Convexity, rearrangements and Brunn-Minkowski inequalities in PDE's

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Introduction

We will deal with general (possibly fully nonlinear) elliptic Dirichlet problems

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$F : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}_n \to \mathbb{R}$ is a continuous proper degenerate elliptic operator:

proper: $F(x,u,p,A) \geq F(x,v,p,A)$ if $u \leq v$,

(degenerate) elliptic: $F(x,u,p,A) \geq F(x,u,p,B)$ if $A \geq B$.

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- **A microscopic technique**

Based on the strong maximum principle and in particular on a smart combination of the so called *constant rank theorems* and *continuity method*.

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G. Alvarez, J.-M. Lasry and P.-L. Lions, *Convex viscosity solutions and state constraints*, J. Math. Pures Appl. 76 (1997), 265-288.

The method of the convex envelope by ALL

Let u be a solution of our problem and define its convex envelope u_{**} as the largest convex function $\leq u$ in the *strictly convex* set Ω. More precisely we have:

$$u_{**}(x) = \inf\{\sum_{i=1}^{n+1} t_i u(x_i) \ : \ x_i \in \Omega, \ t \in \Lambda_n, \ \sum_i t_i x_i = x\}$$

where

$$\Lambda_n = \{t = (t_1, \ldots, t_{n+1}) \ : \ t_i \geq 0, \ \sum_i t_i = 1\} .$$

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They prove that this actually happens under the following simple assumption

$$(x, u, A) \to F(x, u, p, A^{-1}) \quad \text{is convex for } (x, u, A) \in \Omega \times \mathbb{R} \times \mathcal{S}_n^{++} \qquad (0.2)$$

for every fixed $p \in \mathbb{R}^n$.

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Everything is based on the notion of *Minkowski addition of convex sets* and the corresponding functional notion of *infimal convolution*, that allow to compare the solutions in two different convex sets Ω_0 and Ω_1 with the solution in the convex set $\Omega_t = (1-t)\Omega_0 + t\,\Omega_1$.

Minkowski addition and the convex hull

Let $t \in [0,1]$ and let Ω_0 and Ω_1 be two (not necessarily convex) subsets of $\mathbb{R}^n$. The Minkowski linear combination of Ω_0 and Ω_1 wit ratio t is simply defined as

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The convex hull of a set A in $\mathbb{R}^n$ is defined as the smallest convex set containing A, more precisely:

$$\mathrm{conv}(A) = \{\sum_{i=1}^{n+1} t_i x_i \;:\; x_i \in A,\; t_i \geq 0,\; \sum_i t_i = 1\}\,.$$

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The possibility to refine the method of ALL is based on the following simple geometric observation:

$$\mathrm{conv}(A) = \bigcup_{t \in \Lambda_n} A_t\,,$$

where

$$A_t = \sum_{i=1}^{n+1} t_i A\,.$$

Infimal convolution and convex envelope

Minkowski combination of sets has a functional equivalent: the so called *infimal convolution*.
Let $t \in \Lambda_m$ and let $u_1, \ldots, u_m$ be functions (defined in $\Omega_1, \ldots, \Omega_m$).
The infimal convolution of $u_1, \ldots, u_m$ with ratio t is defined (in Ω_t) as follows:

$$u_{*t}(x) = \inf\{\sum_{i=1}^{n+1} t_i u(x_i) \ : \ x_i \in \Omega_i,\ t \in \Lambda_n, \sum_i t_i x_i = x\} \, .$$

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It is easily seen that it corresponds to make the Minkowski linear combination of the epigraphs of the functions u_i, that is

$$K_t = \sum_{i=1}^{m} t_i K_i\,,$$

where

$$K_j = \{(x,z) \,:\, u_j(x) \leq z \leq 0,\, x \in \Omega_j\}\,, \quad j = 1, \dots, m, t\,.$$

Infimal convolution and convex envelope

When the involved functions are strictly convex and sufficiently regular (say C^2_+ for simplicity), we have that the infimum in the definition is a minimum. Then, for every point $x \in \Omega_t$, there exist $x_1 \in \Omega_1, \ldots, x_m \in \Omega_m$ such that

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Moreover, if

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then $u_{*t} \in C^2_+(\Omega_t)$ and the following differential relations hold:

$$Du_1(x_1) = Du_2(x_2) = \cdots = Du_m(x_m) = Du_{*t}(x)$$

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As for sets, we have that the convex envelope of a function can be seen as the infimum of all the possible infimal convolution of $n+1$ copies of the same function. Precisely

$$u_{**}(x) = \inf\{u_{*t}(x)\ :\ t \in \Lambda_n\}\,.$$

Application 1: the main theorem

Theorem 1

Let Ω_0 and Ω_1 be two strictly convex sets and assume u_i is a C^2_+ solution of (0.1) in Ω_i, for $i = 0, 1$. Let $t \in (0, 1)$ and set Ω_t and u_{*t} as above. Then, if

$$(F) \quad \{(x, u, A) \in \mathbb{R}^n \times \mathbb{R} \times \mathcal{S}_n^{++} \; : \; F(x, u, p, A^{-1}) \leq 0\} \text{ is convex } \forall p \in \mathbb{R}^n ,$$

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Notice that assumption (F) is weaker than the assumption by ALL.

Corollary

In the same assumption of the previous theorem, if u_t is the solution of (0.1) *in Ω_t and a comparison principle holds, then $u_t \leq u_{*t}$ in Ω_t.*

The way to rearrangements: Hadwiger's Theorem!

By Hadwiger's Theorem there exists a sequence of rotations $\{\rho_N\}$ such that

$$\Omega_N = \frac{1}{N+1}(\rho_0\Omega + ... + \rho_N\Omega)$$

converges in Hausdorff metric to the ball $\Omega^\star$ which has the same mean-width as Ω.

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Denote by u_N the solution of problem (0.1) in Ω_N and for every $N \in \mathbb{N}$, let $\tilde{u}_N$ be the Minkowski combination of the functions

$$u_0(x) = u(\rho_0^{-1}x), \dots, u_N(x) = v(\rho_N^{-1}x)$$

with ratio

$$t = (1/(N+1), \dots, 1/(N+1)) \in \Lambda_N \,.$$

By Theorem 1, $\tilde{u}_N$ is a supersolution of the problem solved by u_N and it holds

$$|u_N| \geq |\tilde{u}_N| \quad \text{in } \Omega_N \,.$$

The way to rearrangements 2

Notice that the functions $\tilde{u}_N$ are uniformly bounded convex functions, then, possibly up to a subsequence, they converge uniformly to function $\tilde{u}$ which is a viscosity supersolution of problem (0.1) in $\Omega^\star$, thanks to the stability of viscosity solution under uniform convergence. Hence

$$|u^\star| \geq |\tilde{u}| \quad \text{in } \Omega^\star\,,$$

whence

$$\|u^\star\|_{L^p(\Omega^\star)} \geq \|\tilde{u}\|_{L^p(\Omega^\star)} \quad \text{for every } p \in (0, +\infty]\,. \tag{0.3}$$

On the other hand, by the definition of $\tilde{v}_N$ and $\tilde{u}_N$, it holds

$$\left|\tilde{u}_N\left(\frac{1}{N+1}\sum_{i=0}^{N} x_i\right)\right| \geq \frac{1}{N+1}\sum_{i=0}^{N}\left|u(\rho_i^{-1}x_i)\right|\,,$$

for every $x_i \in \rho_i\Omega$, $i = 0, \ldots, N$. This yields

$$|\tilde{u}_N(x)| \geq \prod_{i=0}^{N} |u(\rho_i^{-1}x_i)|^{\frac{1}{N+1}}$$

for every $x_0, \ldots, x_N \in \mathbb{R}^3$ such that $x = \frac{1}{N+1}\sum_{i=0}^{N} x_i$, once we extend u_N and u as zero outside of Ω_N and Ω, respectively.

The way to rearrangements 3

Then the *Prékopa-Leindler inequality* implies

$$\|\tilde{u}_N\|^p_{L^p(\Omega_N)} \geq \prod_{i=0}^{N} \left(\int_{\rho_i \Omega} |u(\rho_i^{-1}\xi)|^p \, d\xi \right)^{\frac{1}{N+1}} = \|u\|^p_{L^p(\Omega)} \quad \text{for every } p \in (0, +\infty] \,. \tag{0.4}$$

Passing to the limit as $N \to \infty$, this yields

$$\|\tilde{u}\|_{L^p(\Omega^\star)} \geq \|u\|_{L^p(\Omega)} \,,$$

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Theorem 2

Let $\Omega \subset \mathbb{R}^3$ be a C^2_+ set and let $\Omega^\star$ be a ball with the same mean-width of Ω.
Denote by u the solution of (0.1) in Ω and by $u^\star$ the solution in $\Omega^\star$.
Then, in the same assumption of Theorem 1 + Comparison Pple,

$$\|u\|_{L^p(\Omega)} \leq \|u^\star\|_{L^p(\Omega^\star)} \quad \text{for every } p \in (0, +\infty] \,. \tag{0.5}$$

Moreover, equality holds for any $p \in (0, +\infty)$ if and only if Ω is a ball.

Convexity of solutions

Essentially with the same proof as above, and taking in account that the convex envelope is just the infimum of all the possible infimal convolution of $n+1$ copies of the function u, we obtain the following.

Theorem 3

Let $\Omega \subset \mathbb{R}^3$ be a convex set and u be a (viscosity) solution of (0.1) in Ω. If the operator F satisfies assumption (F) and a comparison principle holds, then u is convex.

Examples

Example of new applications are given for instance by the following Dirichlet problems for Hessian equations.

$$\begin{cases} S_k(D^2u) = \Lambda_k(\Omega)(-u)^k & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \\ u < 0 & \text{in } \Omega\,, \end{cases} \tag{0.6}$$

and

$$\begin{cases} S_k(D^2u) = 1 & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{0.7}$$

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$k = 2$ and $n = 3$

If u solves (0.6), then $-\log(-u)$ is convex.

If u solves (0.7), then $-\sqrt{-u}$ is convex.

P.S., *Convexity of solutions and Brunn-Minkowski inequalities for Hessian equations in* $\mathbb{R}^3$, preprint 2010 (submitted).

Brunn-Minkowski inequalities for variational functionals

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Notice that, thanks to the same procedure based on Hadwiger's theorem, we can obtain form any BM inequality a corresponding Urysohn's inequality, stating that the maximum (or minimum) of the functional among all the convex sets with given mean width is attained when the domain is a ball.

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Precisely, in this case,

$$\Lambda_k(\Omega) \geq \Lambda_k(\Omega^\star)$$

with $=$ if and only if $\Omega = \Omega^\star$