Asymptotic Behavior of Singular Solutions for a Semilinear Parabolic Equation

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We consider singular solutions of the Fujita equation

$$u_t = \Delta u + u^p \quad \text{in } \mathbb{R}^N,$$

where $N > 2$ and $p > 1$.

1. Singular steady states
2. Time-dependent singular solutions
3. Asymptotic behavior of singular solutions

1. Singular steady states

It has been known that if $N > 2$ and $p > p_{sg} := \dfrac{N}{N-2}$, then the equation

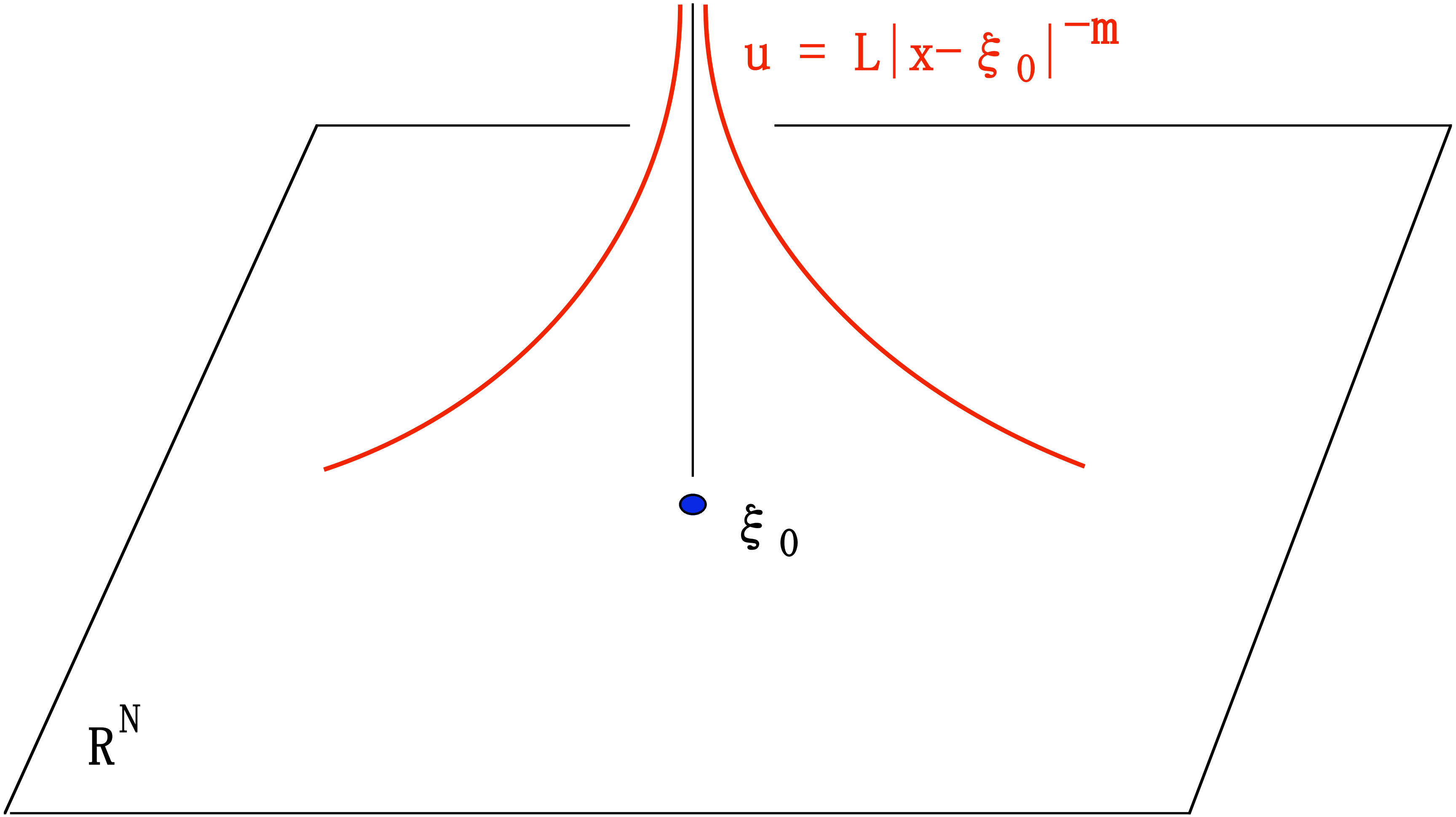
$$u_t = \Delta u + u^p, \qquad x \in \mathrm{R}^N,$$

has a singular steady state

$$u = \varphi_\infty(r) := L r^{-m}, \quad r := |x - \xi_0|,$$

where $\xi_0 \in \mathrm{R}^N$ is arbitrary and

$$m := \frac{2}{p-1}, \qquad L := \left\{m(N-m-2)\right\}^{\frac{1}{p-1}}.$$



Singular steady state

Concerning other singular solutions, the exponents

$$p_* := \frac{N + 2\sqrt{N-1}}{N - 4 + 2\sqrt{N-1}}, \quad N > 2,$$

and

$$p_S := \frac{N+2}{N-2}, \quad N > 2$$

play crucial role.

(i) If $p_{sg} < p < p_S$, then for any $\alpha > 0$, the solution φ_α of

$$\begin{cases} \varphi_{rr} + \dfrac{n-1}{r}\varphi_r + \varphi^p = 0, & r > 0. \\ \lim\limits_{r \to \infty} r^{N-2}\varphi(r) = \alpha. \end{cases}$$

is positive for all $r > 0$ and $\varphi(r) \to \infty$ as $r \to 0$. Then $u = \varphi_\alpha(|x|)$ is a singular steady state.

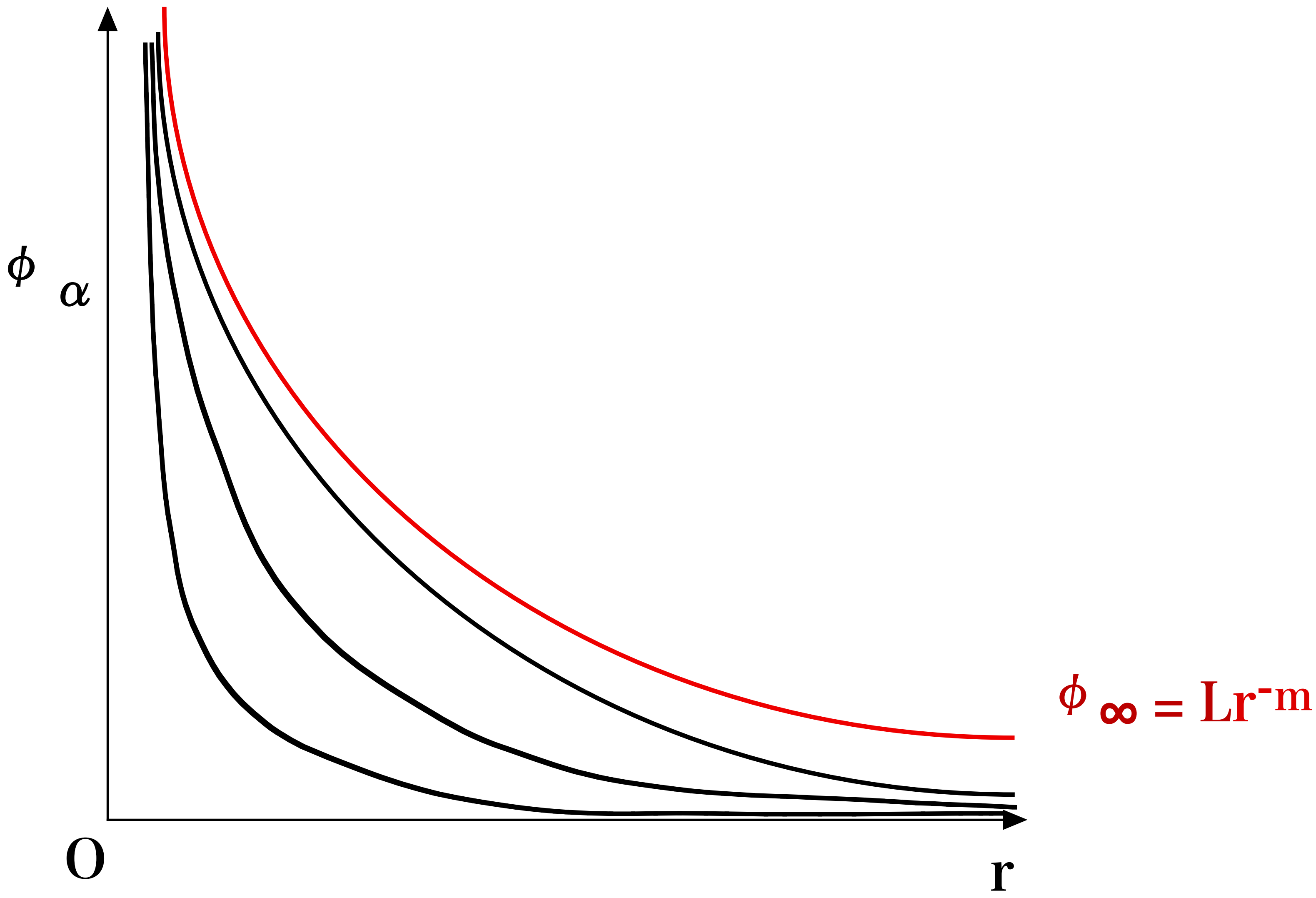
(ii) It was shown by Chen-Lin (1999) that for $p_{sg} < p < p_*$, $\{\varphi_\alpha\}$ the set of singular steady states $\{\varphi_\alpha\}$ has ordered structure (or separation property): $0 < \varphi_{\alpha_1}(r) < \varphi_{\alpha_2}(r) < \varphi_\infty(r)$ for all $0 < \alpha_1 < \alpha_2$ and $r > 0$. Moreover φ_α satisfies

$$\varphi_\alpha(r) = Lr^{-m} - a_\alpha r^{-\lambda_2} + o(r^{-\lambda_2}) \qquad \text{as } r \to 0,$$

where

$$\lambda_1 := \frac{N - 2 - \sqrt{(N-2)^2 - 4pL^{p-1}}}{2},$$
$$\lambda_2 := \frac{N - 2 + \sqrt{(N-2)^2 - 4pL^{p-1}}}{2}.$$

and $0 < \lambda_1 < \lambda_2 < m$. The constant a_α is positive and monotone decreasing in α and satisfies $a_\alpha \to 0$ as $\alpha \to \infty$. We note that $u = \varphi_\infty(|x|)$ and $u = \varphi_\alpha(|x|)$ satisfy the Fujita equation in the distribution sense.



Structure of the singular steady states

2. Time-dependent singular solutions

The singularity of $u = \varphi_\alpha$ and $u = \varphi_\infty$ persists for all $t > 0$, but it does not move in time.

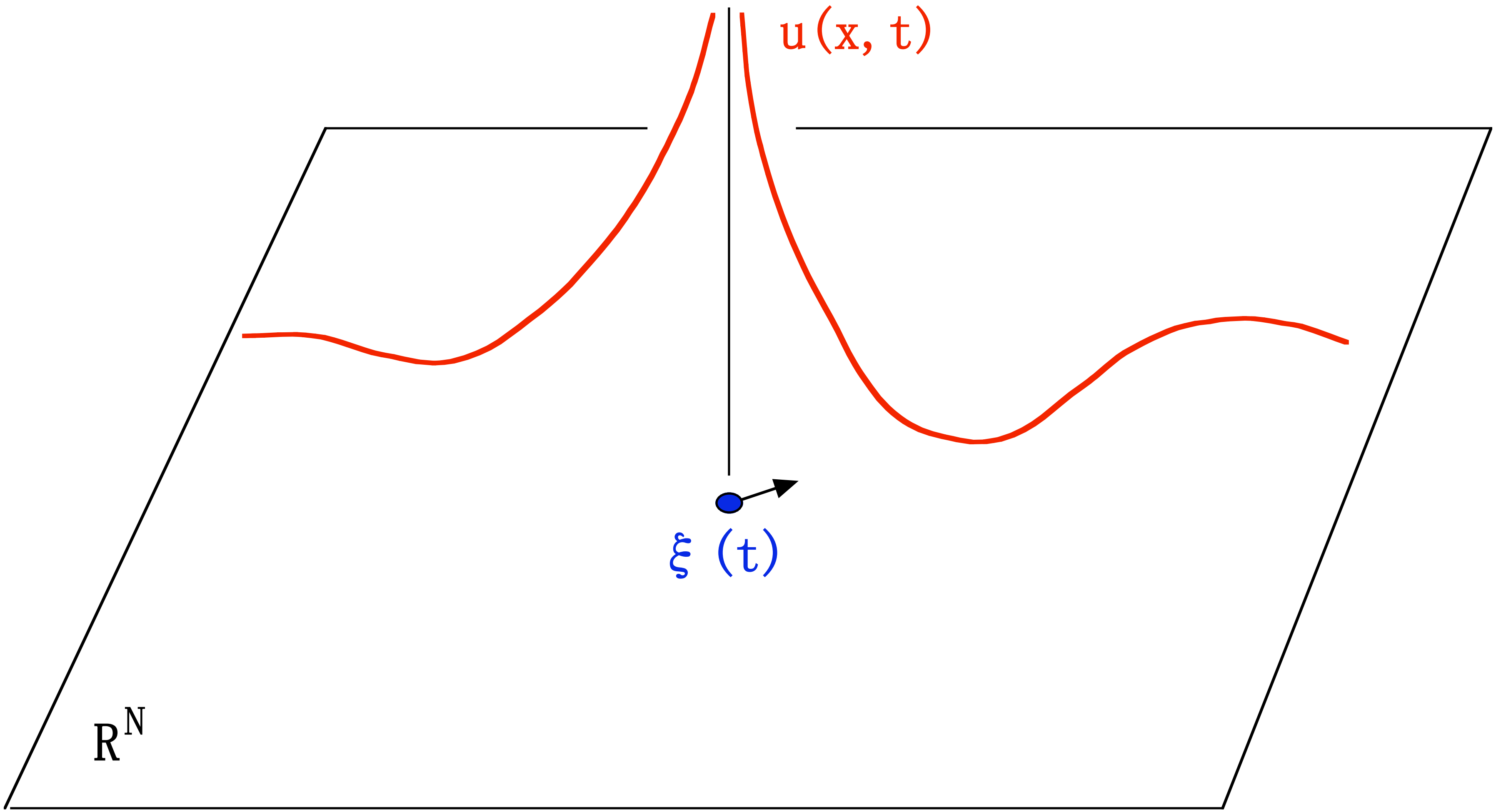
We define a solution with a (moving) singularity as follows.

Definition 1. $u(x,t)$ is a solution of the Fujita equation with a singularity at $\xi(t) \in \mathrm{R}^N$ if the following conditions are satisfied for some $T \in (0,\infty]$:

(i) $u(x,t)$ satisfies the equation in the distribution sense.

(ii) $u(x,t)$ is defined for $(x,t) \in \mathrm{R}^N \setminus \{\xi(t)\} \times [0,T)$, C^2 with respect to x, and C^1 with respect to t.

(iii) $u(x,t) \to \infty$ as $x \to \xi(t)$ for every $t \in [0,T)$.



Solution with a moving singularity

Consider the initial value problem

$$(\mathrm{P}) \qquad \begin{cases} u_t = \Delta u + u^p, & x \in \mathrm{R}^N \setminus \{\xi(t)\}, \quad t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in \mathrm{R}^N \setminus \{\xi(0)\}, \end{cases}$$

where $\xi(t) : [0, \infty) \to \mathrm{R}^N$ is prescribed.

[Assumptions]

(A1) $N \geq 3$ and $\frac{N}{N-2} < p < p_* := \frac{N + 2\sqrt{N-1}}{N - 4 + 2\sqrt{N-1}}$.

(A2) $\xi(t)$ is sufficiently smooth.

(A3) $u_0(x)$ is nonnegative and continuous in $x \in \mathrm{R}^N \setminus \xi(0)$, and is uniformly bounded for $|x - \xi(0)| \geq 1$.

(A4) $u_0(x) = Lr^{-m} + o(r^{-m})$ as $r = |x - \xi(0)| \to 0$.

Under the assumptions (A1) - (A4), the following results are obtained by Sato-Y (2009, 2010, 2011):

(i) (Time-local existence) For some time interval $[0, T)$, there exists a solution u of (P) with a singularity at $\xi(t)$ such that

$$u(x,t) = Lr^{-m} + o(r^{-\lambda_2})$$

as $r = |x - \xi(t)| \to 0$ for all $t \in [0, T)$.

(ii) (Uniqueness) If u_1 and u_2 are two solutions of (P) such that

$$|u_1(x,t) - u_2(x,t)| = o(r^{-\lambda_2})$$

as $r = |x - \xi(t)| \to 0$, then $u_1 \equiv u_2$.

(iii) (Comparison principle) If $u_1 \le u_2$ at $t = t_0$, then $u_1 \le u_2$ for $t > t_0$.

(iv) (Time-global existence) For some $\xi(t) \not\equiv Const.$ and $u_0(x)$, the solution exists globally in time and is asymptotically radially symmetric as $t \to \infty$.

(v) (Appearance of anomalous singularities) At some $t = T < \infty$, the leading term of u at $\xi(t)$ may become different from Lr^{-m}:

$$u(x,t) \simeq L|x - \xi(t)|^{-m} \quad \text{for } t \in (0, T),$$

$$u(x,t) \not\simeq L|x - \xi(t)|^{-m} \quad \text{at } t = T.$$

Why $\frac{N}{N-2} < p < p_*$?

Assume that a solution $u(x,t)$ with a singularity at $\xi(t)$ is close to the singular steady state $u = L|x-\xi(t)|^{-m}$, and formally expand the solution $u(x,t)$ at $r=0$ as follows:

$$u(x,t) = Lr^{-m} + \sum_{i=1}^{[m]} b_i(\omega,t) r^{-m+i} + v(y,t),$$

where

$$m = \frac{2}{p-1}, \quad y = x - \xi(t), \quad r = |y|, \quad \omega = \frac{1}{|y|} y \in S^{N-1}.$$

Substitute this expansion into the equation and equate each power of r to obtain a system of equations for $b_i(\omega,t)$.

These equations are solvable and the remainder term $v(y,t)$ must satisfy

$$v_t = \Delta v + \xi_t \cdot \nabla v + \frac{pL^{p-1}}{|y|^2} v + o(|y|^{-2}).$$

This equation is well-posed if and only if

$$0 < pL^{p-1} < \frac{(N-2)^2}{4}.$$

These inequalities hold if

$$N > 2 \quad \text{and} \quad \frac{N}{N-2} < p < p_* = \frac{N + 2\sqrt{N-1}}{N - 4 + 2\sqrt{N-1}}.$$

$$L := \left\{ \frac{2}{p-1} \left(N - \frac{2}{p-1} - 2 \right) \right\}^{\frac{1}{p-1}}.$$

3. Asymptotic behavior of singular solutions

Next, we investigate the asymptotic behavior of singular solutions that are time-dependent but the singular point is fixed to the origin (i.e., $\xi(t) \equiv 0$).

$$\text{(P)} \qquad \begin{cases} u_t = \Delta u + u^p, & x \in \mathbf{R}^N \setminus \{0\}, \quad t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in \mathbf{R}^N \setminus \{0\}. \end{cases}$$

We shall show

(i) More precise results for the time-local existence.

(ii) Convergence to a singular steady state φ_α.

(iii) Convergence to the singular steady state φ_∞.

[Existence of a singular solution]

Theorem 1. Let $N > 2$, $p_{sg} < p < p_*$ and $a(t) \in C^1([0,\infty))$ be given. Assume that

$u_0(x)$ is continuous and positive for $x \neq 0$,

$u_0(x)$ is uniformly bounded for $|x| > 1$,

$u_0(x) = L|x|^{-m} + O(|x|^{-\lambda})$ as $|x| \to 0$ for ${}^{\exists}\lambda < \min\{m, \lambda_2 + 2\}$.

Then there exist $T > 0$ and a positive solution $u(x,t)$ of (P) defined on $\mathrm{R}^N \setminus \{0\} \times (0,T)$ with the following properties:

(i) $u(x,t)$ satisfies the equation in the distribution sense.

(ii) $u(x,t)$ is C^2 with respect to x and C^1 with respect to t.

(iii) $u(x,t) = L|x|^{-m} - a(t)|x|^{-\lambda_2} + o(|x|^{-\lambda_2})$ as $|x| \to 0$.

(iv) If u_1 and u_2 are solutions of (P) satisfying (i)-(iii) and $u_1 - u_2 = o(|x|^{-\lambda_2})$ as $|x| \to 0$, then $u_1 \equiv u_2$.

(iv) More general comparison principle holds:

If $u_1 \geq u_2$ at $t = 0$ and $a_1(t) \leq a_2(t)$ for $t > 0$, then $u_1 \geq u_2$ for $t > 0$.

Remark. $a(t)$ can be regarded as Dirichlet data. For solutions with a moving singularity, we only considered the case $a(t) \equiv 0$.

Outline of the proof

[Step 1] Find a supersolution and a subsolution that are suitable for our purpose.

[Step 2] Construct a sequence of approximate solutions between the supersolution and the subsolution by solving an initial-boundary value problem on an annular domain

$$D_n := \{x \in \mathrm{R}^N : \frac{1}{n} < |x| < n\}.$$

[Step 3] Extract a convergent subsequence by the Ascoli-Arzelà theorem. (The supersolution and the subsolution give an a priori bound of the approximate solutions.)

[Step 4] Show that the limiting function becomes a solution of (P) with desired properties.

[Convergence to the singular steady state φ_α]

Theorem 2. Let $N \geq 3$ and $p_{sg} < p < p_*$. Assume that the initial value $u_0(x) \in C(\mathrm{R}^N \setminus \{0\})$ satisfies

$$0 \leq u_0(x) \leq \varphi_\infty(|x|) \text{ for } x \in \mathrm{R}^N \setminus \{0\},$$

$$u_0(x) = L|x|^{-m} - a_\alpha |x|^{-\lambda_2} + O(|x|^{-\lambda}) \text{ as } |x| \to 0 \text{ for}$$

some $\lambda < \lambda_2$ and $\alpha > 0$.

Then the singular solution $u(x,t)$ of (P) with $a(t) \equiv a_\alpha$ exists globally in time and has the following properties:

(i) $0 < u(x,t) < \varphi_\infty(|x|)$ for all $(x,t) \in \mathrm{R}^N \setminus \{0\} \times (0,\infty)$.

(ii) $u(x,t) \to \varphi_\alpha(|x|)$ as $t \to \infty$ uniformly on any compact set in $\mathrm{R}^N \setminus \{0\}$.

Proof. The proof is based on the comparison method. We look for a supersolution and a subsolution of the form

$$u^+(x,t) := \min\{\varphi_\alpha(r) + CU(r,t),\ \varphi_\infty(r)\}$$

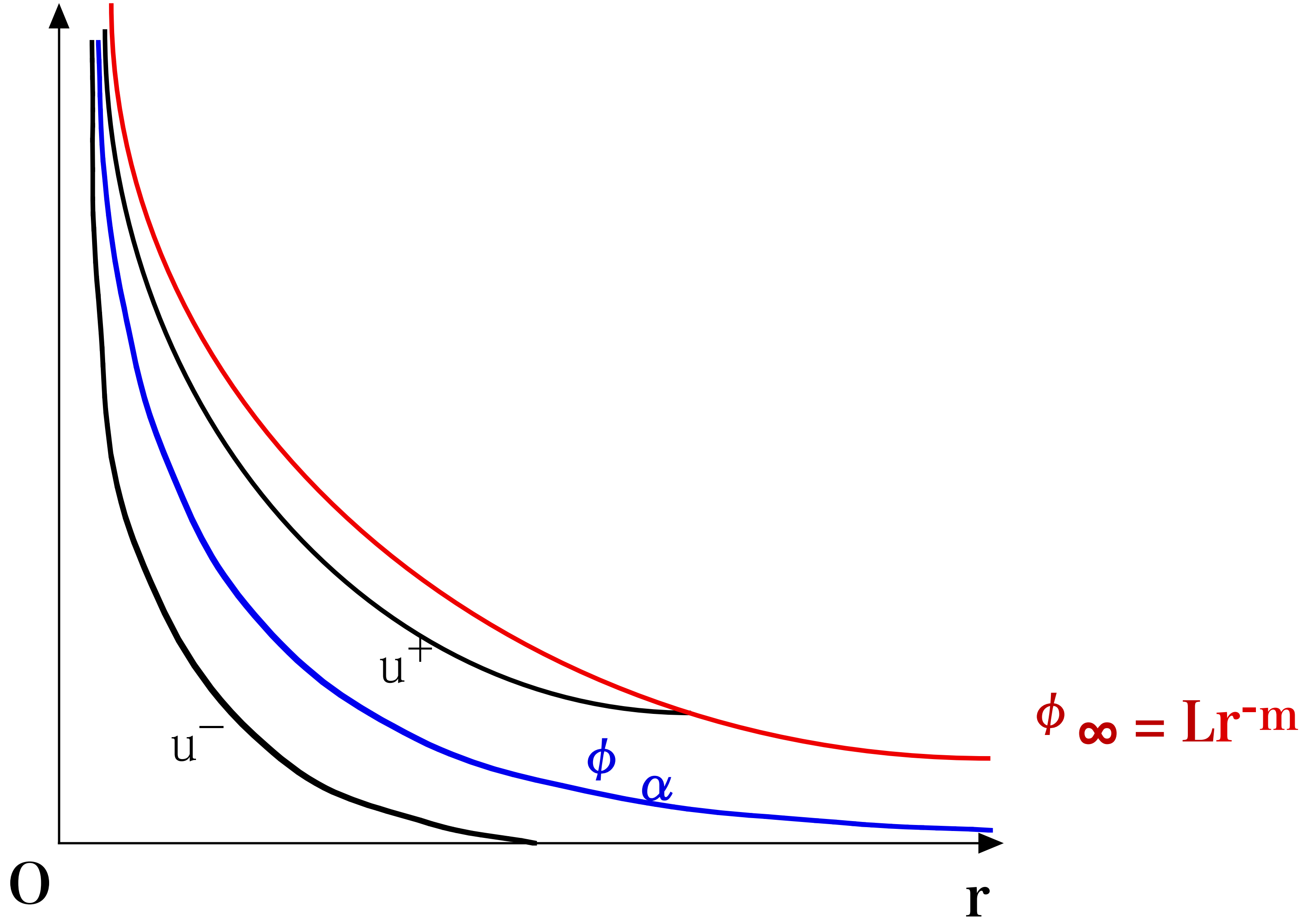
and

$$u^-(x,t) := \max\{\varphi_\alpha(r) - CU(r,t),\ 0\}$$

respectively, with some constant $C > 0$. Then the linearized equation at φ_α will appear naturally, and it turns out to be sufficient if U satisfies

$$U_t = U_{rr} + \frac{N-1}{r}U_r + \frac{l}{r^2}U, \qquad r > 0,\ t > 0$$

with some $0 < l < pL^{p-1}$. (Note that $l = pL^{p-1}$ corresponds to the linearized equation at φ_∞.)



Structure of the singular steady states

Let $\lambda_1(l)$ and $\lambda_2(l)$ be defined by

$$\lambda_1(l) := \frac{N-2-\sqrt{(N-2)^2-4l}}{2}, \quad \lambda_2(l) := \frac{N-2+\sqrt{(N-2)^2-4l}}{2}.$$

Setting $U(r,t) := r^{-\lambda_1(l)}V(r,t)$, the equation

$$U_t = U_{rr} + \frac{N-1}{r}U_r + \frac{l}{r^2}U, \qquad r>0,\ t>0$$

is rewritten as a generalized radial heat equation

$$V_t = V_{rr} + \frac{d-1}{r}V_r, \qquad r>0,\ t>0,$$

where

$$d := N - 2\lambda_1(l) = \lambda_2(l) - \lambda_1(l) + 2 > 2.$$

The generalized radial heat equation has been extensively studied in 1960's. Among others, we use a result by Bragg (1966).

Lemma. Let $d > 2$. For $0 < \gamma < 1$,

$$V(r,t) = \frac{(4t)^{1-\frac{d}{2}-\gamma} \exp\left(-\frac{r^2}{4t}\right)}{\Gamma\left(\frac{d}{2}+\gamma-1\right)} \int_0^1 \sigma^{-\gamma}(1-\sigma)^{\frac{d}{2}+\gamma-2} \exp\left(\frac{r^2}{4t}\sigma\right) d\sigma$$

is a solution of the radial heat equation with the following properties :

(i) $V(r,t) \in C^{2,1}([0,\infty) \times (0,\infty))$.

(ii) $V_r(0,t) = 0$ for all $t > 0$.

(iii) $V(r,t) \to r^{2-d-2\gamma}$ as $t \to 0$ locally uniformly in $r \in (0,\infty)$.

(iv) $r^{d+2\gamma-2} V(r,t) \to 1$ as $r \to \infty$ locally uniformly in $t \in [0,\infty)$.

(v) There exists a constant $C_1 \geq 1$ such that the inequalities $0 <$

$$V(r,t) \leq C_1 \min\{r^{2-d-2\gamma}, t^{1-\frac{d}{2}-\gamma}\}$$ hold for all $r, t > 0$.

Since

$$U(r,t) := r^{-\lambda_1(l)} V(r,t) \to 0 \qquad \text{as } t \to \infty,$$

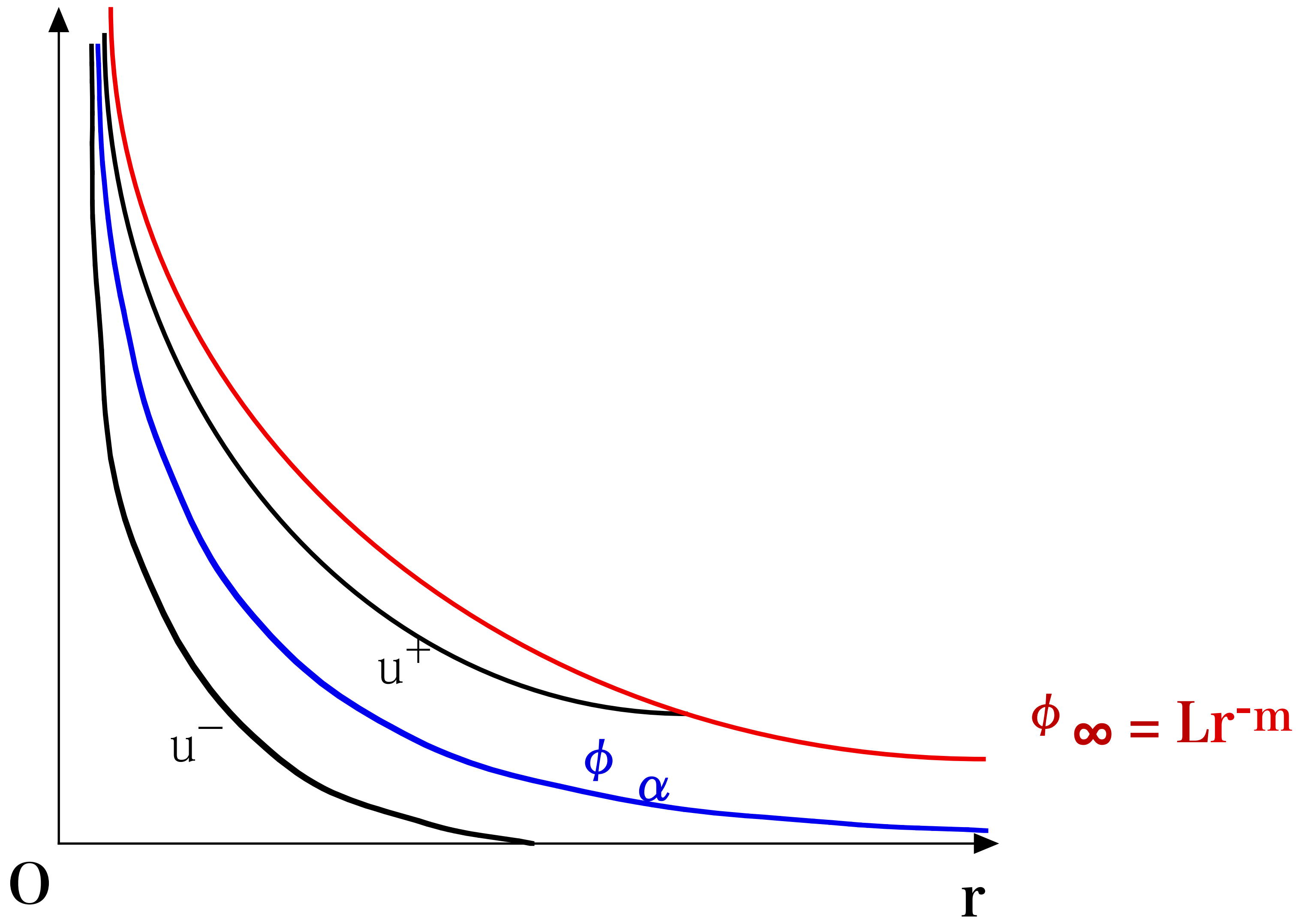
the supersolution

$$u^+(x,t) := \min\{\varphi_\alpha(r) + CU(r,t),\ \varphi_\infty(r)\}$$

and the subsolution

$$u^-(x,t) := \max\{\varphi_\alpha(r) - CU(r,t),\ 0\}$$

converge to φ_α **as** $t \to \infty$**. From the estimates for** V**,** u^+ **and** u^- **have other desired properties.**



Structure of the singular steady states

[Convergence from below to φ_∞]

Theorem 3. Let $N \geq 3$ and $p_{sg} < p < p_*$. Assume that the initial value $u_0(x) \in C(\mathbf{R}^N \setminus \{0\})$ satisfies

$$0 \leq u_0(x) \leq \varphi_\infty(|x|) \qquad \text{for} \quad x \in \mathbf{R}^N \setminus \{0\}$$

and

$$|x|^{\lambda_2}\{u_0(x) - L|x|^{-m}\} \to 0 \quad \text{as} \quad |x| \to 0.$$

Then the singular solution $u(x,t)$ of (P) with $a(t) \equiv 0$ satisfies $u(x,t) \to \varphi_\infty(|x|)$ as $t \to \infty$ uniformly on any compact set in $\mathbf{R}^N \setminus \{0\}$.

Proof. We can find a sequence of subsolutions $\{u_i^-(x,t)\}$ such that each $u_i^-(x,t)$ converges to φ_{α_i} as $t \to \infty$, and $\alpha_i \to \infty$ as $i \to \infty$. Thus, the solution eventually becomes larger than any φ_α. Recalling that $\varphi_\alpha \uparrow \varphi_\infty$ as $\alpha \to \infty$, the solution must satisfy $u(x,t) \to \varphi_\infty(|x|)$ as $t \to \infty$.

[Convergence from above to φ_∞]

Theorem 4. Suppose that

$$p_{sg} < p < \begin{cases} p_* & \text{for } 2 < N \le 10, \\ \dfrac{N+2}{N-1} & \text{for } N > 10. \end{cases}$$

Then there exists $L_1 > L$ such that if the initial value $u_0(x) \in C(\mathrm{R}^N \setminus \{0\})$ satisfies

$$L|x|^{-m} \le u_0(x) \le L_1|x|^{-m} \qquad \text{for } \ x \in \mathrm{R}^N \setminus \{0\},$$

then the singular solution $u(x,t)$ of (P) with $a(t) \equiv 0$ exists globally in time and has the following properties:

(i) $L|x|^{-m} \le u(x,t) \le L_1|x|^{-m}$ for $(x,t) \in \mathrm{R}^N \setminus \{0\} \times (0,\infty)$.

(ii) $u(x,t) \to \varphi_\infty(|x|)$ as $t \to \infty$ uniformly on any compact set in $\mathrm{R}^N \setminus \{0\}$.

We construct a supersolution by using a forward self-similar solution with a singularity at the origin. We have found that such a solution exists above the singular steady state φ_∞ if and only if

$$p_{sg} < p < \begin{cases} p_* & \text{for } N \le 10, \\ \dfrac{N+2}{N-1} & \text{for } N > 10. \end{cases}$$

Future works:

Convergence rate

Behavior of solutions in the case $a(t) \not\equiv Const.$

Asymptotic behavior in the case $\xi(t) \not\equiv Const.$

Bounded domain

Grazie per l'attenzione.