Some identities for Green's function of polyharmonic operaotor under the Navier boundary conditions and its application

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Green's function under the Navier B.C.

- $p \in \mathbb{N}$
- $\Omega \subset \mathbb{R}^N (N \geq 2p)$: a smooth bounded domain
- $(-\Delta)^p$: polyharmonic operator in $\mathbb{R}^N$.

Let $G = G(x, y)$ denote the Green function of $(-\Delta)^p$ under the Navier boundary condition:

Green's function of $(-\Delta)^p$ under the Navier B.C.

$$\begin{cases} (-\Delta)^p G(\cdot, y) = \delta_y & \text{in } \Omega, \\ G(\cdot, y) = (-\Delta)^j G(\cdot, y) = 0 & \text{on } \partial\Omega \ (j = 1, \cdots p-1), \end{cases}$$

Decompose $G(x,y) = \Gamma(x,y) - H(x,y)$, where

fundamental solution of $(-\Delta)^p$

$$\Gamma(x,y) = \begin{cases} \frac{2\Gamma(\frac{N}{2}-p)}{4^p(p-1)!\Gamma(\frac{N}{2})\sigma_N}|x-y|^{2p-N}, & N > 2p, \\ \frac{1}{\{2^{p-1}(p-1)!\}^2\sigma_N}\log\frac{1}{|x-y|}, & N = 2p, \end{cases}$$

- $\sigma_N = \frac{2\pi^{N/2}}{\Gamma(N/2)}$: the volume of the unit sphere in $\mathbb{R}^N$.
- $H(x,y)$: the regular part of the Green function.

$$\begin{cases} (-\Delta)^p H(\cdot,y) = 0 & \text{in } \Omega, \\ (-\Delta)^j H(\cdot,y) = (-\Delta)^j \Gamma(\cdot,y) & \text{on } \partial\Omega \ (j = 0, 1, \cdots p-1). \end{cases}$$

- $R(y) := H(y,y)$: the Robin function of the Green function of $(-\Delta)^p$ with the Navier B.C.

Consider

$$\begin{cases} (-\Delta)^p u = f & \text{in } \Omega, \\ (-\Delta)^j u = g_j & \text{on } \partial\Omega \ (j = 0, 1, \cdots p-1), \end{cases}$$

where f and g_j are smooth functions.

Green's representation formula for $(-\Delta)^p$

$$u(y) = \int_\Omega G(x,y) f(x) dx - \sum_{k=1}^{p} \int_{\partial\Omega} \frac{\partial \overline{G}_{k-1}}{\partial \nu_x}(x,y) g_{p-k}(x) ds_x$$

for $y \in \Omega$.

where $\overline{G}_j(x,y) = (-\Delta_x)^j G(x,y)$ $(j = 0, 1, \cdots, p-1)$.
Note that $\overline{G}_{p-1}$ is the Green function of $-\Delta$ under the Dirichlet boundary condition.

Consider

$$(-\Delta)^p u = f(u) \text{ in } \Omega, \quad u \in C^{2p}(\overline{\Omega})$$

without B.C.

Pohozaev identity for the polyharmonic equation

$$\int_\Omega \left[NF(u) - \left(\frac{N-2p}{2}\right) uf(u) \right] dx = \int_{\partial\Omega} (x\cdot\nu)\left(F(u) - \frac{1}{2}uf(u)\right) ds_x$$
$$+ \frac{(-1)^{p-1}}{2} \sum_{k=1}^{p} \int_{\partial\Omega} \left[\frac{\partial \Delta^{k-1} u}{\partial\nu} \Delta^{p-k}(x\cdot\nabla u) - \Delta^{k-1} u \frac{\partial \Delta^{p-k}(x\cdot\nabla u)}{\partial\nu} \right] ds_x$$

where $F(u) = \int_0^u f(s)ds$.

More general version is known, see Mitidieri, Pucci-Serrin, etc.
Note also that $\Delta^j(x\cdot\nabla u) = 2j\Delta^j u + (x\cdot\nabla\Delta^j u)$.

New Pohozaev identities for the Green function of $(-\Delta)^p$

For any $y \in \Omega$, we have

(1)

$$\sum_{k=1}^{p} \int_{\partial\Omega} (x-y)\cdot\nu \left(\frac{\partial \overline{G}_{k-1}}{\partial \nu}\right)\left(\frac{\partial \overline{G}_{p-k}}{\partial \nu}\right)(x,y)\,ds_x = (N-2p)R(y)$$

when $N > 2p$

$$\sum_{k=1}^{p} \int_{\partial\Omega} (x-y)\cdot\nu \left(\frac{\partial \overline{G}_{k-1}}{\partial \nu}\right)\left(\frac{\partial \overline{G}_{p-k}}{\partial \nu}\right)(x,y)\,ds_x = \frac{1}{\{2^{p-1}(p-1)!\}^2 \sigma_N}$$

when $N = 2p$.

(2)

$$\sum_{k=1}^{p}\int_{\partial\Omega}\left(\frac{\partial\overline{G}_{k-1}}{\partial\nu}\right)\left(\frac{\partial\overline{G}_{p-k}}{\partial\nu}\right)(x,y)\nu_i ds_x=\frac{\partial R}{\partial y_i}(y)$$

when $N\geq 2p$ and for $i=1,\cdots,N$.

(3)

$$2\sum_{k=1}^{p}\int_{\partial\Omega}\left(\frac{\partial\overline{G}_{k-1}}{\partial x_i}\right)\frac{\partial}{\partial y_j}\left(\frac{\partial\overline{G}_{p-k}}{\partial\nu_x}\right)(x,y)ds_x=$$

$$2\sum_{k=1}^{p}\int_{\partial\Omega}\left(\frac{\partial\overline{G}_{p-k}}{\partial x_i}\right)\frac{\partial}{\partial y_j}\left(\frac{\partial\overline{G}_{k-1}}{\partial\nu_x}\right)(x,y)ds_x=\frac{\partial^2 R}{\partial y_i\partial y_j}(y)$$

for $1\leq i,j\leq N$. Here $\nu=\nu(x)$ is the outer unit normal at $x\in\partial\Omega$.

Cf. Brezis-Peletier ($p=1$), Chou-Geng ($p=2$)

1. Nondegeneracy of critical points of the Robin function on symmetric domains

Let $\Omega \subset \mathbb{R}^N, (N \geq 2p)$ be a smooth bounded GNN domain:

(H1) Ω is symmetric w.r.t. hyperplanes $\{x_i = 0\}(i = 1, \cdots, N)$.

(H2) Ω is convex w.r.t. x_i-directions $(i = 1, \cdots, N)$.

Theorem (nondegeneracy of the critical point of R)

Let $\Omega \subset \mathbb{R}^N, (N \geq 2p)$ be a smooth bounded domain with (H1), (H2). Then we have

$$\nabla_x R(0) = 0, \quad \frac{\partial^2 R}{\partial x_i \partial x_j}(0) = \begin{cases} 0 & (i \neq j), \\ a_i > 0 & (i = j) \end{cases}$$

holds true.

Sketch of proof Assume Ω: symmetric w.r.t. $\{x_1 = 0\}$ and fix $y_0 \in \Omega \cap \{x_1 = 0\}$.

- (H1),(H2) $\Rightarrow \overline{G}_j(\cdot, y_0)$: even functions w.r.t. x_1 $(j = 0, 1, \cdots, p-1)$.
- $\frac{\partial \overline{G}_j}{\partial x_1}(\cdot, y_0)$: odd functions w.r.t. x_1 $(j = 0, 1, \cdots, p-1)$.

Let $u = u(x)$ be the unique solution of

$$\begin{cases} (-\Delta)^p u = 0 \quad \text{in } \Omega, \\ (-\Delta)^j u = -\left(\frac{\partial \overline{G}_j}{\partial x_1}\right)(\cdot, y_0) \quad \text{on } \partial\Omega. \ (j = 0, 1, \cdots, p-1) \end{cases}$$

- (H1)(H2) $\Rightarrow \left(\frac{\partial u}{\partial y_1}\right)(y_0) > 0, \left(\frac{\partial u}{\partial y_j}\right)(y_0) = 0, (j = 2, \cdots, N)$

Representation formula $\Rightarrow$

$$u(y) = \sum_{k=1}^{p} \int_{\partial\Omega} \frac{\partial \overline{G}_{k-1}}{\partial \nu_x}(x,y) \left(\frac{\partial \overline{G}_{p-k}}{\partial x_1} \right)(x,y_0) ds_x,$$

$$\therefore \frac{\partial u}{\partial y_j}(y) = \sum_{k=1}^{p} \int_{\partial\Omega} \frac{\partial}{\partial y_j} \left(\frac{\partial \overline{G}_{k-1}}{\partial \nu_x} \right)(x,y) \left(\frac{\partial \overline{G}_{p-k}}{\partial x_1} \right)(x,y_0) ds_x.$$

Pohozaev identities $\Rightarrow$

$$\frac{1}{2} \frac{\partial^2 R}{\partial y_1 \partial y_j}(y) = \sum_{k=1}^{p} \int_{\partial\Omega} \frac{\partial}{\partial y_j} \left(\frac{\partial \overline{G}_{k-1}}{\partial \nu_x} \right)(x,y) \left(\frac{\partial \overline{G}_{p-k}}{\partial x_1} \right)(x,y) ds_x$$

- $\therefore \frac{1}{2}\left(\frac{\partial^2 R}{\partial y_1^2}\right)(y_0) = \left(\frac{\partial u}{\partial y_1}\right)(y_0) > 0,$
  $\frac{1}{2}\left(\frac{\partial^2 R}{\partial y_1 \partial y_j}\right)(y_0) = \left(\frac{\partial u}{\partial y_j}\right)(y_0) = 0 \ (j = 2, \cdots, N).$

This proves Theorem. (Cf. M. Grossi (C.R.Acad, 2002) ($p = 1$))

Convexity of the Robin function

Theorem

Assume $\Omega \subset \mathbb{R}^N$ be a bounded convex domain. Then the Robin function of $-\Delta$ under the Dirichlet B.C. is strictly convex on Ω.

- Caffarelli-Friedman (Duke Math.1985): $N = 2$
- Cardaliaguet-Tahraoui (J. Math Purs. Appl. 2002): $N \geq 3$

Question

Does the same convexity hold true for the Robin function of $(-\Delta)^p$ under the Navier B.C.?

2. Nonexistence of multi-bubble solutions on convex domains

2p-th order mean field equation

$$(MFE)_p \begin{cases} (-\Delta)^p u = \rho \frac{V(x)e^u}{\int_\Omega V(x)e^u dx} & \text{in } \Omega \subset \mathbb{R}^{2p}, \\ (-\Delta)^j u = 0 & \text{on } \partial\Omega \ (j = 0, 1, \cdots p-1). \end{cases}$$

Proposition [C.S.Lin-J.C.Wei] ASNSP (2007)]

Assume $V \in C^{2,\beta}(\overline{\Omega}), \inf_\Omega V > 0$. Let u_{ρ_n} be a solution sequence s.t. $\|u_{\rho_n}\|_{L^\infty(\Omega)} \to \infty$ for $0 < \rho_n = O(1)$.
Then there exists a subsequence ρ_n and m-points set $\mathcal{S} = \{a_1, \cdots, a_m\} \subset \Omega$ (blow up set) s.t.

$$\rho_n \to 2\alpha_0(p)m, \ m \in \mathbb{N}. \quad \text{(mass quantization)}$$

$$u_{\rho_n} \to 2\alpha_0(p) \sum_{j=1}^{m} G(\cdot, a_j) \quad \text{in } C^{2p}_{loc}(\overline{\Omega} \setminus \mathcal{S}),$$

$$\rho_n \frac{V(x)e^{u_{\rho_n}}}{\int_\Omega V(x)e^{u_{\rho_n}} dx} \rightharpoonup 2\alpha_0(p) \sum_{i=1}^{m} \delta_{a_i}$$

in the sense of measures.
Furthermore, the set of blow up points $(a_1, \cdots, a_m)$ satisfies

$$\frac{1}{2}\nabla R(a_i) - \sum_{j=1, j\neq i}^{m} \nabla_x G(a_i, a_j) - \frac{1}{2\alpha_0(p)} \nabla \log V(a_i) = \vec{0} \ (i = 1, \cdots, m)$$

(Characterization of blow up points)

Here $\alpha_0(p)$ is the best constant for the Adams version Trudinger-Moser inequality:

$$\int_\Omega e^{\alpha u^2} dx \leq C(\Omega), \quad \forall \alpha \leq \alpha_0(p), \ \forall u \in C_0^\infty(\Omega)$$

with

$$\|\nabla^p u\|_{L^2(\Omega)} \leq 1,$$

where

$$\nabla^p u = \begin{cases} \nabla \Delta^{\frac{p-1}{2}} u \quad (p: \text{ odd}) \\ \Delta^{\frac{p}{2}} u \quad (p: \text{ even}) \end{cases}$$

Note: $\alpha_0(1) = 4\pi, \alpha_0(2) = 32\pi^2$.

Proposition (Existence of m-points blow up solutions ($p = 2$))

Let $p = 2$ and $m \geq 2$ be an integer. Set $\Omega^m = \Omega \times \cdots \times \Omega$ (m times) and $\Delta = \{(\xi_1, \cdots, \xi_m) \in \Omega^m | \xi_i = \xi_j \text{ for some } i \neq j\}$.
Define the Hamiltonian function

$$\mathcal{F}(\xi_1, \cdots, \xi_m) = \sum_{i=1}^{m} \left(R(\xi_i) - \frac{1}{32\pi^2} \log V(\xi_i) \right) - \sum_{\substack{i \neq j \\ 1 \leq i,j \leq m}} G(\xi_i, \xi_j)$$

on $\Omega^m \setminus \Delta$. If $\mathcal{F}$ has a

- nondegenerate critical point ($V \equiv 1$ case) [Baraket-Dammak-Ouni-Pacard (AIHP 2007)], or
- "minimax value in an appropriate subset" [Clapp-Munõz-Musso (AIHP 2008)]

i.e., if $(a_1, \cdots, a_m)$ satisfies

$$\frac{1}{2}\nabla R(a_i) - \sum_{j=1, j\neq i}^{m} \nabla_x G(a_i, a_j) - \frac{1}{64\pi^2}\nabla \log V(a_i) = \vec{0}$$

for $i = 1, 2, \cdots, m$ and "$+\alpha$" conditions,
then there exists a solution sequence $\{u_\rho\}$ which blows up exactly on $\mathcal{S} = \{a_1, \cdots, a_m\}$.

Remark

- If the cohomology group $H^d(\Omega) \neq 0$ for some $d \in \mathbb{N}$
$\Rightarrow \exists$ m-blow up solution sequence to (MFE) for any $m \geq 1$ and for any smooth $V > 0$.
- $\exists$ m-blow up solutions on m-dumbell shaped domain.
[Clapp-Munõz-Musso (AIHP 2008)]

$$(MFE)_p \quad (-\Delta)^p u = \rho \frac{V(x)e^u}{\int_\Omega V(x)e^u dx} \quad \text{in } \Omega, \quad (-\Delta)^j u = 0 \quad \text{on } \partial\Omega \ (j =$$

Theorem (Nonexistence of multi-bubble solutions)

$\Omega \subset \mathbb{R}^{2p}$ be a bounded convex domain. Let $\{u_{\rho_n}\}$ be a solution sequence to (MFE) s.t. $\|u_{\rho_n}\|_{L^\infty(\Omega)}$ is not bounded while $\rho_n > 0$ is bounded. Assume $\inf_\Omega V > 0$ and $R - \frac{1}{\alpha_0(p)} \log V$ is a strictly convex function on Ω.

Then the only accumulation point of $\{\rho_n\}$ is $2\alpha_0(p)$, and for some $a \in \Omega$,

$$\rho_n \frac{V(x)e^{u_{\rho_n}}}{\int_\Omega V(x)e^{u_{\rho_n}} dx} \rightharpoonup 2\alpha_0(p)\delta_a \quad (n \to \infty).$$

Key lemma

New Pohozaev identity for the Green function of $(-\Delta)^p$

$\Omega \subset \mathbb{R}^N$ $(N \geq 2p)$ be a smooth bounded domain. For any $P \in \mathbb{R}^N$ and $a, b \in \Omega, a \neq b$, it holds

$$\sum_{k=1}^{p} \int_{\partial\Omega} (x - P) \cdot \nu(x) \left(\frac{\partial(-\Delta)^{p-k} G_a}{\partial \nu_x} \right) \left(\frac{\partial(-\Delta)^{k-1} G_b}{\partial \nu_x} \right) ds_x$$
$$= (2p - N) G(a, b) + (P - a) \cdot \nabla_x G(a, b) + (P - b) \cdot \nabla_x G(b, a),$$

where $G_a(x) = G(x, a), G_b(x) = G(x, b)$ and $\nu(x)$ is the unit outer normal at $x \in \partial\Omega$.

Proof of Key Lemma

- $G_a(x) = G(x,a), G_b(x) = G(x,b)$
- $w(x) = (x-P)\cdot\nabla G_a(x)$

$\Rightarrow$

$$\begin{cases}(-\Delta)^p w(x) & = (x-P)\cdot\nabla\delta_a(x) + 2p\delta_a(x),\\ (-\Delta)^p G_b(x) & = \delta_b(x).\end{cases}$$

Multiplying $G_b(x)$, $w(x)$ respectively, and subtracting $\Rightarrow$

$$\int_\Omega ((-\Delta)^p w(x))\, G_b(x) - ((-\Delta)^p G_b(x))\, w(x) dx$$
$$= \int_\Omega \{(x-P)\cdot\nabla\delta_a(x)G_b(x) + 2p\delta_a(x)G_b(x) - \delta_b(x)w(x)\}\, dx.$$

$$
\begin{aligned}
\text{LHS} &= (-1)^p \sum_{k=1}^{p} \int_{\partial\Omega} \left(\frac{\partial \Delta^{p-k} w}{\partial \nu} \underbrace{\Delta^{k-1} G_b(x)}_{=0} ds_x \right) \\
&\quad - (-1)^p \sum_{k=1}^{p} \int_{\partial\Omega} \left(\frac{\partial \Delta^{k-1} G_b}{\partial \nu} \underbrace{\Delta^{p-k} w(x)}_{=2(p-k)\Delta^{p-k} G_a + (x-P)\cdot\nabla\Delta^{p-k} G_a} \right) ds_x \\
&= (-1)^{p+1} \sum_{k=1}^{p} \int_{\partial\Omega} (x-P) \cdot \nabla \Delta^{p-k} G_a \frac{\partial \Delta^{k-1} G_b}{\partial \nu}(x) ds_x \\
&= \sum_{k=1}^{p} \int_{\partial\Omega} (x-P) \cdot \nu(x) \left(\frac{\partial (-\Delta)^{p-k} G_a}{\partial \nu_x} \right) \left(\frac{\partial (-\Delta)^{k-1} G_b}{\partial \nu_x} \right) ds_x .
\end{aligned}
$$

$$
\begin{aligned}
\text{RHS} &= 2pG_b(a) - w(b) + \int_\Omega (x - P)\cdot\nabla\delta_a(x)G_b(x)dx \\
&= 2pG_b(a) - w(b) + \sum_{i=1}^{N}\int_\Omega (x_i - P_i)\frac{\partial\delta_a}{\partial x_i}G_b(x)dx \\
&= 2pG_b(a) - w(b) - \sum_{i=1}^{N}\int_\Omega \frac{\partial}{\partial x_i}\{(x_i - P_i)G_b(x)\}\delta_a(x)dx \\
&= 2pG_b(a) - w(b) - \sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left\{(x_i - P_i)G_b(x)\right\}\Big|_{x=a} \\
&= (2p - N)G(a,b) + (P - a)\cdot\nabla_x G(a,b) + (P - b)\cdot\nabla_x G(b,a).
\end{aligned}
$$

Lemma is porved.

Proof of Theorem

Contradiction argument Assume $\exists \{a_1, \cdots, a_m\} \subset \Omega$ $(m \geq 2)$ satisfying

$$(*) \qquad \nabla_x \left(\frac{1}{2} R - \frac{1}{2\alpha_0(p)} \log V \right)(a_i) - \sum_{j=1, j\neq i}^{m} \nabla_x G(a_i, a_j) = \vec{0}$$

Set $K(x) = \frac{1}{2}R(x) - \frac{1}{2\alpha_0(p)} \log V(x)$.

Step 1

$P \in \Omega$ is chosen later. Multiplying $P - a_i$ to $(*)$ and summing up,

$$\sum_{i=1}^{m} (P - a_i) \cdot \nabla K(a_i) = \sum_{i=1}^{m} \sum_{j=1, j\neq i}^{m} (P - a_i) \cdot \nabla_x G(a_i, a_j)$$

$$= \sum_{1 \leq j < k \leq m} \left\{ (P - a_j) \cdot \nabla_x G(a_j, a_k) + (P - a_k) \cdot \nabla_x G(a_k, a_j) \right\}$$

Step 2 By Key lemma,

$$(P - a_j) \cdot \nabla_x G(a_j, a_k) + (P - a_k) \cdot \nabla_x G(a_k, a_j)$$

$$= \sum_{l=1}^{p} \int_{\partial\Omega} (x - P) \cdot \nu(x) \left(\frac{\partial(-\Delta)^{p-l} G(x, a_j)}{\partial \nu_x} \right) \left(\frac{\partial(-\Delta)^{l-1} G(x, a_k)}{\partial \nu_x} \right) ds_x$$

$$(+(N - 2p) G(a_j, a_k))$$

RHS is positive by the convexity of Ω, since

- $(x - P) \cdot \nu(x) > 0$
- $\frac{\partial(-\Delta)^{p-l} G(x,a_j)}{\partial \nu_x} < 0$, $\frac{\partial(-\Delta)^{l-1} G(x,a_k)}{\partial \nu_x} < 0$ $(x \in \partial\Omega)$

Thus

$$\sum_{i=1}^{m} (a_i - P) \cdot \nabla K(a_i) < 0 \qquad (1)$$

Step 3 By Assumption, $K(x) = \frac{1}{2}R(x) - \frac{1}{2\alpha_0(p)} \log V(x)$ is strictly convex.

Thus, all level sets of K is strictly star-shaped w.r.t. its unique minimum point $P \in \Omega$:

$$(a - P) \cdot \nabla K(a) \geq 0, \quad \forall a \in \Omega \setminus \{P\}.$$

In particular,

$$(2) \qquad \sum_{i=1}^{m} (a_i - P) \cdot \nabla K(a_i) \geq 0.$$

(1) and (2) implies a Contradiction!

3. Asymptotic nondegeneracy of multi-bubble solutions to a 4-th order problem

4-th order Liouville-Gel'fand problem

$$(LG)\begin{cases}\Delta^2 u = \lambda e^u & \text{in } \Omega,\\ u = \Delta u = 0 & \text{on } \partial\Omega\end{cases}$$

- $\Omega \subset \mathbb{R}^4$ is a smooth bounded domain,
- $\lambda > 0$ is a parameter.
- $\{u_n\}$ be a solution sequence to (LG) for $\lambda = \lambda_n \downarrow 0$

Result of Lin-Wei (2007) is applicable:

If $\lambda_n \int_\Omega e^{u_n} dx \to \alpha \in (0, +\infty)$ as $n \to \infty$

$\Rightarrow \exists m \in \mathbb{N}$ s.t. $\alpha = 64\pi^2 m$ and $\exists \mathcal{S} = \{a_1, \cdots, a_m\} \subset \Omega$

1. $u_n \to 64\pi^2 \sum_{i=1}^m G(\cdot, a_i)$
2. $a_i \in \Omega$ (interior point) and satisfies

$$\frac{1}{2}\nabla R(a_i) - \sum_{\substack{j=1\\ j\neq i}}^{m} \nabla_x G(a_i, a_j) = 0, \quad (i = 1, \cdots, m)$$

i.e. $(a_1, \cdots, a_m)$ be a crtical point of the Hamiltonian

$$\mathcal{F}(\xi_1, \cdots, \xi_m) = \sum_{i=1}^{m} R(\xi_i) - \sum_{\substack{1\leq i,j\leq m,\\ i\neq j}} G(\xi_i, \xi_j)$$

for $(\xi_1, \cdots, \xi_m) \in \Omega^m$,

Theorem [Ohtsuka-TF] (in preparation)

Let $\{u_n\}$ be a blowing-up solution sequence to (LG) for $\lambda = \lambda_n$ whose set of blow up points is $\mathcal{S} = \{a_1, \cdots, a_m\}$.
If $(a_1, \cdots, a_m) \in \Omega^m$ is a nondegenerate critical point of the Hamiltonian $\mathcal{F}$, then u_n is nondegenerate for n large.

Cf.

- Gladiali-Grossi (CPDE 2005): 2nd order case, one point blow-up
- Grossi-Ohtsuka-Suzuki (ADE 2011): 2nd order case, multiple blow-up

Sketch of proof

Assume the contrary that $\exists\{v_n\}$: sol's to

$$\begin{cases} \Delta^2 v_n = \lambda_n e^{u_n} v_n & \text{in } \Omega, \\ v_n = \Delta v_n = 0 & \text{on } \partial\Omega \end{cases}$$

with $\|v_n\|_{L^\infty(\Omega)} \equiv 1$.

- $\mathcal{S} = \{a_1, \cdots, a_m\} \Rightarrow \exists\, \rho > 0$ and l sequences $\{x_n^i\}$ s.t.

$$u_n(x_n^i) = \max_{B_\rho(x_n^i)} u_n(x) \to \infty, \quad x_n^i \to a_i \quad (i = 1, \cdots, m)$$

- Define δ_n^i

$$(\delta_n^i)^4 \lambda_n e^{u_n(x_n^i)} \equiv 1 \quad \text{for } i = 1, \cdots, m \text{ and } n \in \mathbb{N}.$$

- Define the rescaled function

$$\tilde{v}_n^i(y) = v_n(\delta_n^i y + x_n^i), \quad \text{in } y \in B_{\frac{\rho}{\delta_n^i}}(0)$$

We see for $i = 1, \cdots, m$,

$$\tilde{v}_n^i(y) \to V^i(y) = \sum_{k=1}^{4} a_k^i \frac{y_k}{(8\sqrt{6} + |y|^2)} + b^i \left(\frac{8\sqrt{6} - |y|^2}{8\sqrt{6} + |y|^2} \right)$$

as $n \to \infty$ for some $\vec{a}^i = (a_1^i, \cdots, a_4^i) \in \mathbb{R}^4$ and $b^i \in \mathbb{R}$.

Step 1. $\vec{a}^i = \vec{0}$ for all $i = 1, \cdots, m$.

Step 2. $b^i = 0$ for all $i = 1, \cdots, m$.

Step 3. $V^i \equiv 0 (i = 1, \cdots, m)$ leads to a contradiction.

For the proof of Step 1, we need

New integral identity for Green's function of Δ^2

For $a, b, c \in \Omega$ and $r > 0$ small such that $\overline{B_r(a)} \subset \Omega$, $b, c \notin \overline{B_r(a)}$ if $b, c \neq a$, put

$$
\begin{aligned}
& I_{ij}(a,b,c) := \\
& = \int_{\partial B_r(a)} \left\{ G_{x_i}(x,b) \frac{\partial}{\partial \nu} \overline{G}_{z_j}(x,c) - G_{z_j}(x,c) \frac{\partial}{\partial \nu} \overline{G}_{x_i}(x,b) \right\} ds_x \\
& + \int_{\partial B_r(a)} \left\{ \overline{G}_{x_i}(x,b) \frac{\partial}{\partial \nu} G_{z_j}(x,c) - \overline{G}_{z_j}(x,c) \frac{\partial}{\partial \nu} G_{x_i}(x,b) \right\} ds_x
\end{aligned}
$$

for $1 \leq i, j \leq 4$, where $\nu(x) = \frac{x-a}{|x-a|}$ denotes the unit outer normal and $G_{x_i}(x,z) = \frac{\partial G}{\partial x_i}(x,z)$ etc.

Then I_{ij} does not depend on $r > 0$ small and we have

$$\begin{cases} I_{ij}(a,b,c) = 0, & \text{if } a \neq b \text{ and } a \neq c \\ I_{ij}(a,a,a) = \frac{1}{2}R_{x_i x_j}(a), & \\ I_{ij}(a,a,c) = -G_{x_i z_j}(a,c), & \text{if } a \neq c \\ I_{ij}(a,b,a) = -G_{x_i x_j}(a,b), & \text{if } a \neq b. \end{cases}$$

Thus,

$$\sum_{k=1}^{m} I_{ii'}(a_j, a_k, a_l) = \begin{cases} \frac{1}{2}R_{x_i x_{i'}}(a_j) - \sum_{\substack{1 \le k \le m \\ k \neq j}} G_{x_i x_{i'}}(a_j, a_k), & (j = l), \\ -G_{x_i z_{i'}}(a_j, a_l), & (j \neq l). \end{cases}$$
$$= \frac{1}{2} \frac{\partial^2}{\partial(\xi_j)_i \partial(\xi_l)_{i'}} \mathcal{F}(\xi_1, \cdots, \xi_m)\Big|_{(\xi_1,\cdots,\xi_m)=(a_1,\cdots,a_m)}$$

for any $j, l \in \{1, \cdots, m\}, i, i' \in \{1, 2, 3, 4\}$.

By an integral identity and asymptotic estimates

$$
\begin{aligned}
0 &= \int_{\partial B_r(a_j)} \left(\frac{\partial (\overline{u}_n)_{x_i}}{\partial \nu_x}\right) \frac{v_n}{\lambda_n^{1/4}} - \frac{\partial}{\partial \nu_x} \left(\frac{\overline{v}_n}{\lambda_n^{1/4}}\right) (u_n)_{x_i} ds_x \\
&+ \int_{\partial B_r(a_j)} \left(\frac{\partial (u_n)_{x_i}}{\partial \nu_x}\right) \frac{\overline{v}_n}{\lambda_n^{1/4}} - \frac{\partial}{\partial \nu_x} \left(\frac{v_n}{\lambda_n^{1/4}}\right) (\overline{u}_n)_{x_i} ds_x \\
&\to 512\pi^4 \sum_{k=1}^{m} \sum_{l=1}^{m} \sum_{i'=1}^{4} C_l a_{i'}^l I_{ii'}(a_j, a_k, a_l) \\
&= 256\pi^4 \sum_{i'=1}^{4} \sum_{l=1}^{m} C_l a_{i'}^l \frac{\partial^2}{\partial (\xi_j)_i \partial (\xi_l)_{i'}} \mathcal{F}(\xi_1, \cdots, \xi_m)\Big|_{(\xi_1,\cdots,\xi_m)=(a_1,\cdots,a_m)}
\end{aligned}
$$

as $n \to \infty$ for $\forall j \in \{1, \cdots, m\}, \forall i \in \{1, 2, 3, 4\}$ and for some $C_l > 0$.
Assumption that $(\mathrm{Hess}\mathcal{F})(a_1, \cdots, a_m)$ is invertible
$\Rightarrow$ all $a_{i'}^l = 0$ ($l = 1, \cdots, m$ and $i' = 1, \cdots, 4$).