Blow-up solutions for critical Trudinger-Moser equations in $\mathbb{R}^2$

Bernhard Ruf
Università degli Studi di Milano

The classical Sobolev embeddings

We have the following well-known Sobolev inequalities:
let $\Omega \subset \mathbb{R}^N$ bounded, and

$$H_0^1(\Omega) : \text{ Sobolev space}$$

Set

$$2^* = \frac{2N}{N-2} \qquad \text{(critical exponent)}$$

Then, for $N > 2$:

$$S_q := \sup_{\|\nabla u\|_2 \leq 1} \int_\Omega |u|^q dx \begin{cases} < +\infty \ , \ \ q < 2^* & \text{compact, attained} \\ < +\infty \ , \ \ q = 2^* & \text{non compact, not attained} \\ = +\infty \ , \ \ q > 2^* & \end{cases}$$

The borderline case $\boxed{N = 2}$

The Trudinger-Moser inequality:

S. Pohozaev (1965), N. Trudinger (1967):

$$u \in H_0^1(\Omega) \quad \Rightarrow \quad \int_\Omega e^{u^2} dx < +\infty$$

Sharpened by J. Moser (1971):

$$T_\alpha := \sup_{\|\nabla u\|_2 \le 1} \int_\Omega e^{\alpha u^2} dx \begin{cases} < +\infty \ , \ \alpha < 4\pi & \text{compact, attained} \\ < +\infty \ , \ \alpha = 4\pi & \text{non compact , attained?} \\ = +\infty \ , \ \alpha > 4\pi & \end{cases}$$

The result of L. Carleson – S.-Y. Alice Chang

For $\Omega = B_R(0)$: $T_{4\pi}$ is **attained**

Surprising, if compared to Sobolev case!

Proof: Assume not attained:

- then maximizing sequence $\{u_n\}$ *concentrates*
- determine $\lim_{n\to\infty} \int_\Omega e^{4\pi u_n^2} = (1+e)|\Omega|$: non-compactness level
- show (by explicit example) that $\sup_{\|\nabla u\|_2 \le 1} \int_\Omega e^{4\pi u^2} > (1+e)|\Omega|$:

contradiction!

Generalizations: Struwe (1988): Ω near $B_R(0)$

Flucher (1992): general bounded $\Omega \subset \mathbb{R}^2$

Associated differential equation

An extremal u for T_α is a **critical point** of the functional

$$\int_\Omega e^{u^2} \quad \text{on} \quad \mathcal{S}_\alpha = \{u \in H_0^1(\Omega) \ ; \ \|\nabla u\|_2^2 = \alpha\}$$

namely

$$D_{\mathcal{S}_\alpha} \left[\int_\Omega e^{u^2} \right] = 0$$

that is

$$\begin{cases} -\Delta u = \lambda \, u \, e^{u^2} & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega \end{cases} , \quad \lambda = \frac{\alpha}{\int_\Omega u^2 \, e^{u^2}}$$

Two related (but not equivalent) problems:

$$(T_\alpha) \qquad -\Delta u = \frac{\alpha u\, e^{u^2}}{\int_\Omega u^2\, e^{u^2}} \ \text{ in } \ \Omega, \ u = 0 \ \text{ on } \ \partial\Omega$$

and

$$(P_\lambda) \qquad -\Delta u = \lambda\, u\, e^{u^2} \ \text{ in } \ \Omega\,, \ u = 0 \ \text{ on } \ \partial\Omega$$

Solutions of (P_λ) correspond to critical points of the free energy

$$J_\lambda(u) = \int_\Omega |\nabla u|^2 - \lambda \int_\Omega e^{u^2} dx$$

Some existence results

- (T_α) has (positive) solution, by Carleson-Chang, Flucher

- Existence of solutions for:

$$(P_\lambda) \qquad -\Delta u = \lambda u e^{u^2} \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

Adimurthi (1990) : a positive solution u_λ exists, for $0 < \lambda < \lambda_1$.

Result valid for more general nonlinearities: suppose that

$$f(t) = h(t)\, e^{t^2} \,,\; h \in \mathcal{C}(\mathbb{R}) \,,\; \frac{h(t)}{e^{t^2}} \to 0$$

and set $F(t) := \int_0^\infty f(s)ds$. Assume

H1) $0 < F(s) \leq \frac{1}{2} f(s)s \,,\ \forall\, s \in \mathbb{R} \setminus \{0\}$

H2) $\limsup_{|s|\to 0} \frac{2F(s)}{s^2} < \lambda_1$

H3) $\liminf_{|s|\to\infty} h(s)s > \frac{2}{d^2}$, d: radius of largest ball $B_d \subset \Omega$

Theorem: (de Figueiredo-Miyagaki-R., 1995).

Assume H1) - H3). Then the critical growth equation

$$-\Delta u = f(u) \ , \quad in \ \ \Omega \ , \ \ u = 0 \ \ on \ \ \partial\Omega$$

has a positive solution

Proof: (similar to Brezis-Nirenberg result for $-\Delta u = \lambda u + u^{2^*-1}$)

▷ use mountain-pass theorem by Ambrosetti-Rabinowitz

▷ determine *non-compactness level*

▷ use H3) and "Moser sequence" to show that mountain-pass level stays below non-compactness level

This existence result is "almost" sharp:

• de Figueiredo-R. (1995):

Let $\Omega = B_1(0)$. There exists $\delta > 0$ such that if

$$\limsup_{|s|\to\infty} h(s)s < \delta$$

then (P_λ) has no positive solution.

On the other hand: "topology helps to get solutions"
(cf. Coron, 1984; Bahri-Coron, 1988):

• Struwe (2000):

For large class of critical nonlinearities (which includes above)

there exists positive solution on suitable non-contractible domains

Loss of compactness:

Quantization for (PS)-sequences

Recall Struwe's result for Brezis-Nirenberg functional:

$$I_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda}{2}\int_\Omega |u|^2 - \frac{1}{2^*}\int_\Omega |u|^{2^*}\ ,\ \Omega \subset \mathbb{R}^N, N \geq 3$$

For a sequence $\lambda_n \to \lambda_0$, and a (PS)-sequence $\{u_n\}$,
i.e. $I'_{\lambda_n}(u_n) \to 0$ and $I_{\lambda_n}(u_n) \to c$, one has

$$I_{\lambda_n}(u_n) = I_{\lambda_0}(u_0) + k\, S_N + o(1)\ ,\quad \text{for some} \quad k \geq 1$$

where u_0 a critical point of I_{λ_0}, and S_N is a positive constant

Bubbling in the Trudinger-Moser case (P_λ)

Druet (2006): Suppose that $\{u_\lambda\}$ is sequence of solutions of (P_λ) with uniformly bounded energy, i.e.

$$(P_\lambda) \qquad -\Delta u_\lambda = \lambda\, u_\lambda e^{u_\lambda^2} \text{ in } \Omega, \quad u_\lambda = 0 \text{ on } \partial\Omega$$

with

$$\Big| J_\lambda(u_\lambda) \Big| = \Big| \int_\Omega |\nabla u_\lambda|^2 - \lambda \int_\Omega e^{u_\lambda^2} \Big| \le c\ ,$$

then for some integer $k \ge 0$

$$(D_k) \qquad J_\lambda(u_\lambda) \to k\, 4\pi\ , \quad \text{as } \lambda \to 0\ ,$$

that is: bubbling in k points.

Previous results on single point bubbling: $k = 1$

Adimurthi-Struwe (2000)

Adimurthi-Druet (2004): in addition:

The solution u_λ has unique isolated maximum which blows up around x_0 (as $\lambda \to 0$), where x_0 is critical point of Robin's function.

Let $G(x, y)$ be Green's function of the problem

$$-\Delta_x G(x, y) = 8\pi\delta_y(x) \ , \ \ x \in \Omega \ ; \ \ G(x, y) = 0 \ , \ \ x \in \partial\Omega$$

and $H(x, y)$ its regular part:

$$H(x, y) = 4 \log \frac{1}{|x - y|} - G(x, y)$$

$H(\xi, \xi)$: Robin's function of Ω

Existence of bubbling solutions

Recall results for: **Liouville equation in $2d$:**

$$-\Delta u = \lambda e^u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega$$

Concentration as $\lambda \to 0$ or $\lambda = \frac{\mu}{\int_\Omega e^u}$

(also on compact manifolds and with singular weights)

Brezis, Merle, Nagasaki, Suzuki, Li, Shafrir, Weston, Ma, Wei, Tarantello, Struwe, Nolasco, Baraket, Pacard, Del Pino, Kowalczyk, Musso, Esposito, Grossi, Pistoia, Bartolucci, Chen, Lin, Chang, Gursky, Yang, Malchiodi, Robert...

Del Pino-Kowalczyk-Musso (2005): If Ω is not simply connected, then solutions with k bubbling points exist for each given $k \geq 1$. Blowing-up takes place at critical points of

$$\psi_k(\xi) \; := \; \sum_{i=1}^{k} H(\xi_j, \xi_j) - \sum_{i \neq j} G(\xi_i, \xi_j).$$

Existence of solutions to (P_λ) with Property (D_k) ?

Bubbling solutions:

For k distinct points $\xi_1, \ldots, \xi_k \in \Omega$ and k positive numbers $m_1, \ldots, m_k \in \mathbb{R}$ consider the function

$$\boxed{\varphi_k(\xi, m) = \sum_{j=1}^{k} 2m_j^2(b + \log m_j^2) + m_j^2 H(\xi_j, \xi_j) - \sum_{i \neq j} m_i m_j \, G(\xi_i, \xi_j)}$$

where $b = 2 \log 8 - 2$

Stable critical point situation (SCS)

We say that φ_k has a *stable critical point situation*, if

for some open set Λ with

$$\overline{\Lambda} \subset \left\{(\xi, m) \in \Omega^k \times \mathbb{R}^k_+ \mid \xi_i \neq \xi_j \text{ if } i \neq j\right\}$$

there exists $\delta > 0$ such that for any $g \in C^1(\overline{\Lambda})$ with $\|g\|_{C^1(\overline{\Lambda})} < \delta$

$$\varphi_k + g$$

has a critical point in Λ.

Important facts:

• $\varphi_1(\xi, m)$ satisfies (SCS), with Λ a neighborhood of its minimum set.

$$\varphi_1(\xi, m) = 2m^2(b + \log m^2) + m^2 H(\xi, \xi)$$

• $\varphi_2(\xi, m)$ satisfies (SCS) whenever Ω is not simply connected.

$$\varphi_2(\xi, m) = \sum_{j=1}^{2} 2m_j^2(b + \log m_j^2) + m_j^2 H(\xi_j, \xi_j) - 2m_1 m_2\, G(\xi_1, \xi_2).$$

We believe that (SCS) holds for any $k \geq 2$ (if Ω is not simply connected).

Theorem 1. (M. del Pino - M. Musso - R., JFA, 2009)
Assume that $\varphi_k(\xi, m)$ has (SCS). Then there exist solutions u_λ to (P_λ) which

- *blow up around k points ξ_j as $\lambda \to 0$, where $\nabla\varphi_k(\xi, m) \to 0$*
- *away from the points ξ_j the solutions u_λ take the form*

$$(1) \qquad u_\lambda(x) = \sqrt{\lambda} \sum_{j=1}^{k} m_j \left[G(x, \xi_j) + o(1) \right]$$

Furthermore

$$J_\lambda(u_\lambda) = 4\pi k + \lambda \left[-|\Omega| + 8\pi \varphi_k(\xi, m) + o(1) \right].$$

This result applies for

- $k = 1$: there exists bubbling solution near minimizer of $H(\xi, \xi)$
- $k = 2$: there exists solution with two bubbles, provided that Ω is not simply connected.

Argument of the proof:

• construction of *approximate solution*, based on "standard bubble"

• linearize equation in this approximate solution

• finite dimensional variational reduction via Lyapunov-Schmidt

• yields finite-dimensional functional f_k which is C^1-close to $\varphi_k(\xi, m)$

• critical points of f_k yield the solution, and the information on the location of the bubbles

This method has been successfully applied by numerous authors to problems in the critical growth range:

Bahri, Coron, Yanyan Li, Rey, Ni, Wei, Takagi, del Pino, Felmer, Musso, Kowalczyk, Pistoia, Grossi, Esposito, Ge, Jing, Baraket, Pacard, ...

Standard bubbles

Let $\xi_j \in \Omega, j = 1, \ldots, k$, with $\xi_j \neq \xi_k$ for $j \neq k$. Fix an index j and define γ_j and ε_j by the relations

$$\gamma_j^2 := \frac{1}{4m_j^2\lambda}, \quad \varepsilon_j^2 := 2m_j^2 e^{-\frac{1}{4m_j^2\lambda}}$$

so that $\gamma_j \to \infty$ and $\varepsilon_j \to 0$, as $\lambda \to 0$. Let us write

$$u(x) \;=\; \gamma_j + \frac{1}{2\gamma_j} v\left(\frac{x-\xi_j}{\varepsilon_j}\right).$$

Then

$$-\Delta u = \lambda u e^{u^2} \quad \Longleftrightarrow \quad -\Delta v = \left(1 + \frac{v}{2\gamma_j^2}\right) e^{\frac{v^2}{4\gamma_j^2}} e^{v}$$

Limiting equation

$$-\Delta v = (1 + \frac{v}{2\gamma_j^2}) e^{\frac{v^2}{4\gamma_j^2}} e^v$$

For $\gamma_j \to +\infty$ we get the *limiting Liouville equation*

$$(2) \qquad -\Delta v = e^v \ , \quad \text{in } \mathbb{R}^2$$

The *radial solutions* of (2) are given by :

$$\omega_\mu(|y|) = \log \frac{8\mu^2}{(\mu^2 + |y|^2)^2}, \quad \mu > 0 \ : \quad \textit{standard bubble}$$

Approximate solution

Let $\xi_1, \ldots, \xi_k \in \Omega$, $m_1, \ldots, m_k > 0$ be given.

We build an *approximate solution* $U_\lambda(\xi, m)$.

First approximation:

Want that *first approximation* U_λ is such that for some $\mu_j > 0$, and for x close to ξ_j

$$U_\lambda(x) \;\approx\; \Big[\, \gamma_j + \frac{1}{2\gamma_j}\, \omega_{\mu_j}\Big(\frac{|x-\xi_j|}{\varepsilon_j}\Big) \Big] =: W_j(x)$$

Add to W_j a harmonic function such that boundary condition zero is satisfied:

$$U_j(x) = W_j(x) - \sqrt{\lambda}\, m_j\, [\log 2m_j^2 + \log 8\mu_j^2 \, + H_j(x)] \;,$$

that is

$$\Delta H_j = 0 \quad \text{in } \Omega, \qquad H_j(x) = \log \frac{1}{(\varepsilon_j^2 \mu_j^2 + |x-\xi_j|^2)^2} \quad \text{on } \partial\Omega \;;$$

Define $U_\lambda(\xi, m) := \sum_{j=1}^{k} U_j$

Choose the μ_j's such that

$$U_\lambda(x) \approx W_j(x) \quad \text{for } x \ \text{ close to } \ \xi_j \ , \quad \text{for all} \ \ j$$

U_λ are approximate solutions, with error

$$R_\lambda(x) = \Delta U_\lambda + \lambda U_\lambda e^{U_\lambda^2}$$

Introduce suitable L^∞-weighted norm so that

$$\|R_\lambda\|_* \leq C\lambda$$

Goal: *find solution u of the form*

$$u = U_\lambda + \phi$$

such that ϕ is small for suitable points ξ_i and m_i

Linearization in U_λ: denote $f(s) = se^{s^2}$

$$\begin{aligned} \Delta\phi + f'(U_\lambda)\phi &= -R - [f(U_\lambda + \phi) - f(U_\lambda) - f'(U_\lambda)\phi] \\ &=: -R - N(\phi) \end{aligned}$$

If $L_\lambda := \Delta + f'(U_\lambda)$ were boundedly invertible:

invert L_λ, use contraction principle: done!

Note:

- far from the points ξ_j : $f'(U_\lambda) = O(\lambda)$, hence

$$L_\lambda = \Delta + f'(U_\lambda) = \Delta + O(\lambda)$$

is "small perturbation" of Δ away from the concentration points

- near the points ξ_j : $f'(U_\lambda) \approx \frac{1}{\varepsilon_j^2}\ e^{\omega_j}$.

Hence L_λ is approximately superposition of the linear operators

$$L_j(\phi) = \Delta\phi + \frac{1}{\varepsilon_j^2}\ e^{\omega_j}\phi$$

Kernel of L

Thus: L_λ is nontrivial perturbation of Δ near the ξ_j, while essentially equal to Δ on most of the domain.

Note:

$$L_j(\phi) = \Delta\phi + \frac{1}{\varepsilon_j^2}\ e^{\omega_j}\phi = 0$$

has **bounded solutions**, due to natural invariances of

$$\Delta\omega + e^{\omega} = 0 \tag{3}$$

Let $\omega_\mu(y)$: explicit solution of (3), then

$$z_{0j}(x) = \partial_\mu \omega_{\mu_j}(y)\,, \quad z_{1j} = \partial_{x_1}\omega_{\mu_j}(x)\,, \quad z_{2j} = \partial_{x_2}\omega_{\mu_j}(x)\,,\ j = 1, \ldots, k$$

satisfy

$$\Delta Z + e^{\omega_{\mu_j}} Z = 0$$

Projection onto kernel of L

Hence:

$$Z_{ij}(x) := z_{ij}\left(\frac{x-\xi_j}{\varepsilon_j}\right)\ ,\ \ i = 0,1,2$$

are bounded solutions of $L_j(Z) = 0$ on $\mathbb{R}^2$; in fact, they are all bounded solutions (Bakaret-Pacard, 1998)

Consider "projected version" of equation $L_\lambda \phi = -R - N(\phi)$, i.e. projection onto the "almost kernel" given by above functions, multiplied by cut-off functions ζ_j

$$(4) \qquad \begin{cases} L_\lambda(\phi) = -R - N(\phi) + \sum_{i=0}^{2}\sum_{j=1}^{k} c_{ij} Z_{ij}\zeta_j\ , & \text{in } \ \Omega \\ \phi = 0\ , & \text{on } \ \partial\Omega \\ \int_\Omega Z_{ij}\zeta_j\,\phi = 0\ , & \text{for all } \ i,j \end{cases}$$

Then: (4) has unique solution $\phi(\xi, m)$, with $\|\phi\|_\infty \le C\lambda$.

Final step:

Find $(\tilde{\xi}, \widetilde{m})$ such that

$$(5) \qquad c_{ij}(\tilde{\xi}, \widetilde{m}) = 0 \ , \quad \text{for all} \quad i = 0, 1, 2; \ j = 1, \cdots, k$$

Put problem in variational form:

$$(6) \qquad (\xi, m) \longmapsto J_\lambda\Big(\sqrt{\lambda}\big(U_\lambda(\xi, m) + \phi(\xi, m)\big)\Big) =: T_\lambda(\xi, m)$$

Fact: if $(\tilde{\xi}, \widetilde{m})$ is critical point of (6), then (5) holds

Energy computation at first approximation:

$$T_\lambda(\xi, m) = 4\pi k - \lambda\,|\Omega| + \lambda\, 8\pi\, \varphi_k(\xi, m) + O(\lambda^2) \ ,$$

where φ_k is the functional in statement of Theorem 1.

By the (SCS) property: T_λ has a critical point. $\square$

Back to the Moser functional (i.e. the case (T_α))

$$J_\alpha(u) := \int_\Omega e^{u^2}\,dx \quad \text{on } \mathcal{S}_\alpha = \{u \in H_0^1(\Omega) \ : \ \|\nabla u\|_2^2 = \alpha\}$$

Recall:

$$\sup_{u \in \mathcal{S}_\alpha} J_\alpha(u) \quad \textbf{attained}\text{, if} \quad \alpha \le 4\pi$$

with associated equation

$$(T_\alpha) \qquad\qquad -\Delta u = \frac{\alpha\, u\, e^{u^2}}{\int_\Omega u^2\, e^{u^2}} \ \text{ in } \ \Omega, \ \ u = 0 \ \text{ on } \ \partial\Omega$$

Existence of critical points in the Supercritical Trudinger-Moser case

Numerical evidence (Monahan, 1971):

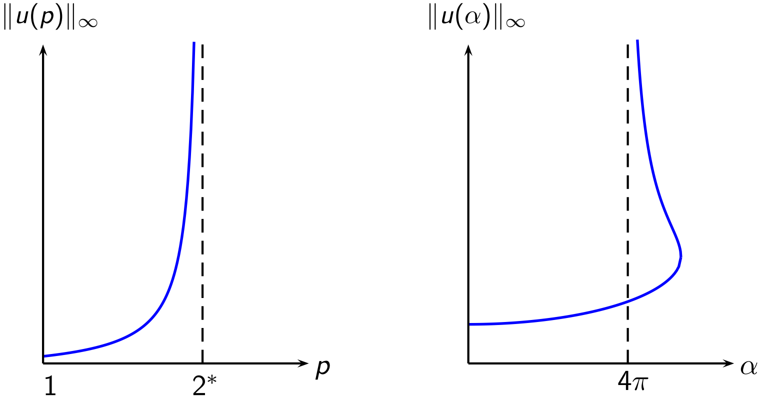
For $\Omega = B_1(0) \subset \mathbb{R}^2$, the Trudinger-Moser functional

$$J_\alpha(u) = \int_\Omega e^{u^2} dx \ , \ \|\nabla u\|_2^2 = \alpha \tag{7}$$

admits local maximum and second critical value in the

supercritical regime, i.e. for $\alpha > 4\pi$ (and near 4π)

Struwe (1988): the global maximum of J_α can be continued as a local maximum for values $4\pi < \alpha < \alpha_1$ and there exists a second solution u_α to (T_α), defined for α in a dense subset of $(4\pi, \alpha_1)$.



Approaching criticality in the Sobolev and the Moser case

Lamm, Robert, Struwe (2009)

A second solution u_α of (T_α) is defined on the entire interval $(4\pi, \alpha_1)$.

Proof:

Asymptotic analysis of a geometric heat flow associated to the Trudinger Moser functional and a min-max argument.

• Little qualitative information on the second solution

• The mentioned numerical evidence (in the case of a ball) suggests existence of a branch of solutions of (T_α) that blows up as $\alpha \downarrow 4\pi$.

Theorem 2. (M. del Pino, M. Musso, R.)

There is a positive solution u_α *for* (T_α) *for* $4\pi < \alpha < \alpha_1$ *such that* u_α *blows-up around a point* ξ_α*, and away from* ξ_α*:*

$$u_\alpha(x) \;=\; (\alpha - 4\pi)^{\frac{1}{4}}\, G(x, \xi_\alpha) + o(1)\,.$$

Besides

$$H(\xi_\alpha, \xi_\alpha) \to \min_{\xi} H(\xi, \xi).$$

Theorem 3. (M. del Pino, M. Musso, R.)

Assume that Ω *is* **not simply connected**. *Then there is a positive solution* u_α *to* (T_α) *for*

$$8\pi < \alpha < \alpha_2$$

which blows up around two points $\xi_1, \xi_2 \in \Omega$*, and away from them:*

$$u_\alpha(x) \;=\; a\,(\alpha - 8\pi)^{\frac{1}{4}}\,[\,m_1 G(x,\xi_1) + m_2 G(x,\xi_1)\,] + o(1).$$

where

$$\nabla\varphi_2(\xi, m) \to 0 \quad \text{as } \alpha \downarrow 8\pi,$$

with

$$\varphi_2(\xi, m) = \sum_{j=1}^{2} 2(b + m_j^2)\log m_j^2 + m_j^2 H(\xi_j, \xi_j) - 2m_1 m_2\, G(\xi_1, \xi_2)$$

$b = 2\log 8 - 2.$

We believe: If Ω is not simply connected and

$$k\,4\pi < \alpha < \alpha_k(\Omega) \ ,$$

then there is a positive solution u_α of

$$\Delta u + \alpha \, \frac{u\, e^{u^2}}{\int_\Omega u^2\, e^{u^2}} = 0 \quad \text{in } \Omega, \ \ u = 0 \quad \text{on } \partial\Omega$$

with k blow-up points ξ_j. Away from them:

$$u_\alpha(x) \ = \ a\,(\alpha - 4k\pi)^{\frac{1}{4}} \sum_{j=1}^{k} m_j\, G(x, \xi_j) \, + \, o(1)$$

where

$$\nabla \varphi_k(\xi, m) \to 0 \quad \text{as } \alpha \downarrow 8\pi$$

with

$$\varphi_k(\xi, m) = \sum_{j=1}^{k} 2m_j^2(b + \log m_j^2) + m_j^2 H(\xi_j, \xi_j) - \sum_{i \neq j} m_i m_j \, G(\xi_i, \xi_j).$$

This is true if $\Omega \subset \mathbb{R}^2$ is an annulus:

Theorem 4. (M. del Pino, M. Musso, R.)

If $\Omega = \{x \in \mathbb{R}^2 : 0 < a < |x| < b\}$, then for

$$4\pi\, k < \alpha < \alpha_k$$

there exists positive solution u_α to (T_α)

- invariant under rotations by angle $\frac{2\pi}{k}$

- blows up around k points (lying on vertices of regular polygon), as $\alpha \to 4\pi k$

Proofs of Theorems 2 - 4:

Idea similar to proof of Theorem 1, however different in technical details and estimates