Existence and uniqueness of the n-dimensional Helfrich flow

Takeyuki Nagasawa

(Saitama University)

§ 1 The Helfrich Variational Problem

$\Sigma \subset \mathbb{R}^{n+1}$: Hypersurface

$$A(\Sigma) = \text{Area},$$

$$V(\Sigma) = -\frac{1}{n+1}\int_{\Sigma} \boldsymbol{f} \cdot \boldsymbol{\nu}\, dS \text{ ([Enclosed] Volume)}$$

§ 1 The Helfrich Variational Problem

$\Sigma \subset \mathbb{R}^{n+1}$: Hypersurface

$$A(\Sigma) = \text{Area},$$

$$V(\Sigma) = -\frac{1}{n+1}\int_\Sigma \boldsymbol{f} \cdot \boldsymbol{\nu}\, dS \text{ ([Enclosed] Volume)}$$

$$H = \text{ Mean curvature}, \quad c_0 = \text{constant}$$

$$W(\Sigma) = \frac{n}{2}\int_\Sigma (H - c_0)^2 dS$$

The Helfrich Variational Problem

For given constants c_0, A_0, and V_0,

find critical points of W under the constraints

$$A(\Sigma) = A_0, \quad V(\Sigma) = V_0.$$

Backgrounds

- $n = 2$
  A model of shape transformation of human red blood cell (Helfrich, 1973)
  (c_0: spontaneous curvature of cell membrane)

Backgrounds

- $n = 2$
A model of shape transformation of human red blood cell (Helfrich, 1973)
(c_0: spontaneous curvature of cell membrane)

- $n = 1$, $c_0 = 0$
A model of shape transformation of a closed loop of plastic tape between two parallel flat plates (Brailsford, 1983)

• $n = 1$, $c_0 = 0$

Trace formula of heat kernel ($\Sigma = \partial\Omega$)

As $t \to +0$,

$$\int_\Omega G(x,x,t)\,dx = \frac{1}{4\pi t}\left(a_0 + a_1 t^{1/2} + a_2 t + a_3 t^{3/2} + \cdots\right)$$

$$a_0 = V, \quad a_1 = -\frac{\sqrt{\pi}}{2}A, \quad a_2 = \frac{1}{3}\int_\Sigma H\,dS, \quad a_3 = \frac{\sqrt{\pi}}{64}\int_\Sigma H^2 dS$$

For given a_0, a_1, and a_2, find Ω which minimizes a_3 (Watanabe, 2000/2002).

Note. When $n = 1$, we may assume $c_0 = 0$.

Note. When $n = 1$, we may assume $c_0 = 0$.

$\because$

$$W(\Sigma) = \frac{1}{2}\int_\Sigma H^2 dS - c_0 \int_\Sigma H\, dS + \frac{1}{2}c_0^2 \int_\Sigma dS$$

Note. When $n = 1$, we may assume $c_0 = 0$.

$\because$

$$W(\Sigma) = \frac{1}{2}\int_\Sigma H^2 dS - c_0 \int_\Sigma H\, dS + \frac{1}{2}c_0^2 \int_\Sigma dS$$

The 2nd term is a topologically invariant.

The 3rd term is constant because of constraint $A = A_0$.

§ 2 Known results $(n = 2)$

- Spheres are critical points.

§ 2 Known results ($n = 2$)

- Spheres are critical points.
- Bifurcating solutions from spheres and their stability/instability
  (Jenkins (1977), Peterson (1985), Ou-Yang-Helfrich (1989))
  $\cdots$ numerical and/or formal calculus
  (N.-Takagi (2003))

§ 2 Known results ($n = 2$)

- Spheres are critical points.
- Bifurcating solutions from spheres and their stability/instability (Jenkins (1977), Peterson (1985), Ou-Yang-Helfrich (1989))
  $\cdots$ numerical and/or formal calculus
  (N.-Takagi (2003))
- Critical points far from spheres (Au-Wan (2003))

§ 2 Known results ($n = 2$)

- Spheres are critical points.
- Bifurcating solutions from spheres and their stability/instability
(Jenkins (1977), Peterson (1985), Ou-Yang-Helfrich (1989))
$\cdots$ numerical and/or formal calculus
(N.-Takagi (2003))
- Critical points far from spheres
(Au-Wan (2003))
- Critical points without rotational symmerty
(N.-Takagi, unpublished)

§ 3 The Helfrich flow

The Helfrich flow:
Gradient Flow associated to
the Helfrich Variational Problem

$$v = -P(\Sigma)\delta W(\Sigma)$$

§ 3 The Helfrich flow

The Helfrich flow:
Gradient Flow associated to
the Helfrich Variational Problem

$$v = -P(\Sigma)\delta W(\Sigma)$$

v : normal velocity, δ : the first variation,

$P \, : \, L^2(\Sigma) \to (\mathrm{span}\,\{\delta A, \delta V\})^\perp$: projection

$$v = -\Delta_g H - (H - c_0)\left\{\frac{n^2}{2}H(H + c_0) + R\right\}$$

$$+ \lambda_1 n H + \lambda_2$$

$$v = -\Delta_g H - (H - c_0)\left\{\frac{n^2}{2}H(H + c_0) + R\right\} + \lambda_1 nH + \lambda_2$$

Δ_g : Laplace-Beltrami operator,

R : scalar curvature

λ_i : Lagrange multipliers

λ_1's are determined by

$$0 = \frac{d}{dt}A(\Sigma) = \langle \delta A(\Sigma), v\rangle, \quad 0 = \frac{d}{dt}V(\Sigma) = \langle \delta V(\Sigma), v\rangle$$

λ_1's are determined by

$$0 = \frac{d}{dt}A(\Sigma) = \langle \delta A(\Sigma), v \rangle, \quad 0 = \frac{d}{dt}V(\Sigma) = \langle \delta V(\Sigma), v \rangle$$

It follows from equation that

$$0 = -\langle \delta A(\Sigma), \delta W(\Sigma) + \lambda_1 \delta A(\Sigma) + \lambda_2 \delta V(\Sigma) \rangle,$$

$$0 = -\langle \delta V(\Sigma), \delta W(\Sigma) + \lambda_1 \delta A(\Sigma) + \lambda_2 \delta V(\Sigma) \rangle,$$

$$\begin{pmatrix} \langle \delta A(\Sigma), \delta A(\Sigma) \rangle & \langle \delta A(\Sigma), \delta V(\Sigma) \rangle \\ \langle \delta V(\Sigma), \delta A(\Sigma) \rangle & \langle \delta V(\Sigma), \delta V(\Sigma) \rangle \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$$

$$= - \begin{pmatrix} \langle \delta A(\Sigma), \delta W(\Sigma) \rangle \\ \langle \delta V(\Sigma), \delta W(\Sigma) \rangle \end{pmatrix}$$

$$\begin{pmatrix} \langle \delta A(\Sigma), \delta A(\Sigma) \rangle & \langle \delta A(\Sigma), \delta V(\Sigma) \rangle \\ \langle \delta V(\Sigma), \delta A(\Sigma) \rangle & \langle \delta V(\Sigma), \delta V(\Sigma) \rangle \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$$

$$= - \begin{pmatrix} \langle \delta A(\Sigma), \delta W(\Sigma) \rangle \\ \langle \delta V(\Sigma), \delta W(\Sigma) \rangle \end{pmatrix}$$

Gramian $:= \det \begin{pmatrix} \langle \delta A(\Sigma), \delta A(\Sigma) \rangle & \langle \delta A(\Sigma), \delta V(\Sigma) \rangle \\ \langle \delta V(\Sigma), \delta A(\Sigma) \rangle & \langle \delta V(\Sigma), \delta V(\Sigma) \rangle \end{pmatrix}$

- If Gramian $\neq 0$,
then λ_i's are uniquely determined from Σ.

- If Gramian $\neq 0$,
then λ_i's are uniquely determined from Σ.

- If Gramian $= 0$,
then λ_i's are not uniquely determined,

• If Gramian $\neq 0$,
then λ_i's are uniquely determined from Σ.

• If Gramian $= 0$,
then λ_i's are not uniquely determined,
but uniquely determined

$$\lambda_1 \delta A(\Sigma) + \lambda_2 \delta V(\Sigma)$$

from Σ.

Theorem 1.

Let $P(\Sigma)$ be the projection

$$L^2(\Sigma) \longrightarrow (\mathrm{span}\,\{\delta A(\Sigma), \delta V(\Sigma)\})^{\perp}\,.$$

Consider the equation

$$v = -P(\Sigma)\delta W(\Sigma).$$

Solutions, if exist, satisfy

$$\frac{d}{dt}W(\Sigma) = -\|v\|_{L^2}^2, \quad \frac{d}{dt}A(\Sigma) \equiv 0, \quad \frac{d}{dt}V(\Sigma) \equiv 0.$$

Ref. Shikhman, V. & O. Stein, *Constrained optimization: projected gradient flows*, J. Optim. Theory Appl. **140** (1) (2009), 117–130.

Ref. Shikhman, V. & O. Stein, *Constrained optimization: projected gradient flows*, J. Optim. Theory Appl. **140** (1) (2009), 117–130.

($\Uparrow$ Gereral theory in case with Gramian $\neq 0$)

Ref. Shikhman, V. & O. Stein, *Constrained optimization: projected gradient flows*, J. Optim. Theory Appl. **140** (1) (2009), 117–130.

($\Uparrow$ Gereral theory in case with Gramian $\neq 0$)

If Gramian $\neq 0$, then it is not so difficult to deal with.

Ref. Shikhman, V. & O. Stein, *Constrained optimization: projected gradient flows*, J. Optim. Theory Appl. **140** (1) (2009), 117–130.

($\Uparrow$ Gereral theory in case with Gramian $\neq 0$)

If Gramian $\neq 0$, then it is not so difficult to deal with.

This is not our case !

Difficulty to deal with the projected gradient flow:

Difficulty to deal with the projected gradient flow:

- Assume
  - Gramian $(t) > 0$ for $t \in [0, T_*)$,
  - Gramian $(T_*) = 0$,

    $\Longrightarrow P(\Sigma(t))$ may be discontinuous at $t = T_*$.

Difficulty to deal with the projected gradient flow:

- Assume
  - Gramian $(t) > 0$ for $t \in [0, T_*)$,
  - Gramian $(T_*) = 0$,

    $\Longrightarrow P(\Sigma(t))$ may be discontinuous at $t = T_*$.

- The condition Gramian $\neq 0$ is a posteriori.

For an integrable function f on Σ:

$$\bar{f} = \frac{1}{A} \int_{\Sigma} f \, dS$$

$$\tilde{f} = f - \bar{f}$$

For an integrable function f on Σ:

$$\bar{f} = \frac{1}{A} \int_{\Sigma} f \, dS$$

$$\tilde{f} = f - \bar{f}$$

For a mathematical object F on $\Sigma(t) \times [0, T)$:

$$F_0 = F\big|_{t=0}$$

Theorem 2 (N.-Yi).

Assume $\Sigma_0 \in h^{3+\alpha}$ (the little Hölder space) $(0 < \alpha < 1)$.

Theorem 2 (N.-Yi).

Assume $\Sigma_0 \in h^{3+\alpha}$ (the little Hölder space) $(0 < \alpha < 1)$.

- If Gramian $\neq 0$ at $t = 0$,

then $^{\exists 1}$ local Helfrich flow with $\Sigma(0) = \Sigma_0$.

Theorem 2 (N.-Yi).

Assume $\Sigma_0 \in h^{3+\alpha}$ (the little Hölder space) $(0 < \alpha < 1)$.

- If Gramian $\neq 0$ at $t = 0$,

then $^{\exists 1}$ local Helfrich flow with $\Sigma(0) = \Sigma_0$.

- Assume Gramian $= 0$ at $t = 0$.
  - If $(\bar{H}_0 - c_0)\tilde{R}_0 \equiv 0$,

    then $^{\exists}$ global Helfrich flow with $\Sigma(0) = \Sigma_0$ (the uniqueness is uncertain expect one-dimensional case).

- Assume Gramian $\neq 0$ at $t \neq 0$.

- Assume Gramian $\neq 0$ at $t \neq 0$.

May assume Gramian$(t) \neq 0$ for small $t > 0$.
$\Longrightarrow$ The equation is quasilinear parabolic of 4th order with non-local terms.

- Assume Gramian $\neq 0$ at $t \neq 0$.

May assume Gramian$(t) \neq 0$ for small $t > 0$.
$\Longrightarrow$ The equation is quasilinear parabolic of 4th order with non-local terms.

$\Longrightarrow$ Gerenal theory (eg. Amann) is applicable.

- Assume Gramian $= 0$ at $t = 0$ and $\left(\bar{H}_0 - c_0\right) \tilde{R}_0 \equiv 0$.

- Assume Gramian $= 0$ at $t = 0$ and $\left(\bar{H}_0 - c_0\right) \tilde{R}_0 \equiv 0$.

$\Longrightarrow \Sigma(t) \equiv \Sigma_0$ is a stationary solution.

Comparison with a known result (1)

Kohsaka-N. (2006) $n = 2$
Analysis of

$$v = -\delta W - \lambda_1 \delta A - \lambda_2 \delta V$$

for given constants λ_i

Comparison with a known result (1)

Kohsaka-N. (2006) $n = 2$
Analysis of

$$v = -\delta W - \lambda_1 \delta A - \lambda_2 \delta V$$

for given constants λ_i

- Local existence and uniqueness under $\Sigma_0 \in h^{2+\alpha}$ $(0 < \alpha < 1)$.

Comparison with a known result (1)

Kohsaka-N. (2006) $n = 2$
Analysis of

$$v = -\delta W - \lambda_1 \delta A - \lambda_2 \delta V$$

for given constants λ_i

- Local existence and uniqueness under $\Sigma_0 \in h^{2+\alpha}$ $(0 < \alpha < 1)$.
- The global existence cannot be expected. ($^\exists$ solutions blowing up in finite/infinite time).

Comparison with a known result (1)

Kohsaka-N. (2006) $n = 2$
Analysis of

$$v = -\delta W - \lambda_1 \delta A - \lambda_2 \delta V$$

for given constants λ_i

- Local existence and uniqueness under $\Sigma_0 \in h^{2+\alpha}$ $(0 < \alpha < 1)$.
- The global existence cannot be expected. ($^\exists$ solutions blowing up in finite/infinite time).
- Existence of center manifold near sphere and an upper estimate of its dimension.

Comparison with a known result (2)

Kurihara-N. (2006/2007) $n = 1$

Comparison with a known result (2)

Kurihara-N. (2006/2007) $n = 1$
In the case $n = 1$: Global existence

Comparison with a known result (2)

Kurihara-N. (2006/2007) $n = 1$
In the case $n = 1$: Global existence

- By penalty method: Construct solutions to

$$v = -\delta W - \frac{1}{2\varepsilon}\delta(A - A_0)^2 - \frac{1}{2\varepsilon}\delta(V - V_0)^2,$$

  and then take the limit as $\varepsilon \downarrow 0$.

Comparison with a known result (2)

Kurihara-N. (2006/2007) $n = 1$

In the case $n = 1$: Global existence

- By penalty method: Construct solutions to

$$v = -\delta W - \frac{1}{2\varepsilon}\delta(A - A_0)^2 - \frac{1}{2\varepsilon}\delta(V - V_0)^2,$$

  and then take the limit as $\varepsilon \downarrow 0$.

- $\Sigma_0 \in C^\infty$

Comparison with a known result (2)

Kurihara-N. (2006/2007) $n = 1$

In the case $n = 1$: Global existence

- By penalty method: Construct solutions to

$$v = -\delta W - \frac{1}{2\varepsilon}\delta(A - A_0)^2 - \frac{1}{2\varepsilon}\delta(V - V_0)^2,$$

  and then take the limit as $\varepsilon \downarrow 0$.
- $\Sigma_0 \in C^\infty$
- The uniqueness is uncertain.

§ 4. A Gramian estimate

- $\Sigma(t) \hookrightarrow \mathbb{R}^{n+1}$: continuous w. r. t. t
  then Gramian$(0) > 0 \Longrightarrow$ Gramian$(t) > 0$.
  (the isoperimetric inequality)

§ 4. A Gramian estimate

- $\Sigma(t) \hookrightarrow \mathbb{R}^{n+1}$: continuous w. r. t. t
  then Gramian$(0) > 0 \Longrightarrow$ Gramian$(t) > 0$.
  (the isoperimetric inequality)

- Non-embedding case ?

§ 4. A Gramian estimate

- $\Sigma(t) \hookrightarrow \mathbb{R}^{n+1}$: continuous w. r. t. t
  then Gramian$(0) > 0 \Longrightarrow$ Gramian$(t) > 0$.
  (the isoperimetric inequality)

- Non-embedding case ?

- Continuity w. r. t. t ?
  (If Gramian$(t) \to +0$ as $t \to T_* - 0$,
  then $P(\Sigma(t))$ may be discontinuous.)

Assume Gramian$(0) > 0$, and define

$$T_* = \sup\{t > 0 \,|\, \text{Gramian}(\tau) > 0 \text{ for } \tau \in [0, t)\}$$

Assume Gramian$(0) > 0$, and define

$$T_* = \sup\{t > 0 \,|\, \text{Gramian}(\tau) > 0 \text{ for } \tau \in [0, t)\}$$

- Do solutions exist with $T_* < \infty$?
- If exists, what happens at $t = T_*$?

Because $\delta A = -nH$ and $\delta V = -1$,

$$\text{Gramian} = \det \begin{pmatrix} \langle \delta A(\Sigma), \delta A(\Sigma) \rangle & \langle \delta A(\Sigma), \delta V(\Sigma) \rangle \\ \langle \delta V(\Sigma), \delta A(\Sigma) \rangle & \langle \delta V(\Sigma), \delta V(\Sigma) \rangle \end{pmatrix}$$

$$= \int_\Sigma n^2 H^2 dS \int_\Sigma dS - \left(\int_\Sigma nH dS \right)^2$$

$$= n^2 A \int_\Sigma \tilde{H}^2 dS.$$

Because $\delta A = -nH$ and $\delta V = -1$,

$$\text{Gramian} = \det \begin{pmatrix} \langle \delta A(\Sigma), \delta A(\Sigma) \rangle & \langle \delta A(\Sigma), \delta V(\Sigma) \rangle \\ \langle \delta V(\Sigma), \delta A(\Sigma) \rangle & \langle \delta V(\Sigma), \delta V(\Sigma) \rangle \end{pmatrix}$$

$$= \int_\Sigma n^2 H^2 dS \int_\Sigma dS - \left(\int_\Sigma nH dS \right)^2$$

$$= n^2 A \int_\Sigma \tilde{H}^2 dS.$$

Let consider the time evolution of $\|\tilde{H}\|_{L^2}^2$.

$$
\begin{aligned}
W &= \frac{n}{2}\|H - c_0\|_{L^2}^2 \\
&= \frac{n}{2}\left\{\|\tilde{H}\|_{L^2}^2 + A\left(\int_\Sigma H\,dS\right)^2\right. \\
&\qquad\left. - 2c_0\int_\Sigma H\,dS + c_0^2 A\right\}
\end{aligned}
$$

$$
\begin{aligned}
W &= \frac{n}{2}\|H - c_0\|_{L^2}^2 \\
&= \frac{n}{2}\left\{\|\tilde{H}\|_{L^2}^2 + A\left(\int_\Sigma H\,dS\right)^2\right. \\
&\qquad\qquad \left. - 2c_0\int_\Sigma H\,dS + c_0^2 A\right\}
\end{aligned}
$$

$$
\frac{d}{dt}\|\tilde{H}\|_{L^2}^2 = \frac{2}{n}\frac{d}{dt}W - \cdots .
$$

$$\text{Gramian} = 0 \iff \text{constant mean curvature}$$
$$\iff \text{Gauss map is harmonic.}$$

$$\text{Gramian} = 0 \iff \text{constant mean curvature}$$
$$\iff \text{Gauss map is harmonic.}$$

Gauss map $\boldsymbol{\nu} : \Sigma \longrightarrow S^n$
The energy

$$E(\boldsymbol{\nu}) = \frac{1}{2}\int_\Sigma \|\nabla \boldsymbol{\nu}\|_g^2 dS$$
$$= \frac{1}{2}\int_\Sigma \left(n^2 H^2 - R\right) = \frac{n^2}{2}\|\tilde{H}\|_{L^2}^2 + \cdots$$

$$\frac{d}{dt}\|\tilde{H}\|^2 = \frac{2}{n^2}\frac{d}{dt}E(\boldsymbol{\nu}) + \cdots$$

Fact. $\delta\boldsymbol{\nu}[\phi] = -\nabla\phi$.

Fact. $\delta\boldsymbol{\nu}[\phi] = -\nabla\phi$.

Corollary.

$$\frac{d}{dt}E(\boldsymbol{\nu}) = \delta E(\boldsymbol{\nu})[\partial_t\boldsymbol{\nu}] = -\delta E(\boldsymbol{\nu})[\nabla v]$$

Fact. $\delta\boldsymbol{\nu}[\phi] = -\nabla\phi$.

Corollary.

$$\frac{d}{dt}E(\boldsymbol{\nu}) = \delta E(\boldsymbol{\nu})[\partial_t \boldsymbol{\nu}] = -\delta E(\boldsymbol{\nu})[\nabla v]$$

Corollary.

$$\frac{1}{2}\frac{d}{dt}\int_\Sigma \left(n^2 H^2 - R\right) dS = -n\langle v, P\Delta_g H\rangle_g$$

$$
\begin{aligned}
0 \leqq n \frac{d}{dt} \|\tilde{H}\|_{L^2}^2 \\
&+ 4c_0 \frac{d}{dt} \int_\Sigma H \, dS - \frac{1}{n} \frac{d}{dt} \int_\Sigma R \, dS \\
&+ \left\| P \left(\frac{n^2}{2} H^3 - \bar{H} R \right) \right\|_{L^2}^2
\end{aligned}
$$

- Assume “$n = 1$”, or “$n = 2$ and $c_0 = 0$”

- Assume “$n = 1$”, or “$n = 2$ and $c_0 = 0$”

$$\Longrightarrow \text{red parts} = 0$$

- Assume "$n = 1$", or "$n = 2$ and $c_0 = 0$"

$$\Longrightarrow \text{red parts} = 0$$

- Assume $\tilde{H} \in L^\infty$, $\tilde{R} \in L^\infty$.

- Assume "$n = 1$", or "$n = 2$ and $c_0 = 0$"

$$\Longrightarrow \text{red parts} = 0$$

- Assume $\tilde{H} \in L^\infty$, $\tilde{R} \in L^\infty$.

$$\Longrightarrow \text{green part} \leq C\|\tilde{H}\|_{L^2}^2$$

- Assume "$n = 1$", or "$n = 2$ and $c_0 = 0$"

$$\Longrightarrow \text{red parts} = 0$$

- Assume $\tilde{H} \in L^\infty$, $\tilde{R} \in L^\infty$.

$$\Longrightarrow \text{green part} \leq C\|\tilde{H}\|_{L^2}^2$$

$$\therefore \quad 0 \leq \frac{d}{dt}\|\tilde{H}\|_{L^2}^2 + \lambda\|\tilde{H}\|_{L^2}^2$$

Theorem 3.

Assume "$n = 1$" or "$n = 2$ and $c_0 = 0$".
Consider the Helfrich flow with Gramian$(0) > 0$.
If $T_* = \sup\{t \,|\, \text{Gramian}(\tau) > 0 \text{ for } \tau \in [0, t)\} < \infty$, then

$$\limsup_{t \to T_* - 0} \text{osc } \tilde{H} \text{ or } \limsup_{t \to T_* - 0} \text{osc } \tilde{R} \text{ blows up.}$$

Corollary. When $n = 1$ and $\Sigma_0 \in C^\infty$,

$$\text{Gramian}(0) > 0 \implies \text{Gramian}(t) > 0$$

Corollary. When $n = 1$ and $\Sigma_0 \in C^\infty$,

$$\text{Gramian}(0) > 0 \implies \text{Gramian}(t) > 0$$

$\because$ We have L^∞-estimate of H.

If Gramian$(t) \longrightarrow 0$ as $t \longrightarrow T_* - 0$,
then (roughly speaking)

$$\|\tilde{H}\|_{L^2} \longrightarrow 0, \quad \|\tilde{H}\|_{L^\infty} \longrightarrow \infty.$$

If Gramian$(t) \to 0$ as $t \to T_* - 0$,
then (roughly speaking)

$$\|\tilde{H}\|_{L^2} \to 0, \quad \|\tilde{H}\|_{L^\infty} \to \infty.$$

$\Longrightarrow$ mean curvature concentrates on a $\mathcal{H}^n$-null set (blow-up set).

Conjecture

- $n \geqq 2 \implies {}^{\exists}$ blow-up solutions
  - $\dim_{\mathcal{H}}(\text{blow-up set}) \leqq n-2$
  - Energy gap $= \dfrac{n}{2} \times 4\pi$

    $\times \mathcal{H}^{n-2}(\text{blow-up sets}) \times$ multiplicity

Thank you very much
for your attention.