Level surfaces, Liouville-type theorems and the hyperplane

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§1. Introduction

Let $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ satisfy

$$\phi \in C^2(\mathbb{R}), \quad \phi(0) = 0,$$

$$0 < {}^{\exists}\delta_1 \leq \phi'(s) \leq {}^{\exists}\delta_2 \quad (s \in \mathbb{R})$$

$$\Phi(s) := \int_1^s \frac{\phi'(\xi)}{\xi} \, d\xi \quad (s > 0)$$

When $\phi(s) \equiv s$, $\Phi(s) = \log s$.

Let $\Omega \subset \mathbb{R}^m$ $(m \geq 2)$ be a domain.

$\partial\Omega$ is not necessarily bounded.

Consider two problems:

$$(\mathrm{IBP})\begin{cases}\partial_t u=\Delta\phi(u) & \text{in } \Omega\times(0,\infty),\\ u=1 & \text{on } \partial\Omega\times(0,\infty),\\ u=0 & \text{on } \Omega\times\{0\}.\end{cases}$$

$$(\mathrm{CP})\begin{cases}\partial_t u=\Delta\phi(u) & \text{in } \mathbb{R}^m\times(0,\infty),\\ u=\chi_{\mathbb{R}^m\setminus\Omega} & \text{on } \mathbb{R}^m\times\{0\}.\end{cases}$$

where $\chi_{\mathbb{R}^m\setminus\Omega}(x)=\begin{cases}1 & \text{if } x\in\mathbb{R}^m\setminus\Omega,\\ 0 & \text{if } x\in\Omega.\end{cases}$

Let $u = u(x,t)$: the unique bounded solution of (IBP) or (CP).

$d(x) = \mathrm{dist}(x, \partial\Omega)$.

Theorem A (Magnanini-S 2010 AIHP, [MS1])

Under $\partial\Omega = \partial(\mathbb{R}^N \setminus \Omega)$,

$$-4t\,\Phi(u(x,t)) \longrightarrow d(x)^2 \quad \text{as } t \to 0^+$$

uniformly on every compact set in Ω.

This is a nonlinear version of Varadhan 1967.

§2 Sliding method.

Let $m = N+1$ with $N \geq 1$.

Let $f \in C(\mathbb{R}^N)$ and

$$\Omega = \{(x, x_{N+1}) \in \mathbb{R}^{N+1} \mid x_{N+1} > f(x)\}.$$

Strong comparison $\Rightarrow$

$$\boxed{\frac{\partial u}{\partial x_{N+1}} < 0 \quad \text{in } \Omega \times (0,\infty) \quad (\text{or in } \mathbb{R}^{N+1} \times (0,\infty)).}$$

Theorem 1. Suppose:

$(\widetilde{BN})$: $\exists \{y^1, \cdots, y^N\} \subset \mathbb{R}^N$: basis

s.t. $\forall j$, $f(x+y^j) - f(x)$ has either a maximum or a minimum in $\mathbb{R}^N$.

If u has a stationary level surface Γ in Ω,

($\exists a(t)$ s.t. $u(x, x_{N+1}, t) = a(t)$ for $(x, x_{N+1}, t) \in \Gamma \times (0, \infty)$)

then f is affine, that is, $\partial\Omega$ must be a hyperplane.

Remark:

- (BN) ([MS1])

$$\left(\forall y \in \mathbb{R}^N \ \exists h(y) \in \mathbb{R} \text{ s.t. } \lim_{|x|\to\infty} (f(x+y) - f(x)) = h(y). \right)$$

We see that

$$(BN) \Longrightarrow (\widetilde{BN}).$$

- Let $N=1$.

	(BN)	$(\widetilde{BN})$	uniformly continuous
$\sin x$	×	○	○
$\tan^{-1} x \cdot \sin x$	×	×	○

Proof of Theorem 1: We use the sliding method due to "Berestycki-Caffarelli-Nirenberg 1997."

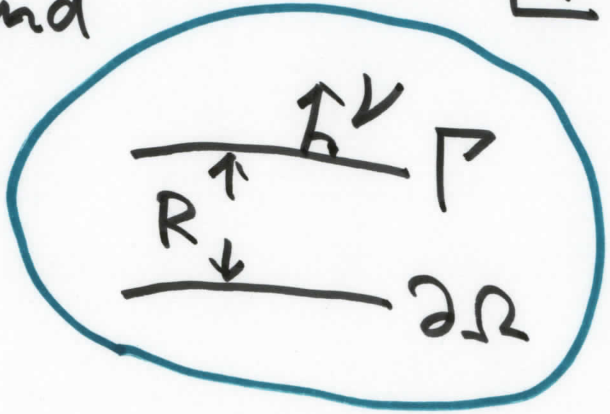
Theorem A and the implicit function theorem

$\Longrightarrow \ \exists g \in C^2(\mathbb{R}^N),\ \exists R > 0$:

$$\begin{cases} \Gamma = \{(x, g(x)) : x \in \mathbb{R}^N\} = \{(x, x_{N+1}) : d(x, x_{N+1}) = R\} \\ g(x) = \sup\limits_{|x-y| \leq R} \left\{ f(y) + \sqrt{R^2 - |x-y|^2} \right\}. \end{cases}$$

Conversely, since g is smooth and

$$(y, g(y)) - R\nu(y) \in \partial\Omega,$$



$$f(x) = \inf_{|x-y| \le R} \left\{ g(y) - \sqrt{R^2 - |x-y|^2} \right\}$$

$$\partial\Omega = \{ (x, x_{N+1}) : \mathrm{dist}((x, x_{N+1}), \{ x_{N+1} \geq g(x) \}) = R \}.$$

Fix j. By $(\widetilde{\mathrm{BN}})$, say

$f(x+y^j) - f(x)$ has a maximum M.

Then $\exists x_0 \in \mathbb{R}^N$ $\forall x \in \mathbb{R}^N$

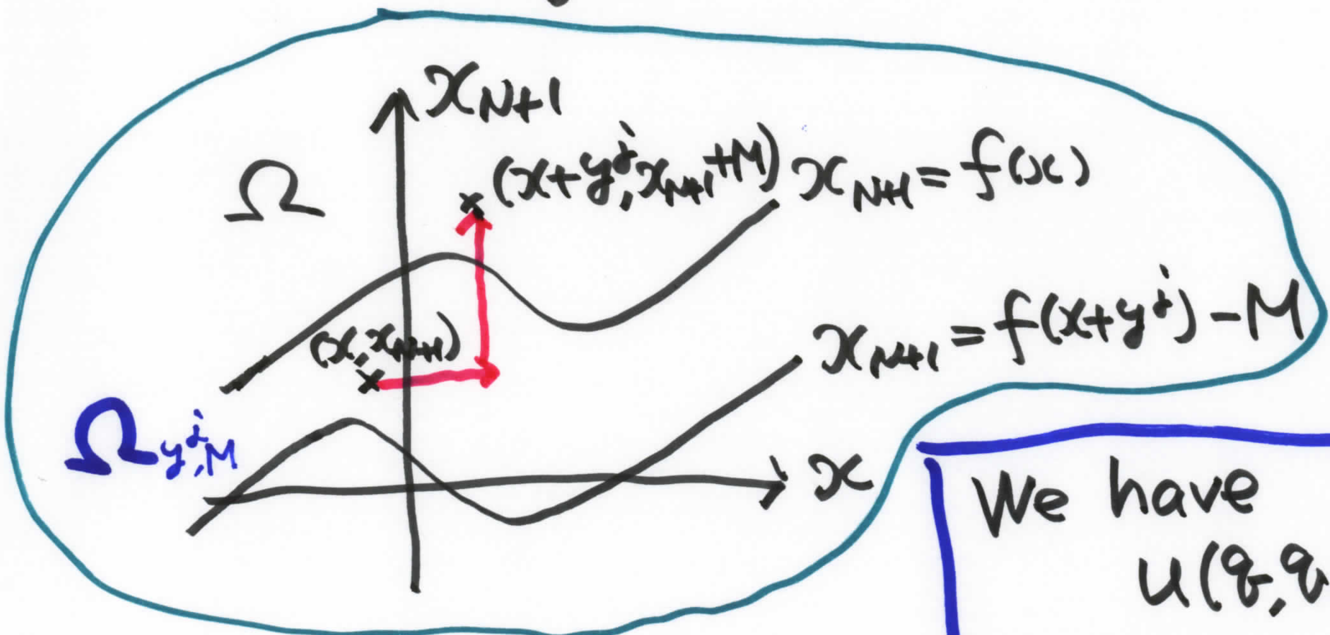
$$\boxed{f(x+y^j) - f(x) \leq M = f(x_0+y^j) - f(x_0).}$$

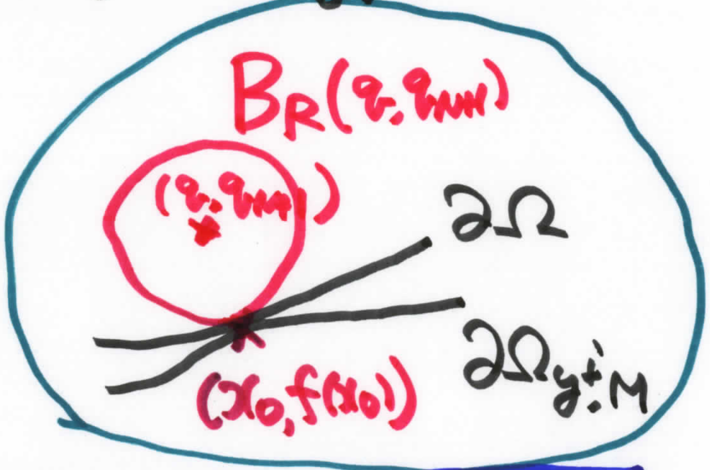
Hence, by the sliding method,

set $\Omega_{y^i,M} = \{ (x, x_{N+1}) : (x+y^i, x_N+M) \in \Omega \}$.

Then $f(x+y^i) - M \leq f(x) \quad (x \in \mathbb{R}^N)$

$\Omega_{y^i,M} \supset \Omega$ and $(x_0, f(x_0)) \in \partial\Omega_{y^i,M} \cap \partial\Omega$.





We have
$$u(q, q_{N+1}, t) = u(q+y^i, q_{N+1}+M, t).$$

Strong comparison principle

$\Rightarrow$ $$f(x+y^j)-M \equiv f(x) \quad (x\in\mathbb{R}^N).$$

Since f is continuous and $\{y^1,\cdots,y^N\}$ is a basis of $\mathbb{R}^N$, we solve these functional equations and get:

$$\boxed{\begin{array}{l}f(x) \text{ is determined by its values} \\ \text{on } \left\{\sum \alpha_j y^j \;\middle|\; 0\le\alpha_j<1,\ j=1,\cdots,N\right\}.\end{array}}$$

Then, we see that
$\forall y \in \mathbb{R}^N$: $f(x+y)-f(x)$ has either
a maximum or a minimum on $\mathbb{R}^N$.

By the sliding method again, we get

$$\boxed{f(x+y)-f(x)=f(z+y)-f(z) \quad (\forall x\ \forall y\ \forall z \in \mathbb{R}^N).}$$

Since f is continuous, we conclude that

$$\boxed{\text{$f(x)$ is affine and $\partial\Omega$ must be a hyperplane.}}$$

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§3 The heat equation.

Let $\phi(s) \equiv s$. Consider

$$
(\mathrm{IBP}) \quad \begin{cases} \partial_t u = \Delta u & \text{in } \Omega \times (0,\infty), \\ u = 1 & \text{on } \partial\Omega \times (0,\infty), \\ u = 0 & \text{on } \Omega \times \{0\}. \end{cases}
$$

Let $f \in C(\mathbb{R}^N)$, $N \geq 1$,

$$
\Omega = \{ (x, x_{N+1}) : x_{N+1} > f(x) \}.
$$

Theorem 2 Suppose that u has a stationary level surface Γ in Ω.
Under one of the following three conditions, f is affine and $\partial\Omega$ must be a hyperplane.

(i) $N=2$

(ii) f is uniformly continuous in $\mathbb{R}^N$.

(iii) $\begin{cases} \sup\limits_{|x-y|\leq 1} |f(x)-f(y)| \leq {}^{\exists}C & \text{—(1)} \\ \Omega \text{ satisfies the uniform exterior sphere condition.} & \text{—(2)} \end{cases}$

Theorem 2.1 Suppose that u has two stationary level surfaces Γ_1 and Γ_2 in Ω. Then f is affine and $\partial\Omega$ must be a hyperplane if f satisfies

$$\sup_{|x-y|\leq 1} |f(x) - f(y)| \leq C \qquad (1)$$

for some constant $C > 0$.

Proof. Let Γ_1 be nearer to $\partial\Omega$. Then, the existence of Γ_2 gives us the uniform exterior sphere condition for Γ_1. So, case (iii) of Theorem 2 works.

Remark. The uniform continuity is weakened, but two stationary level surfaces are strong. Only under (1), one wants the same conclusion.

Lemma [MS2] Under $\partial\Omega = \partial(\mathbb{R}^N \setminus \overline{\Omega})$,

(i) $\exists R > 0$ s.t. $\mathrm{dist}(p, \partial\Omega) = R$ for $p \in \Gamma$.

(ii) Γ and $\partial\Omega$ are analytic and

$\Gamma \,/\!/\, \partial\Omega$ (parallel)

(iii) $\exists C > 0$:

$$\left(\begin{array}{l} \displaystyle\prod_{j=1}^{N} (1 - R\kappa_j(q)) = C \quad \text{for } q \in \partial\Omega, \\ \displaystyle\max_{1 \le j \le N} \kappa_j(q) < \frac{1}{R} \quad \text{〃} \quad . \end{array} \right.$$

where

$\kappa_1, \cdots, \kappa_N$: the principal curv. of $\partial\Omega$ at $q \in \partial\Omega$.

Proof of Theorem 2.

(i) $N=2$ (see §4 of this talk).

(ii) See [S, 2011]. Theory of viscosity solutions and Giga-Ohnuma's strong comparison principle.

"uniformly continuous" was suggested by H. Ishii. (In [S, 2011], "Lipschitz continuous" was treated.)

(iii) Set $\Gamma = \{(x, g(x)) \mid x \in \mathbb{R}^N\}$.

(1) allows us to get

$$\exists\, K^* = \inf\{K > 0 : g \leq f + K \text{ in } \mathbb{R}^N\}$$

This allows us to use the same argument as in [S, 2011].

(2) allows us to get by using (iii) of Lemma.

$$\boxed{\begin{array}{l}\exists\, r_0 > 0 \;\exists\, \tau > 0 \text{ s.t.} \\ \quad -\dfrac{1}{r_0} \leq K_j \leq \dfrac{1}{R} - \tau, \quad j = 1, \cdots, N.\end{array}} \quad (*)$$

Moreover, this (*) also allows us to use the interior gradient estimates due to Y. Y. Li (1991) and N. Korevaar (1987) for elliptic Weingarten equations.

Hence

$$\boxed{|\nabla f| \leq {}^{\exists}C \quad \text{in } \mathbb{R}^N.}$$

Then, we can come back to the line of (ii) in [S, 2011]. //

§4. Bernstein-type theorem and Remark

Let $N=2$.

$s=(s_1, s_2) \in \Omega = \{ s \in \mathbb{R}^2 : s_1>0, s_2>0 \}$.

$F=F(s)$: C^1 symmetric, concave on Ω.

$$F_{s_j} > 0 \quad \text{in } \Omega, \quad j=1,2$$

$G=G(s) = -F(\frac{1}{s_1}, \frac{1}{s_2})$ for $s \in \Omega$.

We assume : G is concave on Ω.

$u=u(x) \in C^2(\mathbb{R}^2)$

$\mathcal{G} = \{ (x, u(x)) \in \mathbb{R}^3 \mid x \in \mathbb{R}^2 \}$

$\kappa_1(x), \kappa_2(x)$: the principal curv. of $\mathcal{G}$ w.r.t. upward normal.

Theorem 3.

$\boxed{\begin{array}{l}\text{Suppose : } \exists R>0.\ \exists c\in\mathbb{R} \text{ s.t. in } \mathbb{R}^2 \\ \qquad F(1-R\kappa_1, 1-R\kappa_2) = c \quad \text{and} \quad \max\limits_{1\le j\le 2} \kappa_j < \dfrac{1}{R} \\ \text{Then } \ c = F(1,1) \text{ and } u \text{ is affine and} \\ \quad g \text{ is a hyperplane.}\end{array}}$

Proof: Let $D = \{(x,t) \in \mathbb{R}^3 : t \leq u(x)\}$.

$\Gamma = \{ p \in \mathbb{R}^3 \mid \mathrm{dist}(p, D) = R \}$.

$\Gamma^* = \{ p \in \mathbb{R}^3 \mid \mathrm{dist}(p, D) = \frac{R}{2} \}$.

Lemma 1. [S. 2011].

$$H_g \leq \frac{1}{2R\delta}\left(F(1,1) - c\right) \leq H_\Gamma \quad \text{in } \mathbb{R}^2$$

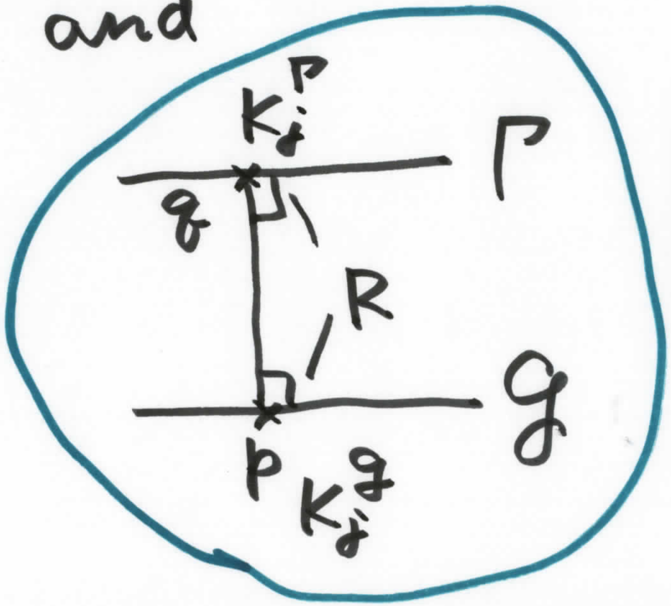
where $\delta = F_{s_1}(1,1) = F_{s_2}(1,1) > 0$,

and H_g, H_Γ : the mean curvature of g, Γ

with respect to upward normal.

Proof: Use concavity of F and G

and

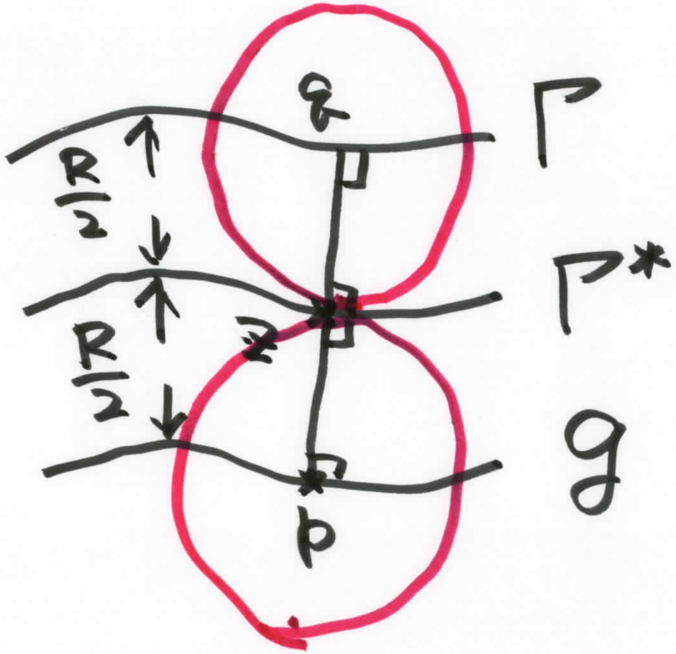


$$\boxed{1 - R K_j^g(p) = \frac{1}{1 + R K_j^\Gamma(q)}}$$

Lemma 2 : [S.2011]

$$\boxed{C = F(1,1) \text{ and } H_g \leq 0 \leq H_\Gamma \text{ in } \mathbb{R}^2}$$

Let κ_j^* be the principal curv. of Γ^*.



$$\Rightarrow \boxed{-\frac{2}{R} < \kappa_j^* < \frac{2}{R} \quad (j=1,2)}$$

Lemma 2 $\Rightarrow$

$$\sum_{j=1}^{2} \frac{K_j^*}{1+\frac{R}{2}K_j^*} = H_g \leqslant 0 \leqslant H_p = \sum_{j=1}^{2} \frac{K_j^*}{1-\frac{R}{2}K_j^*}$$

$\Rightarrow$ $K_1^* + K_2^* + RK_1^*K_2^* \leqslant 0 \leqslant K_1^* + K_2^* - RK_1^*K_2^*$

$\Rightarrow$ $K_1^*K_2^* \leqslant 0$ and

$$RK_1^*K_2^* \leqslant K_1^* + K_2^* \leqslant -RK_1^*K_2^*$$

Then

$$(K_1^*)^2 + (K_2^*)^2 = (K_1^* + K_2^*)^2 - 2K_1^*K_2^*$$
$$\leq R^2 (K_1^*)^2 (K_2^*)^2 - 2K_1^*K_2^*$$
$$\leq 2 \times (-3) K_1^* K_2^*$$

Hence

Gauss Map of P^* is

$(-3, 0)$-quasiconformal on $\mathbb{R}^2$

Gilbarg-Trudinger's book (L. Simon)

Corollary 16.19

$\Longrightarrow$ u is affine. //

Remark:

- When $F = (S_1 S_2)^{\frac{1}{2}}$, we have

$$K_1^* + K_2^* = 0 \quad (\text{just minimal surface})$$

Question:

- When $N = 3$, $F = (S_1 S_2 S_3)^{\frac{1}{3}}$, we have

$$4(K_1^* + K_2^* + K_3^*) + R^2 K_1^* K_2^* K_3^* = 0 \quad \text{in } \mathbb{R}^3$$

$$\left(-\frac{2}{R} < K_i^* < \frac{2}{R},\ i = 1, 2, 3\right)$$

- The conclusion of Theorem 1 holds only under the uniform continuity of f. ??

Another role of condition (1).

Another proof of Theorem 2 (iii) and Theorem 2.1 is as follows: By Theorem A, there exist $g \in C^2(\mathbb{R}^N)$ and $R > 0$ such that

$$\Gamma = \{(x, g(x)) \ : \ x \in \mathbb{R}^N\} \ \text{ and}$$

$$g(x) = \sup_{|x-y| \le R} \{f(y) + \sqrt{R^2 - |x-y|^2}\}.$$

Then (1) implies that $g(x) - f(x)$ is bounded. As in Lemma 2 in Theorem 3, we have

$$\mathcal{M} f \le 0 \le \mathcal{M} g \equiv \ \mathrm{div}\left(\frac{\nabla g}{\sqrt{1 + |\nabla g|^2}}\right) \ \text{ in } \mathbb{R}^N.$$

Let $B_n \subset \mathbb{R}^N$ be a ball with center 0 and radius $n \in \mathbb{N}$, and let f_n, g_n satisfy

$$\mathcal{M} f_n = \mathcal{M} g_n = 0 \ \text{ in } B_n$$

$$f_n = f \ \text{ and } g_n = g \ \text{ on } \partial B_n.$$

Then, by maximum principle,

$$f_n \le f < g \le g_n \ \text{ in } B_n,$$

$\exists z_n \in \partial B_n$ **with** $g_n - f_n \le g(z_n) - f(z_n)$ **in** B_n.

By monotonicity, boundedness of $g - f$, (1), and Moser's theorem,

f_n, g_n converge to affine functions f_∞, g_∞, resp.

Hence, $\quad f_\infty$ and g_∞ are parallel,

$$f_\infty \leq f < g \leq g_\infty, \text{ and}$$

$$g_\infty(z_n) - g(z_n), f(z_n) - f_\infty(z_n) \to 0 \text{ as } n \to \infty.$$

Finally, by using uniform continuity of f, g and the strong comparison principle as in [S, 2011]

$$f \equiv f_\infty \text{ and } g \equiv g_\infty.$$

References

[BCN] H. Berestycki, L. A. Caffarelli, and L. Nirenberg, Monotonicity for elliptic equations in unbounded Lipschitz domains, Comm. Pure Appl. Math. 50 (1997), 1089–1111.

[K] N. J. Korevaar, A priori interior gradient bounds for solutions to elliptic Weingarten equations, Ann. Inst. Henri Poincaré - (C) Anal. Non Linéaire 4 (1987), 405–421.

[L] Y. Y. Li, Interior gradient estimates for solutions of certain fully nonlinear elliptic equations, J. Differential Equations 90 (1991), 172–185.

[MS1] R. Magnanini and S. Sakaguchi, Interaction between nonlinear diffusion and geometry of domain, preprint, arXiv:1009.6131v1.

[MS2] R. Magnanini and S. Sakaguchi, Matzoh ball soup revisited: the boundary regularity issue, preprint, arXiv:1103.6229v1.

[MS3] R. Magnanini and S. Sakaguchi, Stationary isothermic surfaces and some characterizations of the hyperplane in the N-dimensional Euclidean space, J. Differential Equations 248 (2010), 1112–1119.

[MS4] R. Magnanini and S. Sakaguchi, Nonlinear diffusion with a bounded stationary level surface, Ann. Inst. Henri Poincaré - (C) Anal. Non Linéaire 27 (2010), 937–952.

[S] **S. Sakaguchi, A Liouville-type theorem for some Weingarten hypersurfaces, Discrete and Continuous Dynamical Systems - Series S, 4 (2011), 887–895.**