \xi] = -\frac{\mu}{\beta^2} \left(\int_D r^\alpha U e^U dx \right) \left(\int_D \xi^2 dx \right)$$

Proof of Lemma

$$D_u^2F(U)[\psi,\psi]$$
$$= \int \beta^2 r^\alpha e^{\frac{\beta U}{\|U\|}}\left(\frac{\psi}{\|U\|} - \frac{\langle U,\psi\rangle U}{\|U\|^3}\right)^2 dx$$
$$+ \int \beta r^\alpha e^{\beta U/\|U\|}\left(-\frac{2\langle U,\psi\rangle\psi}{\|U\|^3} + \frac{3\langle U,\psi\rangle^2 U}{\|U\|^5} - \frac{\|\psi\|^2 U}{\|U\|^3}\right) dx.$$

Since $\langle U,\xi\rangle = 0$, we have

$$D_u^2F(U)[\xi,\xi]$$
$$= \frac{\beta^2}{\|U\|}\int r^\alpha e^{\beta U/\|U\|}\xi^2 dx - \frac{\beta\|\xi\|^2}{\|U\|^3}\int r^\alpha e^{\beta U/\|U\|} U dx.$$

Multiplying $\Delta\xi + \beta^2 \frac{r^\alpha u}{\int r^\alpha U e^U dx}\xi = -\mu\xi$ by ξ and integrating it, we have

$$\int r^\alpha e^U \xi^2 dx = \frac{1}{\beta^2}\left(\int r^\alpha U e^U dx\right)\left(\int \|\xi\|^2 - \mu\int \xi^2 dx\right).$$

$$\int r^{\alpha} e^{U} \xi^2 dx = \frac{1}{\beta^2}\left(\int r^{\alpha} U e^{U} dx\right)\left(\int \|\xi\|^2 - \mu \int \xi^2 dx\right)$$

$$D_u^2 F(U)[\xi,\xi]$$
$$= \frac{\beta^2}{\|U\|} \int r^{\alpha} e^{\beta U/\|U\|} \xi^2 dx - \frac{\beta \|\xi\|^2}{\|U\|^3} \int r^{\alpha} e^{\beta U/\|U\|} U dx$$

We have

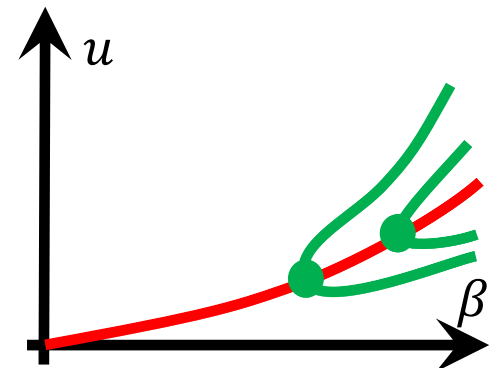
$$D_u^2 F(U)[\xi,\xi] = -\frac{\mu}{\beta^2}\left(\int_D r^{\alpha} U e^{U} dx\right)\left(\int_D \xi^2 dx\right)$$

The proof of the lemma is complete.

Proof of Theorem A

- All radial solutions of (EL) are obtained.

$$\text{(EL)} \quad \begin{cases} \Delta u + \beta^2 \dfrac{r^\alpha e^u}{\int_D r^\alpha u e^u dx} = 0 & \text{in } D \\ u = 0 & \text{on } \partial D \end{cases}$$



- All radial solutions are parameterized by τ.

- For each $\beta > 0$, there exists unique $\tau > 0$.

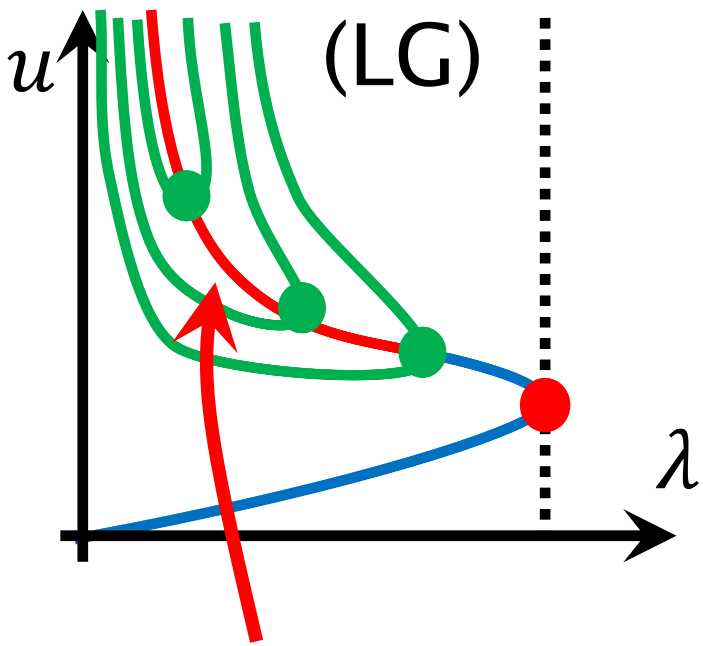
$$\beta(\tau) := \sqrt{8\pi(\alpha + 2)(e^{-\tau/2} - 1 + \tau/2)}$$

- In order to get the nonradiality we show that there is ψ s.t. $D_u^2 F(U)[\psi, \psi] > 0$.

Lemma

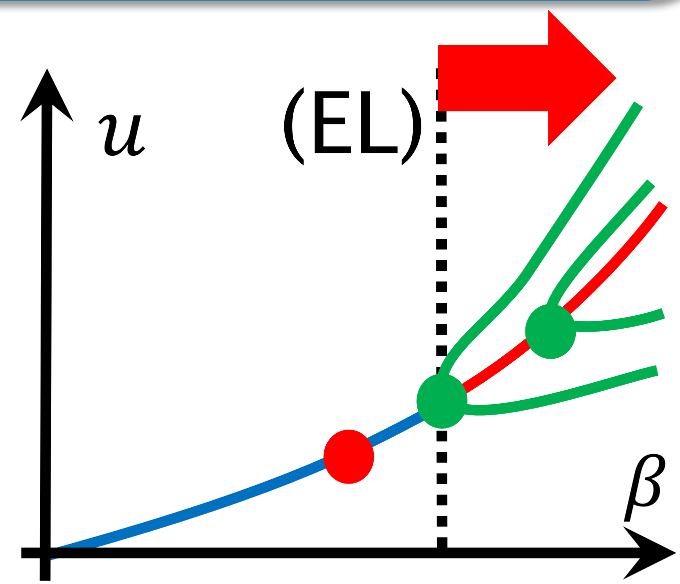
Let (μ, ξ) be an eigenpair of $\Delta\xi + \beta^2 \frac{r^\alpha u}{\int r^\alpha U e^U dx} \xi = -\mu\xi$ with the Dirichlet boundary condition. If $\langle U, \xi \rangle = 0$, then

$$D_u^2 F(U)[\xi, \xi] = -\frac{\mu}{\beta^2}\left(\int_D r^\alpha U e^U dx\right)\left(\int_D \xi^2 dx\right)$$



There is a positive eigenvalue s.t. the associated eigenfunction is

$$\Phi(r, \theta) := \phi(r) \cos\theta$$



$\boldsymbol{\beta > \beta(\tau_1)}$

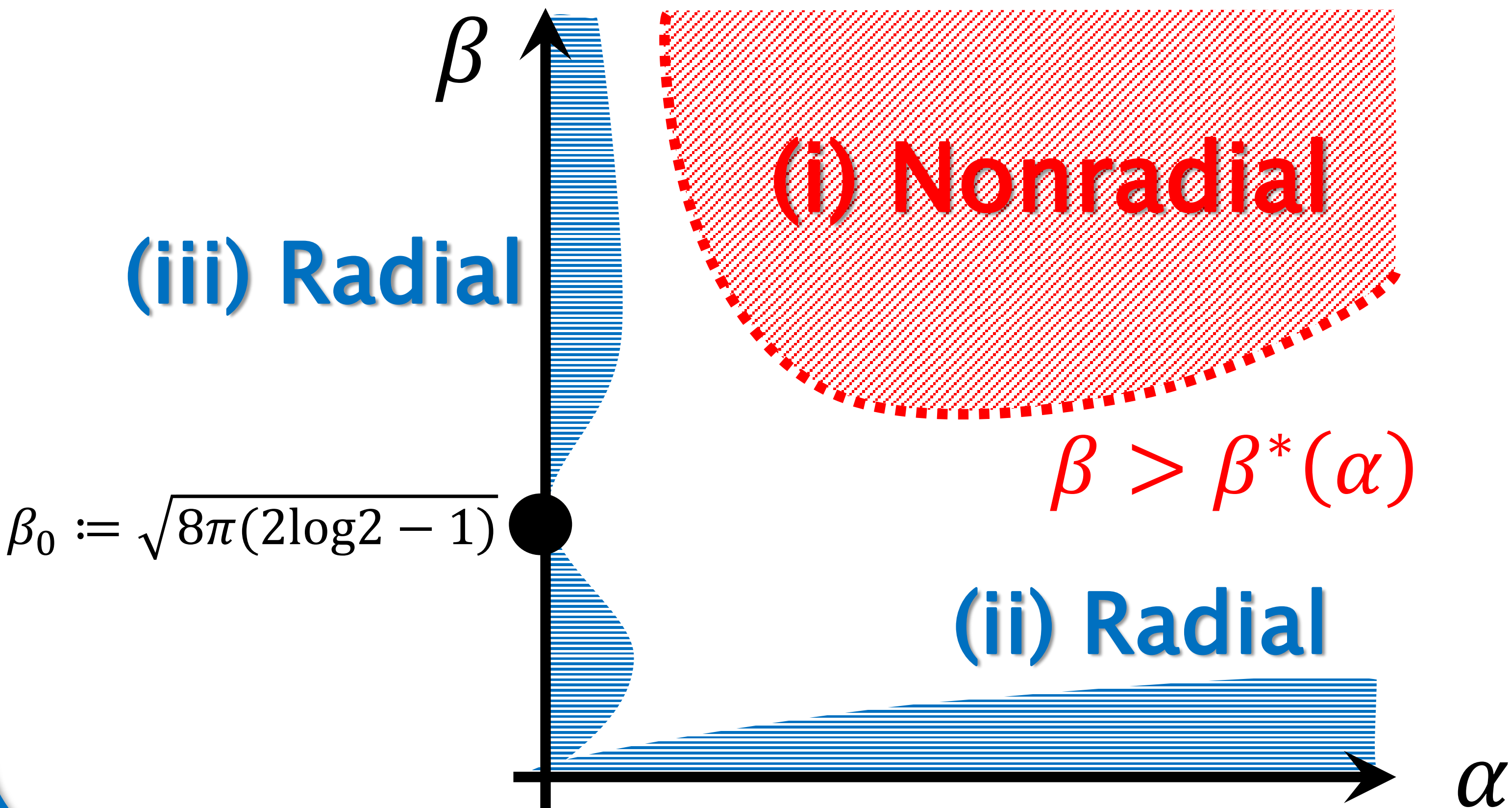
$$= \sqrt{8\pi\left\{(\alpha+2)\log\frac{2\alpha+4}{\alpha} - \frac{\alpha+4}{2}\right\}}$$

Q.E.D.

Theorem A

$$\sup_{u \in H_0^1(D), \|u\| = \beta} \int_D r^\alpha (e^u - 1)\,dx$$

$$\beta^*(\alpha) := \sqrt{8\pi\left\{(\alpha+2)\log\frac{2\alpha+4}{\alpha} - \frac{\alpha+4}{2}\right\}}$$



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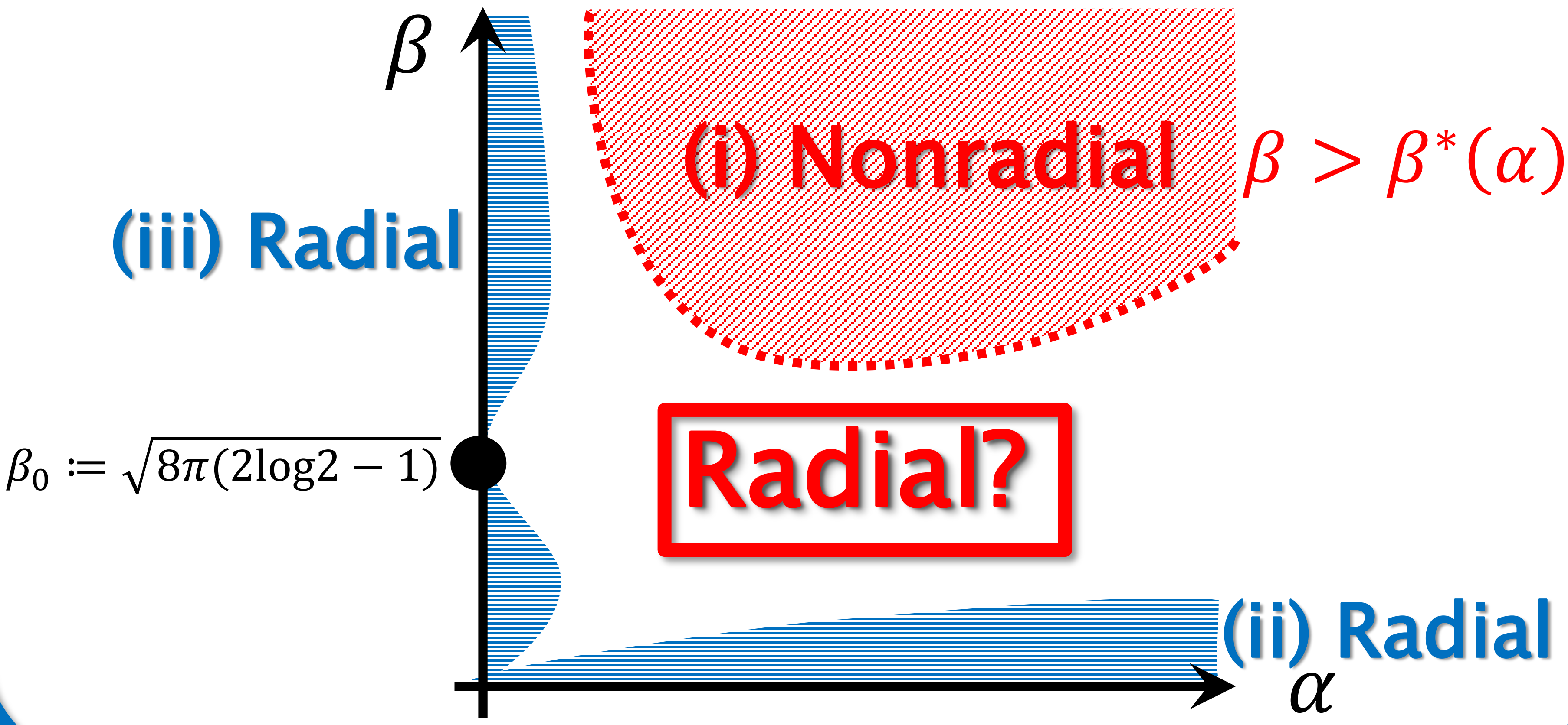
Theorem A

$$\sup_{u\in H_0^1(D),\|u\|=\beta}\int_D r^\alpha(e^u-1)dx$$

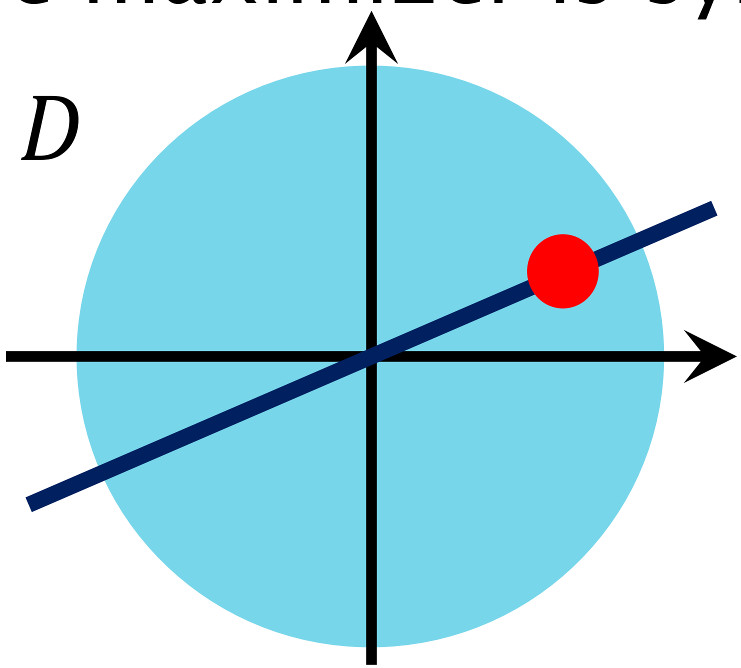
$$\beta^*(\alpha) := \sqrt{8\pi\left\{(\alpha+2)\log\frac{2\alpha+4}{\alpha}-\frac{\alpha+4}{2}\right\}}$$



- We want to know the location of the maximum point of the maximizer of

$$\sup_{u \in H_0^1(D), \|u\| = \beta} \int_D r^\alpha (e^u - 1) dx$$

Using the circular rearrangement, we see that the maximizer is symmetric.



Corollary C

$$\text{(EL)}\quad \begin{cases} \Delta u + \beta^2 \dfrac{r^\alpha e^u}{\int_D r^\alpha u e^u dx} = 0 & \text{in } D \\ u = 0 & \text{on } \partial D \end{cases}$$

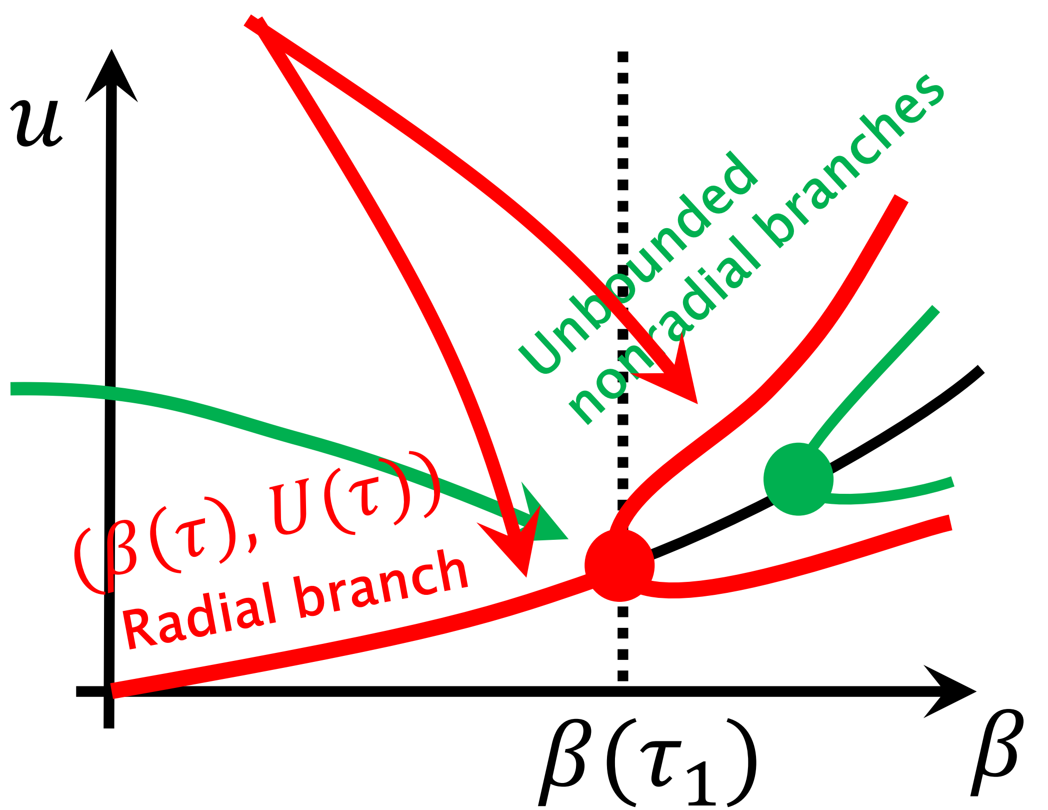
$$\beta(\tau) := \sqrt{8\pi(\alpha+2)(e^{-\tau/2} - 1 + \tau/2)} \qquad U(r;\tau) := \log \frac{e^\tau}{(1 + (e^{\tau/2} - 1) r^{\alpha+2})^2}$$

The branch of the maximizers?

$(\beta(\tau_n), U(r;\tau_n))$

$$\tau_n := 2\log\frac{2\alpha+4}{\alpha+2-2n}$$

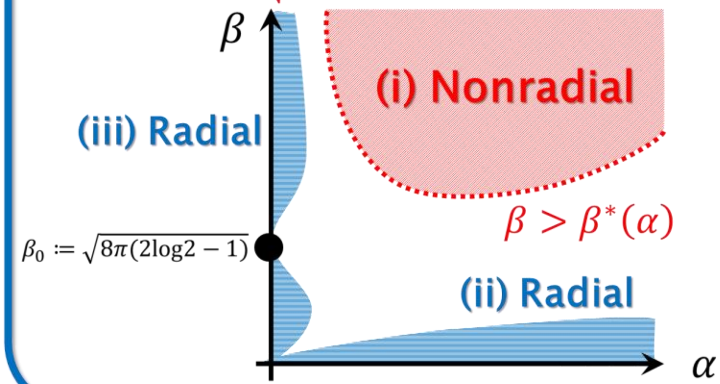
$$\left(0 \le n \le 1 + \frac{\alpha}{2}\right)$$



Theorem A

$$\sup_{u\in H_0^1(D),\|u\|=\beta}\int_D r^\alpha(e^u-1)dx$$

$$\beta^*(\alpha):=\sqrt{8\pi\left\{(\alpha+2)\log\frac{2\alpha+4}{\alpha}-\frac{\alpha+4}{2}\right\}}$$

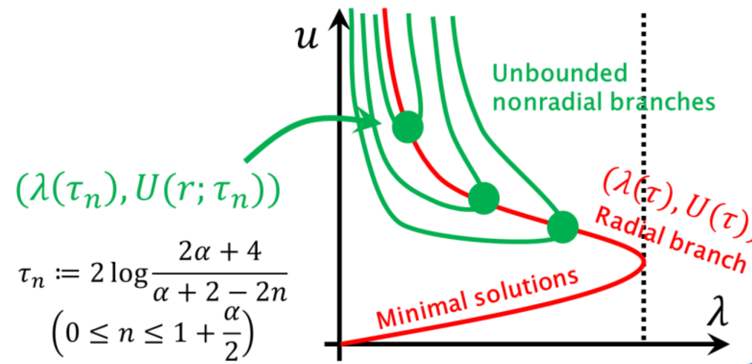


Theorem B

(LG) $\begin{cases}\Delta u+\lambda r^\alpha e^u=0 & \text{in } D\\ u=0 & \text{on } \partial D\end{cases}$

$$\lambda(\tau):=2(\alpha+2)^2(e^{-\tau/2}-e^{-\tau})$$

$$U(r;\tau):=\log\frac{e^\tau}{(1+(e^{\tau/2}-1)r^{\alpha+2})^2}$$



$$\tau_n:=2\log\frac{2\alpha+4}{\alpha+2-2n}$$

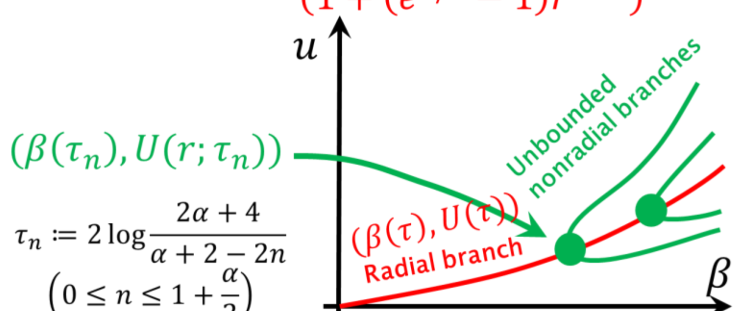
$$\left(0\le n\le 1+\frac{\alpha}{2}\right)$$

Corollary C

(EL) $\begin{cases}\Delta u+\beta^2\dfrac{r^\alpha e^u}{\int_D r^\alpha u e^u dx}=0 & \text{in } D\\ u=0 & \text{on } \partial D\end{cases}$

$$\beta(\tau):=\sqrt{8\pi(\alpha+2)(e^{-\tau/2}-1+\tau/2)}$$

$$U(r;\tau):=\log\frac{e^\tau}{(1+(e^{\tau/2}-1)r^{\alpha+2})^2}$$



$$\tau_n:=2\log\frac{2\alpha+4}{\alpha+2-2n}$$

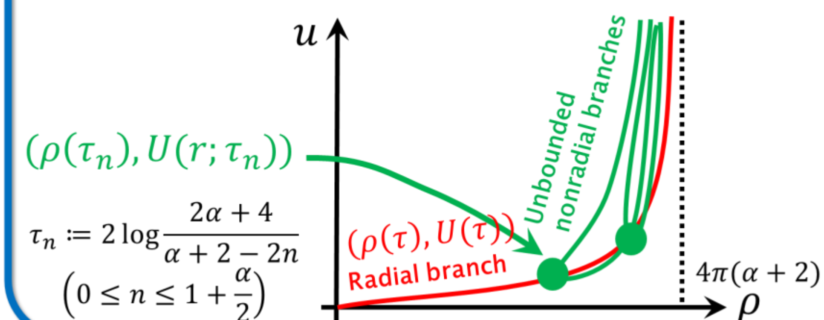
$$\left(0\le n\le 1+\frac{\alpha}{2}\right)$$

Corollary D

(MF) $\begin{cases}\Delta u+\rho\dfrac{r^\alpha e^u}{\int_D r^\alpha e^u dx}=0 & \text{in } D\\ u=0 & \text{on } \partial D\end{cases}$

$$\rho(\tau):=4\pi(\alpha+2)(1-e^{-\tau/2})$$

$$U(r;\tau):=\log\frac{e^\tau}{(1+(e^{\tau/2}-1)r^{\alpha+2})^2}$$



$$\tau_n:=2\log\frac{2\alpha+4}{\alpha+2-2n}$$

$$\left(0\le n\le 1+\frac{\alpha}{2}\right)$$

Thank you for your attention.