On the oriented degree for multivalued compact perturbations of Fredholm maps in Banach spaces

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We discuss the construction of a topological degree for a special class of multivalued locally compact perturbations of Fredholm maps between Banach spaces.

This notion tries to extend and simplify an analogous result given in

V. Obukhovskii, P. Zecca, V. Zvyagin, *An oriented coincidence index for nonlinear Fredholm inclusions with nonconvex-valued perturbations*, Abstr. Appl. Anal., Art. ID 51794, 21 p. (2006).

See also: **M. Väth**, *Topological Analysis: From the Basics to the Triple Degree for Nonlinear Fredholm Inclusions*, de Gruyter, Berlin, New York, 2012.

For the case of multivalued perturbations of the identity in a Banach space, we can see

L. Górniewicz, *Topological Fixed Point Theory of Multivalued Mappings - Second Edition*, Springer, Dordrecht, 2006.

What is the topological degree?

The degree is an integer number, associated to an equation

$$f(x) = y, \quad x \in U, \tag{1}$$

in order to obtain information about the set of solutions. In the above equation, we can imagine that (for example)

i) $f : X \to Y$ is a given function, supposed at least continuous,

ii a) X and Y could be Euclidean spaces or real, finite dimensional, differentiable manifolds or

ii b) Banach spaces or manifolds, possibly of infinite dimension,

iii) y is a fixed element of Y,

iv) U is an open subset of X.

In any of the previous context we determine a family $\mathcal{T}$ of *admissible triples*

$$\mathcal{T} = \{(f, U, y)\},$$

where $f : D(f) \subseteq X \to Y$ is continuous, $U \subseteq D(f)$ is (usually) open and $y \in Y$. A *topological (oriented) degree*, simply a *degree*, is a map

$$\mathrm{deg} : \mathcal{T} \to \mathbb{Z}$$

such that some particular properties are verified. Let us mention here the following two:

1. (*Existence*) given an admissible triple (f, U, y), if

   $$\mathrm{deg}(f, U, y) \neq 0,$$

   then the equation $f(x) = y$ has at least one solution in U.

2. (*Homotopy invariance*) given a continuous map $H : U \times [0, 1] \to Y$ such that $H^{-1}(y)$ is compact, then $\mathrm{deg}(H(\cdot, \lambda), U, y)$ does not depend on $\lambda \in [0, 1]$.

For continuous maps between Euclidean spaces of finite dimension the first construction is due to

– **L.E.J. Brouwer**, *Uber Abbildung von Mannigfaltigkeiten*, Math. Ann. **71** (1912), pp. 97–115.

and, by an analytic approach, to

– **M. Nagumo**, *A theory of degree of mapping based on infinitesimal analysis*, Amer. J. of Math., **73** (1951), 485–496.

It is commonly known as the Brouwer degree.

The Brouwer degree is estended to continuous maps between finite dimensional oriented manifolds.

In the book of **A. Dold**, *Lectures on algebraic topology,* Springer-Verlag, Berlin, 1972, we find an extension to nonorientable manifolds.

In infinite dimension, the first classical constructions are given for special classes of maps (not simply continuous) between Banach spaces, and are due to

J. Leray and **J. Schauder**, *Topologie et équations fonctionnelles*, Ann. Sci. École Norm. Sup., 51 (1934), 45–78,

and (a version of nonoriented degree is due to)

R. Caccioppoli, *Sulle corrispondenze funzionali inverse diramate: teoria generale e applicazioni ad alcune equazioni funzionali non lineari e al problema di Plateau*, Opere scelte, vol. II, Edizioni Cremonese, Roma, 1963, 157–177.

The Leray-Schauder degree is defined for the maps of the form

$$f : D(f) \subseteq E \to E, \qquad f(x) = x - k(x),$$

where E is a real Banach space, k is *completely continuous*.

The *admissible triples* are those (f, U, y) such that f is as above, y belongs to E and $U \subseteq E$ is open with $f^{-1}(y) \cap U$ compact.

The Leray–Schauder degree is based on an (implicit) concept of orientation in infinite dimension.

The subset $GL_c(E) \subseteq L(E)$ of the automorphisms of a Banach space E, of the form $I - K$, with K linear and compact, has two connected components.

(Notice that $GL(E)$ could be connected.)

One of the two components (clearly) contains the identity I. Call it $GL_c^+(E)$ and $GL_c^-(E)$ the other one. One has

$$\deg_{LS}(L, E, y) = 1, \qquad \forall L \in GL_c^+(E), \quad \forall y \in E.$$

In addition,

$$\deg_{LS}(M, E, y) = -1, \qquad \forall M \in GL_c^-(E), \quad \forall y \in E.$$

Other notions of degree

S. Smale (1965): nonlinear (C^2) Fredholm maps between Banach space. Degree in $\mathbb{Z}_2$ (with no use of orientation).

F. Browder and **R. Nussbaum** (1969): noncompact perturbations of the identity in a Banach space (using the Kuratowski measure of noncompacness).

K.D. Elworthy and **A.J. Tromba** (1970): *oriented degree* for nonlinear Fredholm maps of index zero between Banach manifolds (introducing the notion of orientation for an infinite dimensional manifold).

J. Mawhin (1972): *Coincidence degree:* for special perturbations of a linear Fredholm operator between Banach spaces.

V.G. Zvyagin and **N.M. Ratiner** (1991): following the concept of orientation of Elworty and Tromba, they define a degree for completely continuous perturbations of nonlinear Fredholm maps of index zero between Banach spaces.

See also **J. Pejsachowicz** (2007) for a discussion concerning orientation in infinite dimension.

P.M. Fitzpatrick, **J. Pejsachowicz** and **P.J. Rabier** (1991): *orientation of maps instead of spaces*. They introduce a degree for (oriented) nonlinear Fredholm maps of index zero between Banach spaces.

P. B. and **M. Furi** (1997): degree for nonlinear Fredholm maps of index zero between Banach manifolds with *a different notion of orientation maps* with respect to the previous one given by F. P. and R..

P. B. and **M. Furi** (2005): degree for locally compact perturbations (extended to condensing) perturbations of nonlinear Fredholm maps of index zero between Banach spaces.

P. Rabier and **M. Salter** (2005): (with a slight different approach) degree for completely continuous perturbations of nonlinear Fredholm maps of index zero between Banach spaces.

How can we define a degree for the locally compact multivalued perturbations of nonlinear Fredholm maps of index zero between Banach spaces?

The problem of orientation:

we define a concept of orientation for linear Fredholm operators of index zero between Banach spaces.

In particular, given E and F real Banach spaces, $L : E \to F$ linear Fredholm operator of index zero, we are able to associate:

1. an orientation to L,
2. a sign, $+1$ or -1 if L is an isomorphism.

Let $g : \Omega \to F$ be a (nonlinear) Fredholm map of index zero. Assume $Dg(x)$ oriented for any $x \in \Omega$.

By a notion of "continuous transport" of the orientation of $Dg(x)$, moving x in Ω, we define an *orientation* of g as a "continuous" choice of an orientation of $Dg(x)$ for any $x \in E$.

Which kind of multimaps we use?

Let E and F be two real Banach spaces and Ω be an open subset of E. Consider a multimap $K : \Omega \multimap F$.

We assume that:

1) $K(x)$ is a compact subset of F.

2) K is *upper semicontinuous* (*usc*), that is, for every open set $V \subseteq F$ the set $K_+^{-1}(V) = \{x \in \Omega : K(x) \subseteq V\}$ is open in E.

3) For any $x \in \Omega$ $K(x)$ is an R_δ-set, that is, it can be represented as the intersection of a decreasing sequence of compact and contractible spaces.

4) K is locally compact.

Definition 1. Let $\Sigma : X \multimap Z$ be a given multimap. Given a positive ε, a continuous map $f_\varepsilon : X \to Z$ is said to be an ε-*approximation* of Σ if for every $x \in X$ one has

$$f_\varepsilon(x) \in O_\varepsilon(\Sigma(O_\varepsilon(x))) \qquad \text{or} \qquad \Gamma_{f_\varepsilon} \subseteq O_\varepsilon(\Gamma_\Sigma),$$

where $O_\varepsilon(A)$ is the ball with center in (the set) A and radius ε, while Γ_{f_ε} and Γ_Σ denote the graphs of f_ε and Σ respectively.

Proposition 2 (see i.e. Górniewicz, 2006). *Let X be a compact (metric) ANR-space and Z a metric space. Consider a multimap $\Sigma : X \multimap Z$, verifiyng the above first three properties. Then:*

i) Σ is approximable, i.e. for every $\varepsilon > 0$ there exists an ε-approximation f_ε of Σ;

ii) for each $\varepsilon > 0$ there exists $\delta_0 > 0$ such that for every δ $(0 < \delta < \delta_0)$ and for every two δ-approximations f_δ, f'_δ of Σ, there exists a continuous homotopy $f_* : X \times [0,1] \to Z$ s.t.
(a) $f_*(\cdot, 0) = f_\delta$, $f_*(\cdot, 1) = f'_\delta$;
(b) $f_*(\cdot, \lambda)$ is an ε-approximation of Σ for all $\lambda \in [0,1]$.

Degree for locally compact multivalued perturbations of nonlinear Fredholm maps of index zero between Banach spaces.

We call these maps *quasi-Fredholm multimaps*.

Definition. Let $g : \Omega \to F$ be a Fredholm map of index zero and $K : E \to F$ a locally compact multimap (verifying the properties of the previous slide). The map $f : E \to F$, defined by $f = g - K$, is called a *quasi-Fredholm multimap* and g is a *smoothing map* of f.

The following definition provides an extension to quasi-Fredholm multimaps of the concept of orientation of Fredholm maps.

Definition. An *orientation* for a quasi-Fredholm multimap $f = g - K : E \to F$ is an orientation of the smoothing map g.

Let $f = g - K$ be an oriented quasi-Fredholm multimap, (f, U) an *admissible pair* that is, the coincidence set

$$S = \{x \in U : g(x) \in K(x)\}$$

is compact.

The construction of the degree is given in two steps.

Step 1. Suppose $K(U)$ contained in a finite dimensional subspace of F.

Let Z be a finite-dimensional subspace of F, containing $K(U)$, and W an open neighborhood of S in U, with g transverse to Z in W.

Assume Z oriented.

$M := g^{-1}(Z) \cap W$ is a C^1 manifold and $\dim M = \dim Z$.

M can be oriented with an orientation induced by the orientations of g and Z.

M contains S and $f_M : M \multimap Z$ is well defined.

We can obtain a suitable neighborhood V of S in M, where $\cup_{j=1}^{k} V_j$, with every V_j is diffeomorphic to an open ball of $\mathbb{R}^{\dim M}$.

Then, $\overline{V}$ is an ANR since, thanks to a result by W. Haver (1973), it is a finite union of locally contractible, compact, finite dimensional metric spaces.

Therefore, by Proposition 2, the restriction of K to $\overline{V}$ is approximable. More precisely, call d the distance in F

$$d := dist\,(0, f(\partial V)),$$

and consider a continuous map $k : \overline{V} \to Z$ which is a $d/2$-approximation of the restriction of K to $\overline{V}$. Then, is well defined

$$\deg_B(g - k, V, 0), \tag{2}$$

where the right hand side above denotes the Brouwer degree of the triple $(g - k, V, 0)$.

Can we define

$$\deg(f, U) = \deg_B(g - k, V, 0)? \tag{3}$$

It is an *open problem (... at least for us)*

We can prove, by a contradiction method that

$$\text{(4)} \qquad \deg(f, U) = \deg_B(g - h, V, 0),$$

is a correct definition, provided $h : \overline{V} \to Z$ is a continuous map, and ε-approximation of the restriction of K to $\overline{V}$, with ε sufficiently small.

The above definition is well posed in the sense that the right hand side of the formula is independent of the choice of the ANR V, the smoothing map g, the open set W and the subspace Z.

Step 2. We extend the definition of degree to general admissible pairs.

Definition (general definition of degree) Let (f, U) be an admissible pair. Consider:

1. a positively oriented smoothing map g of f;
2. a bounded open neighborhood W of $S = \{x \in U : g(x) \in K(x)\}$ (where $K := g - f$) in U such that $\overline{W} \subseteq U$, g is proper on $\overline{W}$ and $K|_{\overline{W}}$ is compact;
3. a multivalued map $\widetilde{K} : \overline{W} \to F$ having finite dimensional image and obtained as follows:
   a) there exists $\delta > 0$ such that $O_\delta(0) \cap f(\partial W) = \emptyset$ (in F);
   b) consider a continuous map $\phi : K(\overline{W}) \to F$, with finite-dimensional image and such that $\|\phi(x) - x\| < \delta$;
   c) define $\widetilde{K} := \phi \circ K$.

Then, define

$$\deg(f, U) = \deg(g - \widetilde{K}, W). \tag{5}$$

The right hand side of (5) is well defined and does not depend on g, $\widetilde{K}$ and W.

$*\quad *\quad *$

The degree verifies (in particular) the following three *fundamental properties*.

- (*Normalization*) Let $L\colon E \to F$ be a naturally oriented isomorphism. Then

$$\deg(L, E) = 1.$$

- (*Additivity*) Let (f, U) be an admissible multivalued pair, and U_1, U_2 two disjoint open subsets of U such that $S \subseteq U_1 \cup U_2$. Then,

$$\deg(f, U) = \deg(f|_{U_1}, U_1) + \deg(f|_{U_2}, U_2).$$

- (*Homotopy invariance*) Let $H\colon U \times [0,1] \multimap F$ be an oriented multivalued quasi-Fredholm homotopy. If S is compact, then $\deg(H_\lambda, U)$ does not depend on $\lambda \in [0,1]$.

Open problems (... at least, for us)

1) The degree extends the degree for quasi-Fredholm maps, i.e., locally compact perturbations of oriented Fredholm maps in Banach spaces (B.-Furi, Rabier-Salter). Can the degree for quasi-Fredholm multimaps be directly obtained by a generalization of the degree for quasi-Fredholm maps? A sufficient condition could be given by the following result:

Proposition Let $K : X \multimap Z$ be a multimap verifying the first three properties of page 11. Given a positive ε and a compact subset X' of X, then X' admits a neighborhood W such that K has a (single-valued) ε-approximation on W.

Is the above poposition true? (yes, if $K(x)$ is convex, but if not...?)

2) The degree is unique (as a map verifying the fundamental properties)?

3) A very old question: is the contruction of a coincidence degree for maps (or multimaps) in manifolds possible?

4) We are working about applications to nonlinear problems for differential inclusions.