On the existence of a positive solution for the elliptic equation involving the multiple Hardy-Sobolev critical terms on the boundary

Hidemitsu Wadade

Department of Applied Physics, Waseda University (JSPS post-doctoral fellow)

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Brézis-Nirenberg (1983)

$n \geq 3$, $\Omega \subset \mathbb{R}^n$ **: bounded,** $2 < p < \frac{2n}{n-2}$. **Then**

$$\begin{cases} \Delta u + u^{p-1} + u^{\frac{2n}{n-2}-1} = 0, & u > 0 \quad \textbf{in } \Omega, \\ & u = 0 \quad \textbf{on } \partial\Omega \end{cases} \tag{1}$$

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- **Remark** $\begin{cases} \Delta u + u^{\frac{2n}{n-2}-1} = 0, & u > 0 \quad \text{in } \Omega, \\ & u = 0 \quad \text{on } \partial\Omega \end{cases}$

  has no positive solution by Pohozaev's identity if Ω **is star-shaped at some point** $x_0 \in \overline{\Omega}$.

Idea of Brézis-Nirenberg

Energy functional to (1): **for** $u \in H_0^1(\Omega)$**,**

$$\Phi(u\,;\,\Omega) := \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{1}{p}\int_\Omega (u^+)^p dx - \frac{n-2}{2n}\int_\Omega (u^+)^{\frac{2n}{n-2}} dx.$$

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For $u_0 \in H_0^1(\Omega) \setminus \{0\}$ **non-negative with** $\Phi(u_0\,;\,\Omega) < 0$,
define the min-max value :

$$c^*(u_0) := \inf_{P \in \mathcal{P}} \max_{w \in P} \Phi(w\,;\,\Omega),$$

where $\mathcal{P}$ **is the class of continuous paths connecting** 0 **and** u_0**.**

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Brézis-Nirenberg proved :

If $c^*(u_0) < \frac{1}{n} S_n^{\frac{n}{2}}$, **then** $c^*(u_0)$ **becomes a critical value.**

Sobolev-Hardy's inequality

$$S_n := \inf \left\{ \int_\Omega |\nabla u|^2 dx \;\middle|\; u \in H_0^1(\Omega) \text{ and } \int_\Omega |u|^{\frac{2n}{n-2}} dx = 1 \right\}$$

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$n \geq 3$, $\Omega \subset \mathbb{R}^n$ **: bounded with** $0 \in \overline{\Omega}$, $\exists C = C(n, |\Omega|)$ **s.t.**

$$\left(\int_\Omega \frac{|u|^p}{|x|^s} dx\right)^{\frac{1}{p}} \leq C\left(\int_\Omega |\nabla u|^2 dx\right)^{\frac{1}{2}}$$

for all $u \in H_0^1(\Omega)$, $0 \leq s \leq 2$ **and** $2 \leq p \leq 2^*(s) = \frac{2(n-s)}{n-2}$.

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The case $0 \in \Omega$

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Energy functional to (2):

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Property of Hardy-Sobolev best constant $(0 < s \leq 2)$

- **If** $0 \in \Omega$, $\mu_s(\Omega) = \mu_s(\mathbb{R}^n)$ **is independent of** Ω.
- **If** $0 \in \partial\Omega$, $\mu_s(\Omega)$ **depends on** Ω.

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Theorem 1 (Hsia-Lin-W., 2010)

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Theorem 2 (W., 2011)

$n \geq 3$, $\Omega \subset \mathbb{R}^n$ **: bounded with** $0 \in \partial\Omega$, $l \in \mathbb{N}$.

(i) $\{P_i\}_{i=1}^{l} \subset \partial\Omega$, $0 < s < 2$.

$$\begin{cases} \Delta u + u^{\frac{2n}{n-2}-1} + \sum_{i=1}^{l} \frac{u^{2^*(s)-1}}{|x-P_i|^s} = 0, \quad u > 0 \quad \text{in } \Omega, \\ \qquad\qquad\qquad\qquad u = 0 \quad \text{on } \partial\Omega \end{cases}$$

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$H(P_{i_0}) < 0$ **for some** $i_0 \in \{1, \cdots, l\}$.

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(ii) $P \in \partial\Omega$, $\{s_i\}_{i=1}^{l} \subset (0,2)^l$.

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Energy functional to (4) :

$$\Phi_s^*(u\,;\,\Omega) := \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{(n-2)}{2n}\int_\Omega (u^+)^{\frac{2n}{n-2}} dx - \frac{1}{2^*(s)}\int_\Omega \frac{(u^+)^{2^*(s)}}{|x|^s} dx.$$

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Consider the existence of an entire solution to the equation:

$$\begin{cases} \Delta v + v^{\frac{2n}{n-2}-1} + \dfrac{v^{2^*(s)-1}}{|x|^s} = 0, \quad v > 0 \quad \textbf{in } \mathbb{R}^n_+, \\ \qquad\qquad\qquad\qquad\qquad\quad\;\; v = 0 \quad \textbf{on } \partial\mathbb{R}^n_+. \end{cases} \tag{5}$$

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We say $v_0 \in H^1_0(\mathbb{R}^n_+)$ is a least energy solution to (5) if

$$\begin{cases} v_0 \in H^1_0(\mathbb{R}^n_+) \text{ is a solution to } (5) \text{ and} \\ \Phi_s^*(v_0\,;\,\mathbb{R}^n_+) = \inf\{\Phi_s^*(v\,;\,\mathbb{R}^n_+) \mid v \in H^1_0(\mathbb{R}^n_+) \text{ is a solution to } (5)\}. \end{cases}$$

Lemma 1.

$$\begin{cases} \Delta v + v^{\frac{2n}{n-2}-1} + \frac{v^{2^*(s)-1}}{|x|^s} = 0, & v > 0 \quad in\ \mathbb{R}^n_+, \\ & v = 0 \quad on\ \partial\mathbb{R}^n_+ \end{cases}$$

has a least energy solution.

Lemma 2.

$v_0 \in H^1_0(\mathbb{R}^n_+)$ *: a least energy solution by Lemma 1, i.e.,*

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$\Omega \subset \mathbb{R}^n$ *: bounded with* $0 \in \partial\Omega$ *and* $H(0) < 0$*, where* $H(0)$ *: the mean curvature at* 0*. Then* $\exists u \in H^1_0(\Omega)$ *non-negative s.t.* $\Phi^*_s(u\,;\,\Omega) < 0$ *and*

$$c^*(u) \le \max_{0 \le t \le 1} \Phi^*_s(tu\,;\,\Omega) < \Phi^*_s(v_0\,;\,\mathbb{R}^n_+).$$