Dirichlet problem for the Stokes system in Lipschitz domains with unit normal close to VMO

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The papers by V. Maz'ya, M. Mitrea, and T. Shaposhnikova

The inhomogeneous Dirichlet problem for the Stokes system in Lipschitz domains with unit normal close to VMO,

Journal of Functional Analysis and its Applications, **28**, 2009

and

The Dirichlet problem in Lipschitz domains with boundary data in Besov spaces for higher order elliptic systems with rough coefficients,

Journal d'Analyse Mathématique, **110**, 2010.

By $B^l_{p,q}(\mathbb{R}^n)$ we denote the space of functions in $\mathbb{R}^n$ having the finite norm

$$\|u\|_{B^l_{p,q}(\mathbb{R}^n)} = \Big(\int_{\mathbb{R}^n} \|\Delta_h \nabla_{[l]} u\|^q_{L_p(\mathbb{R}^n)} |h|^{-n-q\{l\}}\, dh \Big)^{1/q} + \|u\|_{W^{[l]}_p(\mathbb{R}^n)}$$

where $\{l\} > 0$, p, $q \geq 1$, $\Delta_h v = v(\cdot + h) - v(\cdot)$, and $\nabla_{[l]}$ is the vector of all derivatives of order $[l]$.
There are other ways of characterization of $B^l_{p,q}(\mathbb{R}^n)$ with equivalent norm given in terms of dyadic decompositions of unity (see for example the book by T. Runst and W. Sickel *"Sobolev Spaces of Fractional Order, Nemytzkij Operators and Nonlinear PDEs"*, 1996.

The Besov spaces $B^l_{p,q}(\Omega)$ (and later Triebel-Lizorkin spaces) are defined by restricting the distributions from the corresponding spaces in $\mathbb{R}^n$ to the open set Ω and $B^l_{p,q}(\partial\Omega)$ stands for the Besov class on the Lipschitz manifold $\partial\Omega$ obtained by tranporting(via a partition of unity and pull-back) the standard class $B^l_{p,q}(\mathbb{R}^{n-1})$.

Stokes system

Consider the Stokes system in an arbitrary bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$,

$$\begin{aligned} &\Delta \vec{u} - \nabla \pi = \vec{f} \in B_{p,q}^{s+\frac{1}{p}-2}(\Omega), \quad \operatorname{div} \vec{u} = g \in B_{p,q}^{s+\frac{1}{p}-1}(\Omega), \\ &\vec{u} \in B_{p,q}^{s+\frac{1}{p}}(\Omega), \quad \pi \in B_{p,q}^{s+\frac{1}{p}-1}(\Omega), \quad \operatorname{Tr} \vec{u} = \vec{h} \in B_{p,q}^{s}(\partial\Omega), \end{aligned} \tag{1}$$

subject to the (necessary) compatibility condition

$$\int_{\partial\mathcal{O}} \langle \nu, \vec{h} \rangle \, d\sigma = \int_{\mathcal{O}} g(X)\, dX, \quad \text{for every component } \mathcal{O} \text{ of } \Omega. \tag{2}$$

Related works

- $\partial\Omega$ is sufficiently smooth (at least C^2): Cattabriga (1961), Solonnikov (1966), Temam (1977), Giga (1981), Dautray & J.-L. Lions (1993), Varnhorn (1994).
- $\partial\Omega \in C^{1,1}$: Amrouche & Girault (1991)
- Lipschitz $\partial\Omega$ with small Lipschitz constant: Galdi, Simader & Sohr (1994)
- polygon in $\mathbb{R}^2$ or polyhedron in $\mathbb{R}^3$: Kozlov, Maz'ya & Rossmann (2001), Maz'ya & Rossmann (2009)
- Lipschitz $\partial\Omega$, $p \sim 2$: Mitrea & Wright (2008)

The main hypothesis requires that, at small scales, the so called local mean oscillations of the unit normal to $\partial\Omega$ are not too large, relative to the Lipschitz constant of the domain Ω, and the indices of the corresponding Besov space.

Local mean oscillation

By the local mean oscillation of $F \in L_1(\Omega)$ we understand

$$\{F\}_{\mathrm{Osc}(\Omega)} :=$$

$$\lim_{\varepsilon \to 0} \Big(\sup_{\{B_\varepsilon\}_\Omega} \fint_{B_\varepsilon \cap \Omega} \fint_{B_\varepsilon \cap \Omega} \Big| F(x) - F(x) \Big| dx dy \Big),$$

where $\{B_\varepsilon\}_\Omega$ stands for the family of balls of radius ε centered at points of Ω. Similarly, the local mean oscillation of $f \in L_1(\partial\Omega)$ is

$$\{f\}_{\mathrm{Osc}(\partial\Omega)} :=$$

$$\lim_{\varepsilon \to 0} \Big(\sup_{\{B_\varepsilon\}_{\partial\Omega}} \fint_{B_\varepsilon \cap \partial\Omega} \fint_{B_\varepsilon \cap \partial\Omega} \Big| f(x) - f(y) \Big| ds_x ds_y \Big),$$

where $\{B_\varepsilon\}_{\partial\Omega}$ is the collection of n-dimensional balls of radius ε with centers on $\partial\Omega$.

A locally integrable function g in $\mathbb{R}^n$ belongs to the space $BMO(\mathbb{R}^n)$ if

$$\|g\|_{BMO(\mathbb{R}^n)} := \sup_B \fint_B \left|g(x) - \fint_B g(y)dy\right| dx$$

is finite, where the supremum is taken over all balls B in $\mathbb{R}^n$. The above supremum defines a seminorm in $BMO(\mathbb{R}^n)$.

Remark

Smallness of the local mean oscillation $\{\nu\}_{\mathrm{Osc}(\partial\Omega)}$ does not imply smallness of the Lipschitz constant.

Let

$$\Omega = \{(x, y) \in \mathbb{R}^2,\ y > \varphi_\varepsilon(x)\},$$

where

$$\varphi_\varepsilon(x) = x\ \sin(\varepsilon \log|x|^{-1}).$$

Then $\|\varphi'_\varepsilon\|_{L_\infty(\mathbb{R})} \sim 1$, while $\|\varphi'_\varepsilon\|_{BMO(\mathbb{R})} \le C\,\varepsilon$.

Main theorem

Theorem

Assume that $\frac{n-1}{n} < p \leq \infty$, $0 < q \leq \infty$, $(n-1)\big(\frac{1}{p} - 1\big)_+ < s < 1$. *Then there exists* $\delta > 0$ *which depends only on the Lipschitz character of* Ω *and the exponent* p, *with the property that if* $\{\nu\}_{\mathrm{Osc}(\partial\Omega)} < \delta$, *then the problem (1) is well-posed (with uniqueness modulo locally constant functions in* Ω *for the pressure). There exists a finite, positive constant* $C = C(\Omega, p, q, s, n)$ *such that*

$$\|\vec{u}\|_{B^{s+\frac{1}{p}}_{p,q}(\Omega)} + \inf_{c} \|\pi - c\|_{B^{s+\frac{1}{p}-1}_{p,q}(\Omega)}$$

$$\leq C\|\vec{f}\|_{B^{s+\frac{1}{p}-2}_{p,q}(\Omega)} + C\|g\|_{B^{s+\frac{1}{p}-1}_{p,q}(\Omega)} + C\|\vec{h}\|_{B^{s}_{p,q}(\partial\Omega)},$$

with the infimum taken over all locally constant functions c *in* Ω.

Remark

The smallness condition in the Theorem

$$\{\nu\}_{\mathrm{Osc}(\partial\Omega)} :=$$

$$\lim_{\varepsilon\to 0}\Big(\sup_{\{B_\varepsilon\}_{\partial\Omega}} \fint_{B_\varepsilon\cap\partial\Omega} \fint_{B_\varepsilon\cap\partial\Omega} \Big|\nu(x)-\nu(y)\Big| ds_x ds_y\Big),$$

was first introduced by V. Maz'ya, M. Mitrea, and T. Shaposhnikova in the paper

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published in Journal d'Analyse Mathématique, **110**, 2010.

Let BMO and VMO stand, respectively, for the space of functions of bounded mean oscillation and the space of functions of vanishing mean oscillation (considered on $\partial\Omega$). Recall that $f \in \mathrm{VMO}$ if

$$\fint_{B_r} \left| f(x) - \fint_{B_r} f(y)dy \right| dx \to 0 \quad \text{as } r \to 0.$$

The space *VMO* is equivalently defined as the closure in *BMO* of the space of uniformly continuous functions.
It can be proved that

$$\{F\}_{\mathrm{Osc}} \sim \mathrm{dist}\,(F, \mathrm{VMO})$$

where the distance is taken in BMO. Thus the small oscillation condition in the theorem holds if and only if

$$\mathrm{dist}\,(\nu, \mathrm{VMO}) < \delta.$$

This is trivially the case if $\nu \in \mathrm{VMO}(\partial\Omega)$ irrespective of p and Ω.

A class of domains satisfying the hypotheses of the main Theorem is:

Lipschitz domains with a sufficiently small Lipschitz constant, relatively to the exponent p.

In particular:

Lipschitz polyhedral domains with dihedral angles sufficiently close to π, depending on p.
Polygonal domains with angles sufficiently close to π, depending on p.

The proof is based on the following mapping properties of the hydrostatic layer potentials: single $\mathcal{S}$ and double $\mathcal{D}$ layer potentials for the velocity and single $\mathcal{Q}$ and double $\mathcal{P}$ layer potentials for the presure. These properties are of independent interest.

Recall that in the case $n = 3$ these potentials are:

$$\mathcal{S}_i(x, \vec{\psi}) = \frac{1}{8\pi} \int_{\partial\Omega} \Big(\frac{\delta_{ik}}{|x-y|} + \frac{(x_i - y_i)(x_k - y_k)}{|x-y|^3} \Big) \psi_k(y) d\sigma(y),$$

$$\mathcal{Q}(x, \vec{\psi}) = \frac{1}{4\pi} \int_{\partial\Omega} \frac{x_k - y_k}{|x-y|^3} \psi_k(y) d\sigma(y),$$

$$\mathcal{D}_k(x, \vec{\varphi}) = -\frac{3}{4\pi} \int_{\partial\Omega} \frac{(x_i - y_i)(x_j - y_j)(x_k - y_k)}{|x-y|^5} \varphi_i(y) \nu_j(y) d\sigma(y),$$

$$\mathcal{P}(x, \vec{\varphi}) = \frac{1}{2\pi} \frac{\partial}{\partial x_j} \int_{\partial\Omega} \frac{x_k - y_k}{|x-y|^3} \varphi_k(y) \nu_j(y) d\sigma(y),$$

where $i, k = 1, 2, 3$.

Boundedness of layer potentials

Theorem

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$, and assume that $\frac{n-1}{n} < p \leq \infty$, $(n-1)(\frac{1}{p}-1)_+ < s < 1$, and $0 < q \leq \infty$. Then

$$\mathcal{S} : B^{s-1}_{p,q}(\partial\Omega) \longrightarrow B^{s+\frac{1}{p}}_{p,q}(\Omega),$$

$$\mathcal{Q} : B^{s-1}_{p,q}(\partial\Omega) \longrightarrow B^{s+\frac{1}{p}-1}_{p,q}(\Omega),$$

$$\mathcal{D} : B^{s}_{p,q}(\partial\Omega) \longrightarrow B^{s+\frac{1}{p}}_{p,q}(\Omega),$$

$$\mathcal{P} : B^{s}_{p,q}(\partial\Omega) \longrightarrow B^{s+\frac{1}{p}-1}_{p,q}(\Omega),$$

are well-defined, bounded operators.

Remark

If $|p-2|$ is small, $0 < q \leq \infty$, and $0 < s < 1$, the Main Theorem is valid in any Lipschitz domain without assumption on the size of the oscillation of the outward unit normal. This follows from a paper by E. Fabes, C. Kenig and G. Verchota of 1988, containing L_p-estimates of the nontangential maximal function for solutions of the Stokes system when p is close to 2 in any Lipschitz domain.

A well-posedness result, analogous to the Main Theorem, holds on the Triebel-Lizorkin scale, i.e. for the problem

$$\begin{aligned} &\Delta \vec{u} - \nabla \pi = \vec{f} \in F_{p,q}^{s+\frac{1}{p}-2}(\Omega), \quad \operatorname{div} \vec{u} = g \in F_{p,q}^{s+\frac{1}{p}-1}(\Omega), \\ &\vec{u} \in F_{p,q}^{s+\frac{1}{p}}(\Omega), \quad \pi \in F_{p,q}^{s+\frac{1}{p}-1}(\Omega), \quad \operatorname{Tr} \vec{u} = \vec{h} \in B_{p,p}^{s}(\partial\Omega), \end{aligned} \tag{3}$$

This time, in addition to the previous conditions imposed on the indices p, q, it is also assumed that $p, q < \infty$.