Sharp estimates for a nonlocal eigenvalue problem

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*joint work with B. Brandolini, P. Freitas, and C. Trombetti
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Let $\alpha \in \mathbb{R}$ and $\Omega \subset \mathbb{R}^n$ an open bounded set we consider the eigenvalue problem

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\begin{aligned}
-\Delta u + \alpha \int_\Omega u \, dx &= \lambda u \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
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A non local eigenvalue problem

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Sperb (1981)
Henrot & Freitas (2004)
Caffarelli & Lin (2009)
Pinsky (2009)
A non local eigenvalue problem

Let $\alpha \in \mathbb{R}$ and $\Omega \subset \mathbb{R}^n$ an open bounded set we consider the eigenvalue problem

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\begin{cases}
-\Delta u + \alpha \int_{\Omega} u \, dx = \lambda u & \text{in } \Omega \\
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\end{cases}
$$

We shall consider the first eigenvalue $\lambda_1(\alpha, \Omega)$ and more precisely the related shape optimization problem:

**Problem**

*Find the optimal domain which minimize $\lambda_1$ among the domains with given measure.*
Some known facts when $\alpha = 0$

$\Omega \subset \mathbb{R}^n$

$$
\begin{cases}
-\Delta u = \lambda u & \text{in } \Omega \\
 u = 0 & \text{on } \partial \Omega.
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Some known facts when $\alpha = 0$

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The first eigenfunction has constant sign on connected domains.
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- The first eigenfunction has constant sign on connected domains.
- It is always possible to choose a positive first eigenfunction.
Some known facts when $\alpha = 0$

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- The first eigenfunction has constant sign on connected domains.
- It is always possible to choose a positive first eigenfunction.
- The ball is the unique set achieving the minimum first eigenvalue among the domains with given measure.

**Faber-Krahn inequality**

\[
\lambda_1(\Omega) = \frac{\int_{\Omega} |Du|^2 \, dx}{\int_{\Omega} u^2 \, dx} \geq \frac{\int_{\Omega^\#} |Du^\#|^2 \, dx}{\int_{\Omega^\#} u^\#^2 \, dx} \geq \lambda_1(\Omega^\#)
\]
Some known facts when $\alpha = 0$

$\Omega \subset \mathbb{R}^n$

\[
\begin{cases}
-\Delta u = \lambda u & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]

- The first eigenfunction has constant sign on connected domains.
- It is always possible to choose a positive first eigenfunction.
- The ball is the unique set achieving the minimum first eigenvalue among the domains with given measure.
- On the union of two disjoint balls of equal radii the first and the second eigenvalues coincide and this domain is the unique open bounded set achieving the minimum second eigenvalue among the domains with given measure.
The nonlocal problem

$\Omega \subset \mathbb{R}^n$ open bounded, $\alpha \in \mathbb{R}$

\[
\begin{aligned}
-\Delta u + \alpha \int_{\Omega} u \, dx &= \lambda u \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]

- Is the ball the set achieving the minimum first eigenvalue among the domains with given measure?
The nonlocal problem

\[ \Omega \subset \mathbb{R}^n \text{ open bounded, } \alpha \in \mathbb{R} \]

\[
\begin{cases}
-\Delta u + \alpha \int_{\Omega} u \, dx = \lambda u & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega.
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- Is the ball the set achieving the minimum first eigenvalue among the domains with given measure?
- Is the sign of the first eigenfunction constant on connected domains?
The nonlocal problem

\[ \Omega \subset \mathbb{R}^n \text{ open bounded, } \alpha \in \mathbb{R} \]

\[ \begin{cases} 
-\Delta u + \alpha \int_{\Omega} u \, dx = \lambda u & \text{in } \Omega \\
 u = 0 & \text{on } \partial \Omega.
\end{cases} \]

- Is the ball the set achieving the minimum first eigenvalue among the domains with given measure?
- Is the sign of the first eigenfunction constant on connected domains?
- For positive \( \alpha \), on the union of two disjoint balls of equal radii, is the first eigenvalue simple?
The variational formulation

\[ \Omega \subset \mathbb{R}^n \text{ open bounded, } \alpha \in \mathbb{R} \]

\[ H(u, \Omega, \alpha) = \frac{\int_{\Omega} |Du|^2 \, dx + \alpha \left( \int_{\Omega} u \, dx \right)^2}{\int_{\Omega} u^2 \, dx}, \]

then the first eigenvalue is given by the associated minimization problem,

\[ \lambda_1(\alpha, \Omega) = \min_{u \in H_0^1(\Omega)} H(u, \Omega, \alpha). \]
Some basic properties of the nonlocal problem

The first eigenvalue $\lambda_1(\alpha, \Omega)$ is non decreasing in $\alpha$

$$\lambda_1(\alpha, \Omega) = \frac{\int_{\Omega} |Du|^2 \, dx + \alpha \left( \int_{\Omega} u \, dx \right)^2}{\int_{\Omega} u^2 \, dx} \geq \frac{\int_{\Omega} |Du|^2 \, dx + (\alpha - \varepsilon) \left( \int_{\Omega} u \, dx \right)^2}{\int_{\Omega} u^2 \, dx} \geq \lambda_1(\alpha - \varepsilon, \Omega)$$

Formally $\frac{d\lambda_1}{d\alpha} = \frac{\left( \int_{\Omega} u \, dx \right)^2}{\int_{\Omega} u^2 \, dx} > 0$ unless $\int_{\Omega} u \, dx = 0$

Limiting Cases

Laplacian $0 \leftarrow \alpha \rightarrow \infty$ Twisted Laplacian
Some basic properties of the nonlocal problem

The following properties are equivalent:

1. $\bar{\lambda}$ is eigenvalue for all $\alpha \in \mathbb{R}$
2. There exist two values $\bar{\alpha} \neq \bar{\alpha}$ such that $\bar{\lambda}$ is an eigenvalue
3. There exists an eigenfunction $\bar{u}$, corresponding to the eigenvalue $\bar{\lambda}$, such that $\int_{\Omega} \bar{u} \, dx = 0$
Some basic properties of the nonlocal problem

The following properties are equivalent:

1. \( \bar{\lambda} \) is eigenvalue for all \( \alpha \in \mathbb{R} \)
2. There exist two values \( \bar{\alpha} \neq \bar{\alpha} \) such that \( \bar{\lambda} \) is an eigenvalue
3. There exists an eigenfunction \( \bar{u} \), corresponding to the eigenvalue \( \bar{\lambda} \), such that
   \[ \int_{\Omega} \bar{u} \, dx = 0 \]

\[-\Delta u_1 + \bar{\alpha} \int_{\Omega} u_1 \, dx = \bar{\lambda} u_1 \]
\[-\Delta u_2 + \bar{\alpha} \int_{\Omega} u_2 \, dx = \bar{\lambda} u_2.\]

Multiplying the first equation by \( u_2 \) and the second by \( u_1 \), integrating over \( \Omega \) and subtracting yields

\[(\bar{\alpha} - \bar{\alpha}) \int_{\Omega} u_1 \, dx \int_{\Omega} u_2 \, dx = 0.\]

We conclude that either \( u_1 \) or \( u_2 \) must have zero average.
Some basic properties of the nonlocal problem

The following properties are equivalent:

1. \( \bar{\lambda} \) is eigenvalue for all \( \alpha \in \mathbb{R} \)
2. There exist two values \( \bar{\alpha} \neq \bar{\alpha} \) such that \( \bar{\lambda} \) is an eigenvalue
3. There exists an eigenfunction \( \bar{u} \), corresponding to the eigenvalue \( \bar{\lambda} \), such that \( \int_{\Omega} \bar{u} \, dx = 0 \)

Impossible
Some basic properties of the nonlocal problem

The following properties are equivalent:

1. \( \lambda \) is eigenvalue for all \( \alpha \in \mathbb{R} \)
2. There exist two values \( \bar{\alpha} \neq \bar{\alpha} \) such that \( \lambda \) is an eigenvalue
3. There exists an eigenfunction \( \tilde{u} \), corresponding to the eigenvalue \( \lambda \), such that \( \int_{\Omega} \tilde{u} \, dx = 0 \)

Possible
The Main result

**Theorem (Saturation)**

For every $n \geq 2$ there exists a positive value

$$\alpha_c = \frac{2^{3/n} \omega_n^{2/n} j_{n/2-1,1} J_{n/2-1} (2^{1/n} j_{n/2-1,1})}{2^{1/n} j_{n/2-1,1} J_{n/2-1} (2^{1/n} j_{n/2-1,1}) - nJ_{n/2} (2^{1/n} j_{n/2-1,1})}$$

such that, for every bounded, open set $\Omega$ in $\mathbb{R}^n$ and for every real number $\alpha$, it holds

$$\lambda(\alpha, \Omega) \geq \begin{cases} 
\lambda(\alpha, \Omega^\#) & \text{if } \alpha|\Omega|^{1+2/n} \leq \alpha_c \\
2^{2/n} \omega_n^{2/n} j_{n/2-1,1}^2 / |\Omega|^{2/n} & \text{if } \alpha|\Omega|^{1+2/n} \geq \alpha_c.
\end{cases}$$

If equality sign holds when $\alpha|\Omega|^{1+2/n} < \alpha_c$ then $\Omega$ is a ball, while if equality sign holds when $\alpha|\Omega|^{1+2/n} > \alpha_c$ then $\Omega$ is the union of two disjoint balls of equal measure.
For $\alpha \leq 0$ using Pólya-Szegö

$$\lambda(\alpha, \Omega) = \frac{\int_{\Omega} |Du|^2 \, dx + \alpha (\int_{\Omega} u \, dx)^2}{\int_{\Omega} u^2 \, dx}$$

$$\geq \frac{\int_{\Omega^\#} |Du^\#|^2 \, dx + \alpha (\int_{\Omega^\#} u^\# \, dx)^2}{\int_{\Omega^\#} u^\#^2 \, dx} \geq \lambda(\alpha, \Omega^\#)$$
Sketch of the proof

Let $\alpha > 0$ and $u$ be an eigenfunction. Set $\Omega_+ = \{ x \in \Omega : u(x) > 0 \}$, $\Omega_- = \{ x \in \Omega : u(x) < 0 \}$, $u^+(x) = \max\{ u(x), 0 \}$, $u^-(x) = \min\{ u(x), 0 \}$, using Pólya-Szegö

\[
\lambda(\alpha, \Omega) = \frac{\int_{\Omega} |Du|^2 \, dx + \alpha \left( \int_{\Omega} u \, dx \right)^2}{\int_{\Omega} u^2 \, dx}
\]

\[
\geq \frac{\int_{\Omega^+_\#} |D(u^+)\#|^2 \, dx + \int_{\Omega^-_\#} |D(u^-)\#|^2 \, dx + \alpha \left( \int_{\Omega^+_\#} (u^+)\# \, dx - \int_{\Omega^-_\#} (u^-)\# \, dx \right)^2}{\int_{\Omega^+_\#} (u^+)\#^2 \, dx + \int_{\Omega^-_\#} (u^-)\#^2 \, dx}
\]

\[
\geq \inf_{A \in B(|\Omega|)} \lambda(\alpha, A),
\]

where $\Omega^+_\#$ and $\Omega^-_\#$ are two balls with the same measure as $\Omega_+$, $\Omega_-$ respectively, and $B(t)$ is the set of all the possible domains union of two disjoint balls and having measure $t$. 
Sketch of the proof

\[
\frac{1}{\alpha_{\lambda}} = \frac{\omega_n (R_1^n + R_2^n)}{\lambda} - \frac{n\omega_n}{\lambda^{3/2}} \left[ R_1^{n-1} \frac{J_{n/2}(\sqrt{\lambda}R_1)}{J_{n/2-1}(\sqrt{\lambda}R_1)} + R_2^{n-1} \frac{J_{n/2}(\sqrt{\lambda}R_2)}{J_{n/2-1}(\sqrt{\lambda}R_2)} \right]
\]