Sharp estimates for a nonlocal eigenvalue problem*

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Università degli Studi di Napoli "Federico II"

20 giugno 2011

*joint work with B. Brandolini, P. Freitas, and C. Trombetti

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A non local eigenvalue problem

Let $\alpha \in \mathbb{R}$ and $\Omega \subset \mathbb{R}^n$ an open bounded set we consider the eigenvalue problem

$$\begin{cases} -\Delta u + \alpha \displaystyle\int_\Omega u \, \mathrm{d}x = \lambda u & \text{in } \Omega \\ \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

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SPERB (1981)
HENROT & FREITAS (2004)
CAFFARELLI & LIN (2009)
PINSKY (2009)

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We shall consider the first eigenvalue $\lambda_1(\alpha, \Omega)$ and more precisely the related shape optimization problem:

Problem

Find the optimal domain which minimize λ_1 among the domains with given measure.

Some known facts when $\alpha = 0$

$\Omega \subset \mathbb{R}^n$

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- The first eigenfunction has constant sign on connected domains.

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- It is always possible to choose a positive first eigenfuction.

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- It is always possible to choose a positive first eigenfuction.
- The ball is the unique set achieving the minimum first eigenvalue among the domains with given measure.

Faber-Krahn inequality

$$\lambda_1(\Omega) = \frac{\displaystyle\int_\Omega |Du|^2 \, \mathrm{d}x}{\displaystyle\int_\Omega u^2 \, \mathrm{d}x} \geq \frac{\displaystyle\int_{\Omega^\sharp} |Du^\sharp|^2 \, \mathrm{d}x}{\displaystyle\int_{\Omega^\sharp} {u^\sharp}^2 \, \mathrm{d}x} \geq \lambda_1(\Omega^\sharp)$$

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- The first eigenfunction has constant sign on connected domains.
- It is always possible to choose a positive first eigenfuction.
- The ball is the unique set achieving the minimum first eigenvalue among the domains with given measure.
- On the union of two disjoint balls of equal radii the first and the second eigenvalues coincide and this domain is the unique open bounded set achieving the minimum second eigenvalue among the domains with given measure.

The nonlocal problem

$\Omega \subset \mathbb{R}^n$ open bounded, $\alpha \in \mathbb{R}$

$$\begin{cases} -\Delta u + \alpha \displaystyle\int_\Omega u \,\mathrm{d}x = \lambda u & \text{in } \Omega \\ \\ u = 0 \qquad \text{on } \partial\Omega. \end{cases}$$

- Is the ball the set achieving the minimum first eigenvalue among the domains with given measure?

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- Is the sign of the first eigenfunction constant on connected domains?

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- Is the ball the set achieving the minimum first eigenvalue among the domains with given measure?
- Is the sign of the first eigenfunction constant on connected domains?
- For positive α, on the union of two disjoint balls of equal radii, is the first eigenvalue simple?

The variational formulation

$\Omega \subset \mathbb{R}^n$ open bounded, $\alpha \in \mathbb{R}$

$$H(u, \Omega, \alpha) = \frac{\displaystyle\int_\Omega |Du|^2 \, dx + \alpha \left(\int_\Omega u \, dx \right)^2}{\displaystyle\int_\Omega u^2 \, dx},$$

then the first eigenvalue is given by the associated minimization problem,

$$\lambda_1(\alpha, \Omega) = \min_{u \in H_0^1(\Omega)} H(u, \Omega, \alpha).$$

Some basic properties of the nonlocal problem

The first eigenvalue $\lambda_1(\alpha, \Omega)$ is non decreasing in α

$$\lambda_1(\alpha,\Omega) = \frac{\displaystyle\int_\Omega |Du|^2 \,\mathrm{d}x + \alpha \left(\int_\Omega u \,\mathrm{d}x \right)^2}{\displaystyle\int_\Omega u^2 \,\mathrm{d}x} \geq \frac{\displaystyle\int_\Omega |Du|^2 \,\mathrm{d}x + (\alpha - \varepsilon) \left(\int_\Omega u \,\mathrm{d}x \right)^2}{\displaystyle\int_\Omega u^2 \,\mathrm{d}x}$$

$$\geq \lambda_1(\alpha - \varepsilon, \Omega)$$

Formally $\dfrac{d\lambda_1}{d\alpha} = \dfrac{\left(\int_\Omega u \,\mathrm{d}x \right)^2}{\int_\Omega u^2 \,\mathrm{d}x} > 0$ unless $\int_\Omega u \,\mathrm{d}x = 0$

Limiting Cases

Laplacian $0 \longleftarrow \quad \alpha \quad \longrightarrow \infty$ Twisted Laplacian

Some basic properties of the nonlocal problem

The following properties are equivalent:

1. $\bar{\lambda}$ is eigenvalue for all $\alpha \in \mathbb{R}$
2. There exist two values $\bar{\alpha} \neq \bar{\bar{\alpha}}$ such that $\bar{\lambda}$ is an eigenvalue
3. There exists an eigenfunction $\bar{u}$, corresponding to the eigenvalue $\bar{\lambda}$, such that $\int_\Omega \bar{u}\,\mathrm{d}x = 0$

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$$-\Delta u_1 + \bar{\alpha} \int_\Omega u_1\,\mathrm{d}x = \bar{\lambda} u_1$$

$$-\Delta u_2 + \bar{\bar{\alpha}} \int_\Omega u_2\,\mathrm{d}x = \bar{\lambda} u_2.$$

Multiplying the first equation by u_2 and the second by u_1, integrating over Ω and subtracting yields

$$(\bar{\alpha} - \bar{\bar{\alpha}}) \int_\Omega u_1\,\mathrm{d}x \int_\Omega u_2\,\mathrm{d}x = 0.$$

We conclude that either u_1 or u_2 must have zero average.

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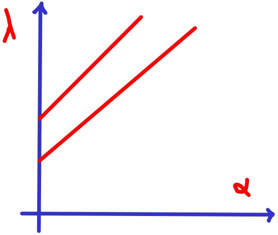
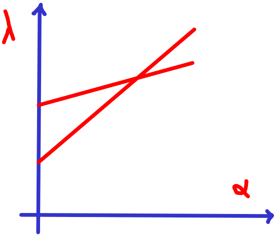
Impossible





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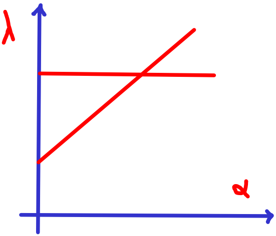
Possible



The Main result

Theorem (Saturation)

For every $n \geq 2$ there exists a positive value

$$\alpha_c = \frac{2^{3/n}\omega_n^{2/n} j_{n/2-1,1}^3 J_{n/2-1}(2^{1/n} j_{n/2-1,1})}{2^{1/n} j_{n/2-1,1} J_{n/2-1}(2^{1/n} j_{n/2-1,1}) - n J_{n/2}(2^{1/n} j_{n/2-1,1})}$$

such that, for every bounded, open set Ω in $\mathbb{R}^n$ and for every real number α, it holds

$$\lambda(\alpha,\Omega) \geq \begin{cases} \lambda(\alpha,\Omega^\sharp) & \text{if} \quad \alpha|\Omega|^{1+2/n} \leq \alpha_c \\ \\ \dfrac{2^{2/n}\omega_n^{2/n} j_{n/2-1,1}^2}{|\Omega|^{2/n}} & \text{if} \quad \alpha|\Omega|^{1+2/n} \geq \alpha_c . \end{cases}$$

If equality sign holds when $\alpha|\Omega|^{1+2/n} < \alpha_c$ then Ω is a ball, while if equality sign holds when $\alpha|\Omega|^{1+2/n} > \alpha_c$ then Ω is the union of two disjoint balls of equal measure.

Sketch of the proof

For $\alpha \leq 0$ using Pólya-Szegö

$$\lambda(\alpha,\Omega) = \frac{\int_\Omega |Du|^2\,\mathrm{d}x + \alpha\left(\int_\Omega u\,\mathrm{d}x\right)^2}{\int_\Omega u^2\,\mathrm{d}x}$$

$$\geq \frac{\int_{\Omega^\sharp} |Du^\sharp|^2\,\mathrm{d}x + \alpha\left(\int_{\Omega^\sharp} u^\sharp\,\mathrm{d}x\right)^2}{\int_{\Omega^\sharp} {u^\sharp}^2\,\mathrm{d}x} \geq \lambda(\alpha,\Omega^\sharp)$$

Sketch of the proof

Let $\alpha > 0$ and u be an eigenfunction.
Set $\Omega_+ = \{x \in \Omega : u(x) > 0\}$, $\Omega_- = \{x \in \Omega : u(x) < 0\}$,
$u^+(x) = \max\{u(x), 0\}$, $u^-(x) = \min\{u(x), 0\}$, using Pólya-Szegö

$$\lambda(\alpha, \Omega) = \frac{\int_\Omega |Du|^2 \, \mathrm{d}x + \alpha \left(\int_\Omega u \, \mathrm{d}x \right)^2}{\int_\Omega u^2 \, \mathrm{d}x}$$

$$\geq \frac{\int_{\Omega_+^\sharp} |D(u^+)^\sharp|^2 \, \mathrm{d}x + \int_{\Omega_-^\sharp} |D(u^-)^\sharp|^2 \, \mathrm{d}x + \alpha \left(\int_{\Omega_+^\sharp} (u^+)^\sharp \, \mathrm{d}x - \int_{\Omega_-^\sharp} (u^-)^\sharp \, \mathrm{d}x \right)^2}{\int_{\Omega_+^\sharp} {(u^+)^\sharp}^2 \, \mathrm{d}x + \int_{\Omega_-^\sharp} {(u^-)^\sharp}^2 \, \mathrm{d}x}$$

$$\geq \inf_{A \in \mathcal{B}(|\Omega|)} \lambda(\alpha, A),$$

where $\Omega_+^\sharp$ and $\Omega_-^\sharp$ are two balls with the same measure as Ω_+, Ω_- respectively, and $\mathcal{B}(t)$ is the set of all the possible domains union of two disjoint balls and having measure t.

Sketch of the proof

$$\frac{1}{\alpha_\lambda} = \frac{\omega_n\,(R_1^n + R_2^n)}{\lambda} - \frac{n\omega_n}{\lambda^{3/2}}\left[R_1^{n-1}\frac{J_{n/2}(\sqrt{\lambda}R_1)}{J_{n/2-1}(\sqrt{\lambda}R_1)} + R_2^{n-1}\frac{J_{n/2}(\sqrt{\lambda}R_2)}{J_{n/2-1}(\sqrt{\lambda}R_2)}\right]$$