(Non)local phase transitions and minimal surfaces

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Outline of the talk

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$$(-\Delta)^s u = \mathcal{F}^{-1}\big(|\xi|^{2s}(\mathcal{F}u)\big),$$

where $s \in (0,1)$ and $\mathcal{F}$ is the Fourier transform.
This definition is consistent with the case $s = 1$:

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An equivalent definition may be given by integrating against a singular kernel, which suitably averages a second-order incremental quotient:

$$-(-\Delta)^s u(x) = \int_{\mathbb{R}^n} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}}\,dy.$$

Up to a factor 2, this is the same as defining the operator as an integral in the principal value sense

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Motivation: the fractional Laplacian naturally surfaces in probability, water waves, and lower dimensional obstacle problems (among others). In statistical mechanics it is a way to take into account long-range particle interactions.

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Goal: Understand the geometric properties of the solutions of

$$(-\Delta)^s u + u - u^3 = 0.$$

When $s = 1$, the equation

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is named after Allen-Cahn (or Ginzburg-Landau, or Modica-Mortola...) and it is a model for phase transitions.
The pure phases correspond to $u \sim +1$ and $u \sim -1$.
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Let u be a smooth bounded solution of

$$-\Delta u + u - u^3 = 0$$

in $\mathbb{R}^n$, with

$$\partial_{x_n} u > 0.$$

Is it true that u depends only on one Euclidean variable?

I.e. $\exists u_o : \mathbb{R} \to \mathbb{R}$ and $\omega \in S^{n-1}$ such that $u(x) = u_o(\omega \cdot x)$?

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The answer is YES when $n \leq 3$ and NO when $n \geq 9$.

The answer is also YES when $n \leq 8$ and

$$\lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1.$$

The answer is also YES for any n if

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In this case, Cabré and Solà-Morales proved that the answer is YES when $n = 2$ and $s = 1/2$.

Also YES when $n = 2$ and any $s \in (0, 1)$ (Cabré, Sire and V.) and when $n = 3$ and $s \in [1/2, 1)$ (Cabré and Cinti).

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For the fractional Laplacian, here is a density estimate (Savin and V.):

If u_ϵ minimizes $\mathcal{F}_\epsilon$ in B_r and $u(0)=0$ then

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provided that $\epsilon \leq cr$.

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The scaling of the energy functional $\mathcal{F}_\epsilon$ is chosen to satisfy the following asymptotics:

If $s \in [1/2, 1)$, the functional $\mathcal{F}_\epsilon$ Γ-converges to the perimeter functional.

If $s \in (0, 1/2)$, the functional $\mathcal{F}_\epsilon$ Γ-converges to the nonlocal perimeter functional (as introduced by Caffarelli, Roquejoffre and Savin).

A minimizer u_ϵ converges a.e. to a step function $\chi_E - \chi_{\mathbb{R}^n \setminus E}$, and the level sets of u_ϵ converge to ∂E locally uniformly.

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For Γ-convergence of nonlocal functionals related with phase transitions, see also Alberti, Bellettini, Bouchitté, Garroni, González, Seppecher, etc.

For any $s \in (0, 1/2)$ the s-perimeter of a set E inside a given domain Ω is defined by

$$\begin{aligned} \mathrm{Per}_s(E,\Omega) \quad := \quad & \int_{E\cap\Omega}\int_{(\mathcal{C}E)\cap\Omega}\frac{1}{|x-y|^{n+2s}}\,dy\,dx \\ & + \int_{E\cap\Omega}\int_{(\mathcal{C}E)\cap(\mathcal{C}\Omega)}\frac{1}{|x-y|^{n+2s}}\,dy\,dx \\ & + \int_{E\cap(\mathcal{C}\Omega)}\int_{(\mathcal{C}E)\cap\Omega}\frac{1}{|x-y|^{n+2s}}\,dy\,dx, \end{aligned}$$

where $\mathcal{C}$ means the complement (see Caffarelli, Roquejoffre and Savin).

Then (see Caffarelli and V.), if E is a smooth set,

$$\lim_{s\to(1/2)^-} s(1-2s)\mathrm{Per}_s(E,B_r) = \mathrm{Per}(E,B_r)$$

for a dense set of r's.

Also, if E_k are minimal for Per_{s_k} and $s_k \to (1/2)^-$, then E_k converges to some set E which is minimal for Per.

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It would be desirable to better understand the behavior of nonlocal minimal perimeter sets and to exploit their rigid (?) geometry in order to obtain information on the level sets of u...