A local minimum theorem and nonlinear differential problems

Gabriele Bonanno

University of Messina

FIRENZE 2014

International Workshop on

Topological and Variational Methods for ODEs

Dedicated to Massimo Furi Professor Emeritus at the University of Florence

Florence, Italy June 3 - 4, 2014

A LOCAL MINIMUM THEOREM

Our aim is to present a local minimum theorem for functionals of the type:

Φ $-$ Ψ

$$\Phi - \Psi$$

A LOCAL MINIMUM THEOREM

Let X be a real Banach space and $\Phi, \Psi : X \to \mathbb{R}$ two continuously Gâteaux differentiable functions. Put

$$I = \Phi - \Psi$$

and assume that there are $x_0 \in X$ and $r_1, r_2 \in \mathbb{R}$, with $r_1 < \Phi(x_0) < r_2$, such that

$$\sup_{u \in \Phi^{-1}(]r_1, r_2[)} \Psi(u) \leq r_2 - \Phi(x_0) + \Psi(x_0), \qquad (1)$$

$$\sup_{u \in \Phi^{-1}(]-\infty, r_1])} \Psi(u) \leq r_1 - \Phi(x_0) + \Psi(x_0). \qquad (2)$$

Moreover, assume that I satisfies ${}^{[r_1]}(PS)^{[r_2]}$-condition.

Then, there is $u_0 \in \Phi^{-1}(]r_1, r_2[)$ such that $I(u_0) \leq I(u)$ for all $u \in \Phi^{-1}(]r_1, r_2[)$ and $I'(u_0) = 0$.

BONANNO G., *A critical point theorem via the Ekeland variational principle*, Nonlinear Analysis, **75** (2012), 2992-3007.

It is an existence theorem of a critical point for continuously Gâteaux differentiable functions, possibly unbounded from below.

The approach is based on Ekeland's Variational Principle applied to a non-smooth variational framework by using also a novel type of Palais-Smale condition which is more general than the classical one.

PALAIS-SMALE CONDITION

Let X be a real Banach space, we say that a Gâteaux differentiable functional

$$I : X \to \boldsymbol{R}$$

verifies the Palais-Smale condition (in short (PS)-condition) if any sequence

$\{u_n\}$ *such that*

(α) $\{I(u_n)\}$ is bounded,

(β) $\lim_{n \to +\infty} \|I'(u_n)\|_{X^*} = 0$,

has a convergent subsequence.

Let X be a real Banach space and let $\Phi : X \to \mathbf{R}$, $\Psi : X \to R$ two Gâteaux differentiable functions. Put $I = \Phi - \Psi$.

Fix $r_1, r_2 \in [-\infty;+\infty]$, with $r_1 < r_2$, we say that the function I verifies the Palais-Smale condition cut off lower at r_1 and upper at r_2 (in short ${}^{[r_1]}(PS)^{[r_2]}$-condition) if any sequence $\{u_n\}$ such that

(α) $\{I(u_n)\}$ is bounded,

(β) $\lim\limits_{n\to+\infty} \|I'(u_n)\|_{X^*} = 0$,

(γ) $r_1 < \Phi(u_n) < r_2 \ \forall n \in \mathbf{N}$, *has a convergent subsequence.*

Clearly, if $r_1 = -\infty$ *and* $r_2 = +\infty$ *it coincides with the classical (PS)-condition.* *Moreover, if* $r_1 = -\infty$ *and* $r_2 \in \mathbf{R}$ *we denote it by* $(PS)^{[r_2]}$, *while if* $r_1 \in \mathbf{R}$ *and* $r_2 = +\infty$ *we denote it by* ${}^{[r_1]}(PS)$.

In particular,

If $I = \Phi - \Psi$ **satisfies** (PS)**-condition, then it satisfies** ${}^{[r_1]}(PS)^{[r_2]}$**-condition for all** $r_1, r_2 \in [-\infty, +\infty]$ **with** $r_1 < r_2$**.**

Proposition. *Let X be a reflexive real Banach space; $\Phi : X \to \mathbf{R}$ be a sequentially weakly lower semicontinuous, coercive and continuously Gâteaux differentiable function whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbf{R}$ be a continuously Gâteaux differentiable function whose Gâteaux derivative is compact.*

Then, for all $r_1, r_2 \in [-\infty, +\infty[$, with $r_1 < r_2$, the functional $\Phi - \Psi$ satisfies the ${}^{[r_1]}(PS)^{[r_2]}$-condition.

CLARKE

CHANG

To prove the local minimum theorem we use the theory for locally Lipschitz functionals investigated by K.C. Chang, which is based on the Nonsmooth Analysis by F.H. Clarke, and generalizes the study on the variational inequalities as given by A. Szulkin .

SZULKIN

PANAGIOTOPOULOS

This theory is applied to study variational and variational-hemivariational inequalities . In particular, for instance, differential inclusions and equations with discontinuous nonlinearities are investigated.

MOTREANU

PAPAGEORGIOU

MARANO

Here, by using the nonsmooth theory we obtain results for smooth functions.

THE EKELAND VARIATIONAL PRINCIPLE

Arguing in a classical way of the smooth analysis (as, for instance, Ghossoub), but using the definitions and properties of the non-smooth analysis (as, for instance, Motreanu-Radulescu, the following consequence of the Ekeland variational Principle can be obtained.

A CONSEQUENCE OF THE EKELAND VARIATIONAL PRINCIPLE IN THE NONSMOOTH ANALYSIS FRAMEWORK

Lemma. *Let X be a real Banach space and $I : X \to \mathbf{R}$ a locally Lipschitz function bounded from below. Then, for all minimizing sequence of I, $\{u_n\}_{n\in\mathbf{N}} \subseteq X$, there exists a minimizing sequence of I, $\{v_n\}_{n\in\mathbf{N}} \subseteq X$, such that*

$$I(v_n) \le I(u_n) \quad \forall n \in \mathbf{N},$$

$$I^\circ(v_n; h) \ge -\varepsilon_n \|h\| \quad \forall h \in X, \ \forall n \in \mathbf{N}, \qquad \textit{where } \varepsilon_{\mathbf{n}} \to \mathbf{0}^+.$$

Sketch of Proof. *Put*

$$M = r_2 - \Phi(x_0) + \Psi(x_0),$$

$$\Psi_M(u) = \begin{cases} \Psi(u) & \text{if } \Psi(u) < M \\ M & \text{if } \Psi(u) \geq M, \end{cases}$$

$$\Phi^{r_1}(u) = \begin{cases} \Phi(u) & \text{if } \Phi(u) > r_1 \\ r_1 & \text{if } \Phi(u) \leq r_1, \end{cases}$$

$$J = \Phi^{r_1} - \Psi_M.$$

Clearly, J is locally Lipschitz and bounded from below. Hence, Lemma and a suitable computation ensure the conclusion.

A LOCAL MINIMUM THEOREM

Let X be a real Banach space and $\Phi, \Psi : X \to \mathbb{R}$ two continuously Gâteaux differentiable functions. Put

$$I = \Phi - \Psi$$

and assume that there are $x_0 \in X$ and $r_1, r_2 \in \mathbb{R}$, with $r_1 < \Phi(x_0) < r_2$, such that

$$\sup_{u \in \Phi^{-1}(]r_1, r_2[)} \Psi(u) \leq r_2 - \Phi(x_0) + \Psi(x_0), \qquad (1)$$

$$\sup_{u \in \Phi^{-1}(]-\infty, r_1])} \Psi(u) \leq r_1 - \Phi(x_0) + \Psi(x_0). \qquad (2)$$

Moreover, assume that I satisfies ${}^{[r_1]}(PS)^{[r_2]}$-condition.

Then, there is $u_0 \in \Phi^{-1}(]r_1, r_2[)$ such that $I(u_0) \leq I(u)$ for all $u \in \Phi^{-1}(]r_1, r_2[)$ and $I'(u_0) = 0$.

First

Three consequences

Let X be a real Banach space and $\Phi, \Psi : X \to \mathbb{R}$ two continuously Gâteaux differentiable functions with Φ bounded from below and $\Phi(0) = \Psi(0) = 0$. Fix $r > 0$ such that $\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi < +\infty$ and assume that,

for each $\lambda \in \left]0, \dfrac{r}{\sup\limits_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}\right[$, *the functional* $I_\lambda = \Phi - \lambda\Psi$ *satisfies* $(PS)^{[r]}$*-condition.*

Then, for each $\lambda \in \left]0, \dfrac{r}{\sup\limits_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}\right[$, *there is* $u_1 \in \Phi^{-1}(]-\infty, r[)$ *such that* $I_\lambda(u_1) \leq I_\lambda(u)$ *for all* $u \in \Phi^{-1}(]-\infty, r[)$ *and* $I'_\lambda(u_1) = 0$.

Second

Three consequences

Let X be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functions. Fix $\inf_X \Phi < r < \sup_X \Phi$ and assume that

$$\rho(r) > 0,$$

where $\rho(r) = \displaystyle\sup_{v \in \Phi^{-1}(]r,\infty[)} \frac{\Psi(v) - \displaystyle\sup_{u \in \Phi^{-1}(]-\infty,r])} \Psi(u)}{\Phi(v) - r}$, *and*

for each $\lambda > \dfrac{1}{\rho(r)}$ *the function* $I_\lambda = \Phi - \lambda\Psi$ *is bounded from below and satisfies* $^{[r]}(PS)$*-condition.*

Then, for each $\lambda > \dfrac{1}{\rho(r)}$ *there is* $u_2 \in \Phi^{-1}(]r, +\infty[)$ *such that* $I_\lambda(u_2) \leq I_\lambda(u)$ *for all* $u \in \Phi^{-1}(]r, +\infty[)$ *and* $I'_\lambda(u_2) = 0$.

Third

Three consequences

Let X be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$. Assume that there are $r \in \mathbb{R}$ and $\tilde{u} \in X$, with $0 < \Phi(\tilde{u}) < r$, such that

$$\frac{\sup\limits_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u)}{r} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})}$$

and, for each $\lambda \in \left] \frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup\limits_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u)} \right[$, *the functional* $I_\lambda = \Phi - \lambda \Psi$ *satisfies* $(PS)^{[r]}$*-condition.*
Then, for each $\lambda \in \left] \frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup\limits_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u)} \right[$, *there is* $u_0 \in \Phi^{-1}(]0, r[)$ *(hence, $u_0 \neq 0$) such that* $I_\lambda(u_0) \leq I_\lambda(u)$ *for all* $u \in \Phi^{-1}(]0, r[)$ *and* $I'_\lambda(u_0) = 0$.

BONANNO G., *A critical point theorem via the Ekeland variational principle,* Nonlinear Analysis, **75** (2012), 2992-3007.
BONANNO G., *Relations between the mountain pass theorem and local minima,* Advances in Nonlinear Analysis, **1** (2012), 205-220.

Some examples on third consequence of LOCAL MINIMUM THEOREM

- *Consider*

$$(P_\lambda) \quad \begin{cases} -u'' = \lambda\alpha(x)f(u) & x \in]0,1[\\ u(0) = u(1) = 0. \end{cases}$$

$\alpha \in L^1([0,1]),\ \alpha(x) \geq 0\ a.e\ x \in [0,1],\ \alpha \not\equiv 0,$

$f : \mathbb{R} \to \mathbb{R}$ *is continuous and nonnegative function,*

$F(\xi) = \int_0^\xi f(t)dt\ \forall \xi \in \mathbb{R},$

λ *is a positive parameter.*

Theorem. *Assume that f is nonnegative and there exist two positive constants c, d, with $d < c$, such that*

$$\frac{F(c)}{c^2} < \left(\frac{\int_{\frac{1}{4}}^{\frac{3}{4}} \alpha(x)dx}{2\|\alpha\|_1} \right) \frac{F(d)}{d^2}.$$

Then, for each $\lambda \in \left] \frac{4}{\int_{\frac{1}{4}}^{\frac{3}{4}} \alpha(x)dx} \frac{d^2}{F(d)}, \frac{2}{\|\alpha\|_1} \frac{c^2}{F(c)} \right[$, *the problem (P_λ) admits at least one positive weak solution $\bar{u}$ such that*

$$|\bar{u}(x)| < c \text{ for all } x \in [0, 1].$$

When $\alpha \equiv 1$ the algebraic inequality becomes

there are two positive constants c, d, with $d < c$ such that

$$\frac{F(c)}{c^2} < \frac{1}{4}\frac{F(d)}{d^2}.$$

In this case, the interval is

$$\left]8\frac{d^2}{F(d)}, 2\frac{c^2}{F(c)}\right[$$

and the solution is a classical solution.

Corollary *Assume that*

$$\lim_{t\to 0^+} \frac{f(t)}{t} = +\infty,$$

and put

$$\lambda^* = \frac{2}{\|\alpha\|_1} \sup_{c\in]0,+\infty[} \frac{c^2}{F(c)}.$$

Then, for each $\lambda \in]0, \lambda^*[$, *the problem* (P_λ) *admits at least one positive weak solution* $\bar{u}$.

We have the same situation for other ordinary nonlinear differential problems and for elliptic problems involving the p-laplacian with p>n.

However, the same type of low can be obtained also for $p \leq n$

$$
(\bar{D}_\lambda) \quad \begin{cases} -\Delta u = \lambda f(u) & in\ \Omega \\ u = 0 & on\ \partial\Omega, \end{cases}
$$

Theorem. *Let $f : \mathbb{R} \to \mathbb{R}$ be a nonnegative continuous function satisfying*

(h) *there exist two non-negative constants a_1, a_2 and $q \in \,]1, 2N/(N-2)[$ such that*

$$f(t) \leq a_1 + a_2|t|^{q-1} \; \forall t \in \mathbb{R}.$$

Assume that

$$\lim_{t \to 0^+} \frac{f(t)}{t} = +\infty,$$

and put

$$\lambda^* = \frac{1}{\left(\sqrt{2}c_1a_1 + \frac{2^{q/2}c_q^q a_2}{q}\right)}.$$

Then, for each $\lambda \in \,]0, \lambda^*[$, the problem (D_λ) admits at least one positive weak solution.

BONANNO G. – SCIAMMETTA A., *An existence result of one non-trivial solution for two point boundary value problems,* Bulletin of the Australian Mathematical Society, **84** (2011), 288-299.

BONANNO G. – PIZZIMENTI P.F., *Neumann boundary value problems with not coercive potential,* Mediterranean Journal of Mathematics, **9** (2012), 603-611.

BONANNO G. – SCIAMMETTA A., *Existence and multiplicity results to Neumann problems for elliptic equations involving the p-Laplacian,* Journal of Mathematical Analysis and Application, **390** (2012), 59-67.

BONANNO G. – PIZZIMENTI P.F., *Existence results for nonlinear elliptic problems,* Applicable Analysis, **92** (2013), 411-423.

PHD thesis of Sciammetta:

PHD thesis of Pizzimenti:

Critical nonlinearites

$$\begin{cases} -\Delta u = u^{\frac{N+2}{N-2}} + g(u) \quad \text{in} \quad \Omega, \\ u > 0 \quad \text{in} \quad \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

$$g(u) = \mu|u|^s, \ \mu \geq 0 \ s > 0.$$

The embedding $H_0^1(\Omega)$ in $L^{\frac{2N}{N-2}}(\Omega)$ is not compact.

If $\mu = 0$, the problem has no solution (Pohozaev, 1965).

If $s \geq 1$ the problem has at least one solution for suitable $\mu > 0$ (Brezis-Nirenberg, 1983).

If $s < 1$ the problem has at least two solutions for suitable $\mu > 0$ (Ambrosetti-Brezis-Cerami, 1994).

AMBROSETTI-BREZIS-CERAMI

Fix $0 < s < 1$. *Then, there is* $\Lambda > 0$ *such that for each* $\mu \in]0, \Lambda[$ *problem*

$$\begin{cases} -\Delta u = u^{\frac{N+2}{N-2}} + \mu u^s \quad \text{in} \quad \Omega, \\ u > 0 \quad \text{in} \quad \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

admits at least two weak solutions.

AMBROSETTI-BREZIS-CERAMI

Their proof is a clever combination of topological and variational methods. Precisely, they determine the existence of a first solution by using the method of sub- and super-solutions and then, through a deep reasoning, prove that this solution is the minimum of a suitable functional and apply the mountain pass theorem so ensuring the existence of a second solution. However, in their proof, no numerical estimate of Λ is provided.

- *Owing to the local minimum theorem we obtain the first solution directly as a minimum . Moreover, a precise estimate of parameters is given.*

$$\begin{cases} -\Delta u = \lambda\left(h(u) + \mu g(u)\right) & \text{in} \quad \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

$$h(t) = t^{2^*-1},\ g(t) = t^{q-1} \qquad f(t) = h(t) + \mu g(t) \qquad \Phi(u) = \frac{\|u\|^2}{2} \qquad \Psi(u) = \int_\Omega F(u(x))dx$$

$$\lambda_r^* = \frac{r}{\left(\frac{\mu}{q}c_q^q(2r)^{q/2} + \frac{(2r)^{2^*/2}}{2^*}c_{2^*}^{2^*}\right)}, \qquad \tilde{\lambda}_r = \frac{1}{c_{2^*}^{2^*}(2rN)^{\frac{2}{N-2}}},$$

$$\overline{\lambda}_r = \min\left\{\lambda_r^*, \tilde{\lambda}_r\right\},$$

$$c_{2^*} = \frac{1}{\sqrt{N(N-2)\pi}}\left(\frac{N!}{2\Gamma(1+N/2)}\right)^{1/N},$$

$$c_s \leq \frac{meas(\Omega)^{\frac{2^*-s}{2^*s}}}{\sqrt{N(N-2)\pi}}\left(\frac{N!}{2\Gamma(N/2+1)}\right)^{1/N}$$

$$I_\lambda = \Phi - \lambda\Psi$$

Lemma *Fix $r > 0$. Then, for each $\lambda \in]0, \overline{\lambda}_r[$ the functional $I_\lambda = \Phi - \lambda\Psi$ satisfies the $(PS)^{[r]}-$condition.*

Theorem *Fix $q \in]1, 2[$. Then, there exists $\mu^* > 0$, where*

$$\mu^* = \left(\frac{q}{c_q^q}\frac{1}{2^{\frac{q+2}{2}}}\right)\left(\min\left\{\left(\frac{2^*}{2^{\frac{2^*+2}{2}}c_{2^*}^{2^*}}\right)^{\frac{2}{2^*-2}}; \frac{1}{3N}\left(\frac{1}{c_{2^*}^{2^*}}\right)^{\frac{N-2}{2}}\right\}\right)^{\frac{2-q}{2}}$$

such that for each $\mu \in]0, \mu^*[$ problem

$$(D_\mu) \quad \begin{cases} -\Delta u = u^{2^*-1} + \mu u^{q-1} & \text{in} \quad \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

admits at least one positive weak solution u_μ such that

$$\|u_\mu\| < \left(\frac{2^*}{c_{2^*}^{2^*}}\right)^{\frac{1}{2^*-2}}.$$

Moreover, the mapping

$$\mu \to \frac{1}{2}\int_\Omega |\nabla u_\mu|^2 dx - \int_\Omega \frac{1}{2^*}|u_\mu|^{2^*} dx - \mu\int_\Omega \frac{1}{q}|u_\mu|^q dx$$

is negative and strictly decreasing in $]0, \mu^*[$.

Theorem *Fix $q \in]1,2[$. Then, there exists $\mu^* > 0$, where*

$$\mu^* = \left(\frac{q}{c_q^q}\frac{1}{2^{\frac{q+2}{2}}}\right)\left(\min\left\{\left(\frac{2^*}{2^{\frac{2^*+2}{2}}c_{2^*}^{2^*}}\right)^{\frac{2}{2^*-2}};\frac{1}{3N}\left(\frac{1}{c_{2^*}^{2^*}}\right)^{\frac{N-2}{2}}\right\}\right)^{\frac{2-q}{2}}$$

such that for each $\mu \in]0,\mu^*[$ problem (D_μ) admits at least

two positive weak solutions u_μ and w_μ such that $\|u_\mu\| < \left(\frac{2^*}{c_{2^*}^{2^*}}\right)^{\frac{1}{2^*-2}}$ and $w_\mu > u_\mu$.

Example Fix $N = 3$ and let $\Omega = \{x \in \mathbb{R}^3 : |x| < 1\}$. Then, the problem

$$\begin{cases} -\Delta u = u^5 + \frac{3}{8}\sqrt{u} \quad \text{in} \quad \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

admits at least two positive weak solutions u_μ and w_μ such that $\int_\Omega |\nabla u_\mu(x)|^2 dx < \frac{9\pi^2}{2^{5/2}}$ $w_\mu > u_\mu$.

BONANNO G. – D'AGUI' G., *Critical nonlinearities for elliptic Dirichlet problems*, Dynamic Systems and Applications **22** (2013), 411-418.

BONANNO G. – D'AGUI' G. - O'REGAN D., *A local minimum theorem and critical nonlinearities,* preprint.

Mean curvature operator

$$(P_\lambda) \quad \begin{cases} -\left(\dfrac{u'}{\sqrt{1+|u'|^2}}\right)' = \lambda f(u) \quad \text{in }]0,1[, \\[2ex] u(0) = u(1) = 0, \end{cases}$$

Theorem . *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous and nonnegative function and assume that there exist two positive constants c, d, with $d < c$, such that*

$$\left[\left(\frac{F(c)}{c^2}\right)^2 + (2\max_{[0,c]} f)^2\right]^{1/2} < \frac{1}{4}\frac{F(d)}{d^2}.$$

Then, for every

$$\lambda \in \left]\frac{8d^2}{F(d)}, 2\left[\left(\frac{F(c)}{c^2}\right)^2 + (2\max_{[0,c]} f)^2\right]^{-1/2}\right[$$

problem (P_λ) admits at least one non trivial, nonnegative solution $u_\lambda \in C^{1,\tau}([0,1])$ for some $\tau \in]0,1]$, such that

$$\|u_\lambda\|_{C^0} < c, \ \ \|u'_\lambda\|_{C^0} \leq \frac{2\max_{[0,c]} f}{F(c)} c^2.$$

Theorem *Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous, nonnegative function such that*

$$\limsup_{s\to 0^+}\frac{F(s)}{s^2}=+\infty.$$

Then, for each $\lambda\in\,]0,\lambda^*[$, where

$$\lambda^*=2\left\{[F(1)]^2+[2\max_{[0,1]}f]^2\right\}^{-1/2},$$

problem (P_λ) *admits at least one nontrivial, nonnegative solution $u_\lambda\in C^{1,\tau}([0,1])$ for some $\tau\in\,]0,1]$, such that*

$$\|u_\lambda\|_{C^0}<1,\ \ \|u'_\lambda\|_{C^0}\le\frac{\max_{[0,1]}f}{F(1)}.$$

BONANNO G.- LIVREA R. – MAWHIN J., *Existence results for parametric boundary value problems involving the mean curvature operator*, preprint.

Boundary value problems in the real line

$$\textit{Find } u \in W^{1,p}(\mathbb{R}) \textit{ satisfying}$$

$$(P_\lambda) \quad -(|u'(x)|^{p-2}u'(x))' + B|u(x)|^{p-2}u(x) = \lambda\alpha(x)g(u(x)) \ \textit{a.e. in } \mathbb{R},$$

Theorem *Assume that there exist two positive constants* γ, κ, *with* $\kappa < \gamma$, *such that*

$$\frac{G(\gamma)}{\gamma^p} < R\frac{G(\kappa)}{\kappa^p}.$$

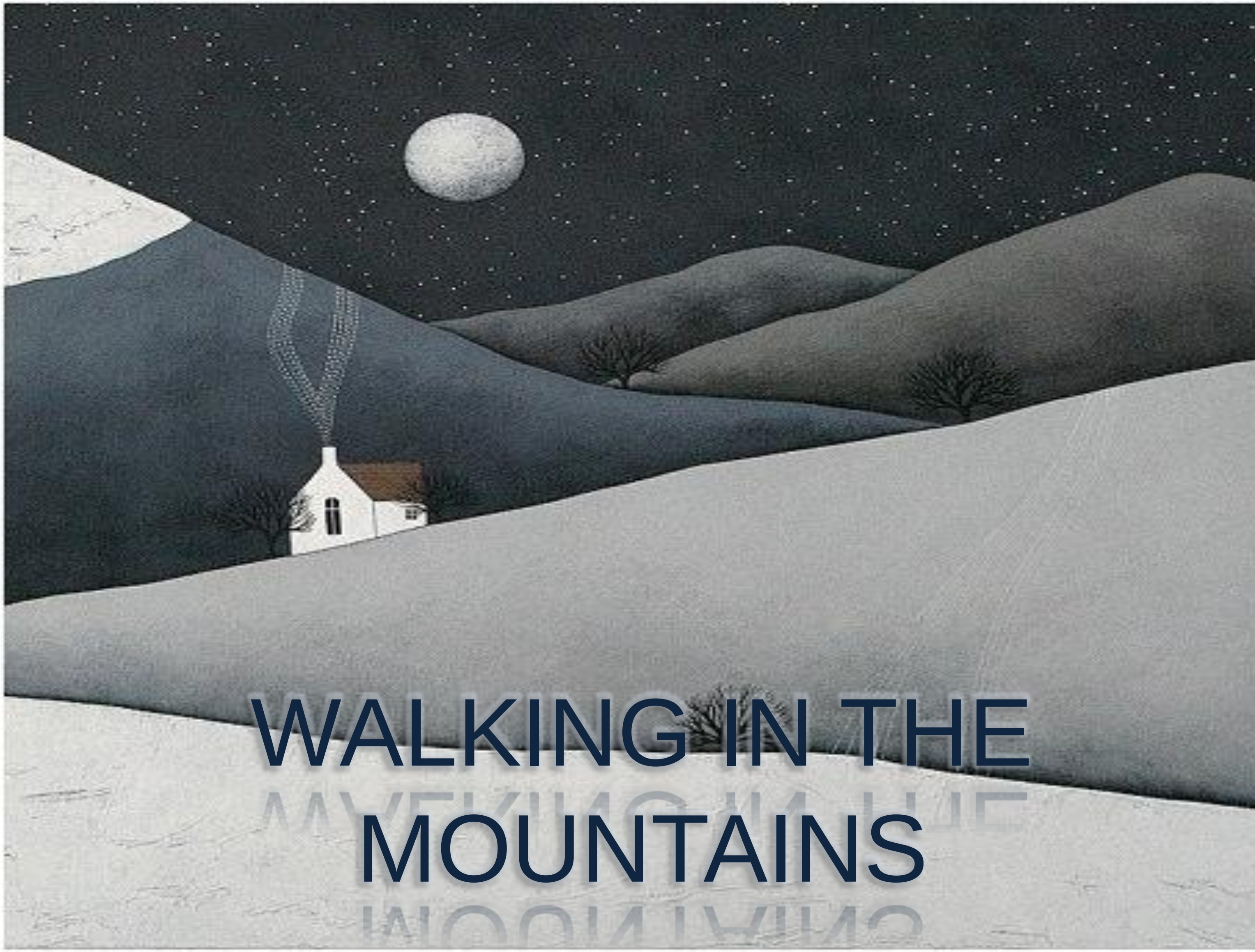
Then, for each $\lambda \in \left]\frac{1}{pc_B^p|\alpha|_1}\frac{1}{R}\frac{\kappa^p}{G(\kappa)}, \frac{1}{pc_B^p|\alpha|_1}\frac{\gamma^p}{G(\gamma)}\right[$, *problem* (P_λ) *admits at least one nontrivial and nonnegative solution* $u_{0,\lambda}$ *such that* $|u_{0,\lambda}|_\infty < \gamma$.

Corollary *Assume that*

$$\lim_{t\to 0^+} \frac{g(t)}{t^{p-1}} = +\infty.$$

Then, for each $\gamma > 0$ *and for each* $\lambda \in \left]0, \frac{1}{pc_B^p|\alpha|_1}\frac{\gamma^p}{G(\gamma)}\right[$ *problem* (P_λ) *admits at least one nontrivial and nonnegative solution* $u_{0,\lambda}$ *such that* $|u_{0,\lambda}|_\infty < \gamma$.

BARLETTA G. – BONANNO G. – O'REGAN D., *A variational approach to multiplicity results for boundary value problems on the real line*, Proceedings of the Royal Society of Edinburgh, Section A, **140**, to appear. .



WALKING IN THE MOUNTAINS

Some remarks on the classical Ambrosetti-Rabinowitz theorem are presented. In particular, it is observed that the geometry of the mountain pass, if the function is bounded from below, is equivalent to the existence of at least two local minima, while, when the function is unbounded from below, it is equivalent to the existence of at least one local minimum.

So, the Ambrosetti-Rabinowitz theorem actually ensures three or two distinct critical points, according to the function is bounded from below or not.

HISTORICAL NOTES

Let X be a real Banach space, $I : X \to \mathbb{R}$ a continuously Gâteaux differentiable function which verifies (PS).

1973

THE AMBROSETTI-RABINOWITZ THEOREM

Assume that

(G) *there are* $u_0, u_1 \in X$ *and* $r \in \mathbb{R}$, *with* $0 < r < \|u_1 - u_0\|$, *such that*

$$\inf_{\|u-u_0\|=r} I(u) > \max\{I(u_0), I(u_1)\}.$$

Then, I *admits a critical value* c *characterized by*

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t))$$

where

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = u_0; \gamma(1) = u_1\}.$$

1985

THE PUCCI-SERRIN THEOREM

(G') *there are* $u_0, u_1 \in X$ *and* $r, R \in \mathbb{R}$, *with* $0 < r < R < \|u_1 - u_0\|$, *such that*

$$\inf_{r<\|u-u_0\|<R} I(u) \geq \max\{I(u_0), I(u_1)\}.$$

Corollary. *If* I *admits two local minima, then* I *admits a third critical point.*

1989

THE GHOUSSOUB-PREISS THEOREM

(MG) *there are* $u_0, u_1 \in X$ *and* $r \in \mathbb{R}$, *with* $0 < r < \|u_1 - u_0\|$, *such that*

$$\inf_{\|u-u_0\|=r} I(u) \geq \max\{I(u_0), I(u_1)\}.$$

A REMARK

Theorem. *Let X be a real Banach space, $I : X \to \mathbb{R}$ a continuously Gâteaux differentiable function which verifies (PS) and it is bounded from below. Then, the following assertions are equivalent:*

(MG) *there are $u_0, u_1 \in X$ and $r \in \mathbb{R}$, with $0 < r < \|u_1 - u_0\|$, such that*

$$\inf_{\|u-u_0\|=r} I(u) \geq \max\{I(u_0), I(u_1)\};$$

(L) *I admits at least two distinct local minima.*

So, the Ambrosetti-Rabinowitz theorem, when the function is bounded from below actually ensures three distinct critical points.

In fact, in this case the mountain pass geometry implies the existence of two local minima and the Pucci-Serrin theorem ensures the third critical point.

In a similar way it is possible to see that, when the function is unbounded from below, the mountain pass geometry is equivalent to the existence of at least one local minimum.

In this case, the following condition is requested:

The function I is bounded from below on every bounded set of X.

REMARK *Let X be a real Banach space and $I : X \to \mathbb{R}$ be a functional of class C^1 satisfying the $(PS)-$condition and the mountain pass geometry (MG). Assume that I is bounded from below on every bounded set of X. Then, I admits two or three distinct critical points according to whether it is unbounded from below or not.*

BONANNO G., *A characterization of the mountain pass geometry for functionals bounded from below,* Differential and Integral Equations **25** (2012), 1135-1142.

BONANNO G., *Relations between the mountain pass theorem and local minima,* Advances in Nonlinear Analysis, **1** (2012), 205-220.

First

Three consequences

Let X be a real Banach space and $\Phi, \Psi : X \to \mathbb{R}$ two continuously Gâteaux differentiable functions with Φ bounded from below and $\Phi(0) = \Psi(0) = 0$. Fix $r > 0$ such that $\sup_{u \in \Phi^{-1}(]-\infty, r[)} \Psi < +\infty$ and assume that,

for each $\lambda \in \left] 0, \dfrac{r}{\sup\limits_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u)} \right[$, *the functional*

$I_\lambda = \Phi - \lambda \Psi$ *satisfies* $(PS)^{[r]}$*-condition.*

Then, for each $\lambda \in \left] 0, \dfrac{r}{\sup\limits_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u)} \right[$, *there is*

$u_1 \in \Phi^{-1}(]-\infty, r[)$ *such that* $I_\lambda(u_1) \leq I_\lambda(u)$ *for all* $u \in \Phi^{-1}(]-\infty, r[)$ *and* $I'_\lambda(u_1) = 0$.

Three consequences

Second

Let X *be a real Banach space and let* $\Phi, \Psi : X \to \mathbb{R}$ *be two continuously Gâteaux differentiable functions. Fix* $\inf_X \Phi < r < \sup_X \Phi$ *and assume that*

$$\rho(r) > 0,$$

where $\rho(r) = \sup\limits_{v \in \Phi^{-1}(]r,\infty[)} \dfrac{\Psi(v) - \sup\limits_{u \in \Phi^{-1}(]-\infty,r])} \Psi(u)}{\Phi(v) - r}$, *and*

for each $\lambda > \dfrac{1}{\rho(r)}$ *the function* $I_\lambda = \Phi - \lambda\Psi$ *is bounded from below and satisfies* $^{[r]}$*(PS)-condition.*

Then, for each $\lambda > \dfrac{1}{\rho(r)}$ *there is* $u_2 \in \Phi^{-1}(]r, +\infty[)$ *such that* $I_\lambda(u_2) \leq I_\lambda(u)$ *for all* $u \in \Phi^{-1}(]r, +\infty[)$ *and* $I'_\lambda(u_2) = 0$.

A THREE CRITICAL POINTS THEOREM

From the preceding two variants of the local minimum theorem, a three critical points theorem is obtained. Here a special case is pointed out.

A THREE CRITICAL POINTS THEOREM

Let X be a real Banach space and $\Phi, \Psi : X \to \mathbb{R}$ two continuously Gâteaux differentiable functionals with Φ bounded from below. Assume that $\Phi(0) = \Psi(0) = 0$ and there are $r > 0$ and $\overline{x} \in X$, with $r < \Phi(\overline{x})$, such that

$$\frac{\sup\limits_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)}{r} < \frac{\Psi(\overline{x})}{\Phi(\overline{x})}. \qquad (3)$$

Further assume that, for each

$$\lambda \in \Lambda := \left] \frac{\Phi(\overline{x})}{\Psi(\overline{x})}, \frac{r}{\sup\limits_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)} \right[,$$

the functional $I_\lambda = \Phi - \lambda\Psi$ is bounded from below and satisfies (PS)-condition.

Then, for each $\lambda \in \Lambda$ the functional I_λ admits at least three critical points.

NONLINEAR DIFFERENTIAL PROBLEMS

A TWO-POINT BOUNDARY VALUE PROBLEM

Consider the following two point boundary value problem

$$(D_\lambda) \qquad \begin{cases} -u'' = \lambda f(u) \ \text{ in } \ ]0,1[\\ u(0) = u(1) = 0, \end{cases}$$

where $f : \mathbf{R} \to \mathbf{R}$ is a continuous function and is λ a positive real parameter.

Moreover, put

$$F(\xi) = \int_0^{\xi} f(t)dt$$

for all $\xi \in \mathbf{R}$ and assume, for clarity, that f is nonnegative.

Theorem. *Assume that*
there are two positive constants c and d, with $c < d$, such that

(1) $$\frac{F(c)}{c^2} < \frac{1}{4}\frac{F(d)}{d^2}$$

and there are two positive constants a and s, with $s < 2$, such that

(2) $$F(\xi) \leq a(1 + |\xi|^s) \qquad \forall \xi \in \mathbb{R}.$$

Then, for each $\lambda \in \left]8\frac{d^2}{F(d)}, 2\frac{c^2}{F(c)}\right[$, *problem* (D_λ) *admits at least three (nonnegative) classical solutions.*

TWO-POINT BOUNDARY VALUE PROBLEMS

NEUMANN BOUNDARY VALUE PROBLEMS

MIXED BOUNDARY VALUE PROBLEMS

STURM-LIOUVILLE BOUNDARY VALUE PROBLEMS

HAMILTONIAN SYSTEMS

FOURTH-ORDER ELASTIC BEAM EQUATIONS

BOUNDARY VALUE PROBLEMS ON THE HALF_LINE

NONLINEAR DIFFERENCE PROBLEMS

ELLIPTIC DIRICHLET PROBLEMS
INVOLVING THE P-LAPLACIAN WITH P>N

ELLIPTIC NEUMANN PROBLEMS
INVOLVING THE P-LAPLACIAN WITH P>N

ELLIPTIC SYSTEMS

ELLIPTIC PROBLEMS INVOLVING THE p(x)-LAPLACIAN

NONLINEAR EIGENVALUE PROBLEMS
IN ORLICS-SOBOLEV SPACES

NONLINEAR ELLIPTIC PROBLEMS
ON THE SIERPI NSKI GASKET

GENERALIZED YAMABE EQUATIONS
ON RIEMANNIAN MANIFOLDS

ELLIPTIC PROBLEMS INVOLVING THE p-LAPLACIAN WITH p≤N

$$\begin{cases} -\Delta u = \lambda f(x,u) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

(h) *There exist two non-negative constants* a_1, a_2 *and* $q \in]1, 2N/(N-2)[$ *such that*

$$|f(x,t)| \leq a_1 + a_2 |t|^{q-1},$$

for every $(x,t) \in \Omega \times \mathbb{R}$.

(j) *There exist two positive constants* c *and* d, *with* $d > c\kappa$ *such that*

$$\frac{\inf_{x\in\Omega} F(x,d)}{d^2} > a_1 \frac{K_1}{c} + a_2 K_2 c^{q-2}, \qquad (1)$$

where a_1, a_2 *are given in* (h) *and* κ, K_1, K_2 *are given by*

$$\kappa := \frac{D\sqrt{2}}{2\pi^{N/4}} \left(\frac{\Gamma(1+N/2)}{D^N - (D/2)^N} \right)^{1/2}, K_1 := \frac{2\sqrt{2}c_1(2^N - 1)}{D^2}, \qquad K_2 := \frac{2^{\frac{q+2}{2}} c_q^q (2^N - 1)}{qD^2}.$$

BONANNO G. – CANDITO P., *Nonlinear difference equations investigated via critical point methods,* Nonlinear Analysis **70** (2009), 3180-3186.

BONANNO G. – LIVREA R., *Multiple periodic solutions for Hamiltonian systems with not coercive potential,* Journal of Mathematical Analysis and Applications, **363** (2010) 627–638.

BONANNO G. – D'AGUI' G., *A critical point theorem and existence results for a nonlinear boundary value problem,* Nonlinear Analysis, **72** (2010) 1977-1982.

BONANNO G. – MARANO S.A., *On the structure of the critical set of non-differentiable functions with a weak compactness condition,* Applicable Analysis, **89** (2010), 1-10.

BONANNO G. – CHINNI' A., *Discontinuous elliptic problems involving the p(x)-laplacian,,* Mathematische Nachrichten **284** (2011), 639-652.

BONANNO G. – DI BELLA B., *A boundary value problem for fourth-order elastic beam equations,* Journal of Mathematical Analysis and Applications **343** (2008), 1166-1176.

BONANNO G. – TORNATORE E., *Existence results for nonlinear elliptic Dirichlet systems,* Electronic Journal of Differential Equations, **2012** n.183 (2012), 1-11.

…

BONANNO G. – HEIDARKHANI S. – O'REGAN D., *Multiple solutions for a class of Dirichlet quasilinear elliptic systems driven by a (p,q)-Laplacian operator,* Dynamic Systems and Applications, **20** (2011), 89-100.

BONANNO G. – MOTREANU D. – WINKERT P., *Variational-hemivariational inequalities with small perturbations of nonhomogeneous Neumann boundary conditions*, Journal of Mathematical Analysis and Application, **381** (2011), 627-637.

BONANNO G. – DI BELLA B. – HENDERSON J., *Existence of solutions to second-order boundary-value problem with small perturbations of impulses,* Electronic Journal of Differential Equations, **2013** n.126 (2013), 1-14.

BONANNO G. - JEBELEAN P. – SERBAN C., *Three solutions for discrete anisotropic periodic and Neumann problems*, Dynamic Systems and Applications **22** (2013), 183-196.

…and others

A TWO CRITICAL POINTS THEOREM

Theorem 1 . *Let X be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that Φ is bounded from below and $\Phi(0) = \Psi(0) = 0$. Fix $r > 0$ such that $\sup_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u) < +\infty$ and assume that, for each*

$$\lambda \in \left]0, \frac{r}{\sup_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u)}\right[,$$

the functional $I_\lambda = \Phi - \lambda\Psi$ satisfies the (PS)*-condition and it is unbounded from below. Then, for each*

$$\lambda \in \left]0, \frac{r}{\sup_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u)}\right[,$$

the functional I_λ admits two distinct critical points.

A TWO CRITICAL POINTS THEOREM

Theorem 2 . *Let X be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that* $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$. *Assume that there are $r \in \mathbb{R}$ and $\tilde{u} \in X$, with $0 < \Phi(\tilde{u}) < r$, such that*

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}{r} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})}$$

and, for each

$$\lambda \in \left]\frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}\right[,$$

the functional $I_\lambda = \Phi - \lambda\Psi$ satisfies the (PS)*-condition and it is unbounded from below. Then, for each*

$$\lambda \in \left]\frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}\right[,$$

the functional I_λ admits two distinct critical points u_1 and u_2 such that

$$\Phi(u_1) < r, \quad u_1 \neq 0 \quad \text{and} \quad I_\lambda(u_2) = \inf_{\gamma\in\Gamma_{u_1}^{\bar{x}}} \max_{t\in[0,1]} I_\lambda(\gamma(t)),$$

where $\bar{x}$ is such that $I_\lambda(\bar{x}) \leq I_\lambda(u_1)$.

A TWO CRITICAL POINTS THEOREM

Theorem 3 . *Let X be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that* $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$. *Assume that there are $r \in \mathbb{R}$ and $\tilde{u} \in X$, with $0 < \Phi(\tilde{u}) < r$, such that*

$$\frac{\sup_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u)}{r} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})}$$

and, for each

$$\lambda \in \left] \frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u)} \right[,$$

the functional $I_\lambda = \Phi - \lambda\Psi$ satisfies the (PS)*-condition and it is unbounded from below. Then, for each*

$$\lambda \in \left] \frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u)} \right[,$$

the functional I_λ admits two distinct critical points u_1 and u_2 such that

$$\Phi(u_1) < r, \quad u_1 \neq 0 \quad \text{and} \quad u_2 \neq 0 \quad ,$$

[redacted]

1

$$(P_\lambda) \qquad \begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

Theorem. *Let $f : \mathbb{R} \to \mathbb{R}$ be a nonnegative continuous function satisfying*

(h) *there exist two non-negative constants a_1, a_2 and $q \in]1, 2N/(N-2)[$ such that*

$$f(t) \le a_1 + a_2|t|^{q-1} \ \forall t \in \mathbb{R}.$$

Put

$$\lambda^* = \frac{1}{\left(\sqrt{2}c_1 a_1 + \frac{2^{q/2} c_q^q a_2}{q}\right)}.$$

Assume that

$$0 < \mu F(t) \le t f(t)$$

(A-R)

for all $|t| \ge r$, for some $r > 0$ and for some $\mu > 2$. Then, for each $\lambda \in]0, \lambda^*[$, the problem (P_λ) admits at least two weak solutions, whose at least one is positive.

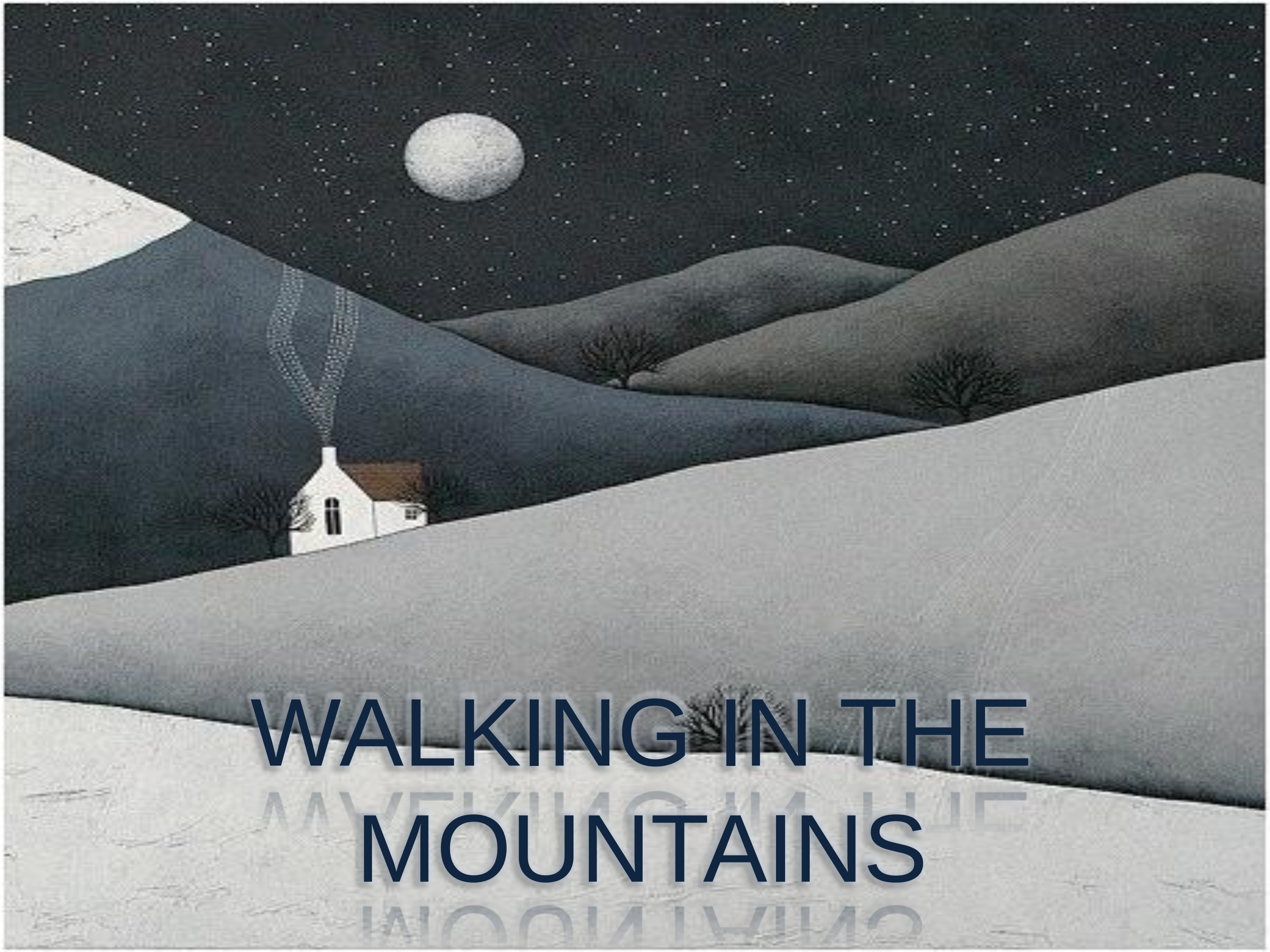
Further, if in addition, assume

$$\lim_{t \to 0^+} \frac{f(t)}{t} = +\infty, \qquad (1)$$

Then, for each $\lambda \in]0, \lambda^*[$, *the problem* (P_λ) *admits at least two positive weak solutions.*

BONANNO G., *Relations between the mountain pass theorem and local minima,* Advances in Nonlinear Analysis, **1** (2012), 205-220.

BONANNO G. - D'AGUI' G., *Two non-zero solutions for elliptic Dirichlet problems,* in preparation.



WALKING IN THE MOUNTAINS

1. *Mountain pass geometry*
2. *Palais-Smale condition*

Then, there is a critical point.

1. *Strong mountain pass geometry*
2. *Weak Palais-Smale condition*

Then, there is a critical point near to local minimum.

AVERNA D. – BONANNO G., *A mountain pass theorem for a suitable class of functions,* Rocky Mountain Journal of Mathematics **39** (2009), 707-727.

BONANNO G. – CANDITO P., *Non-differentiable functionals and applications to elliptic problems with discontinuous nonlinearities,* Journal of Differential Equations **244** (2008), 3031-3059.

A VARIANT OF THREE CRITICAL POINTS THEOREM FOR FUNCTIONS UNBOUNDED FROM BELOW

If we apply two times the first special case of the local minimum theorem and owing to a novel version of the mountain pass theorem where the (PS) cut off upper at r is assumed we can give a variant of the three critical theorem. In the applications it became

Theorem. *Assume that there are three positive constants* c_1, d *and* c_2, *with* $c_1 < d < \frac{\sqrt{2}}{2} c_2$, *such that*

$$\frac{F(c_1)}{c_1^2} < \frac{1}{6} \frac{F(d)}{d^2} \qquad (1)$$

and

$$\frac{F(c_2)}{c_2^2} < \frac{1}{12} \frac{F(d)}{d^2}. \qquad (2)$$

Then, for each $\lambda \in \left] 12 \frac{d^2}{F(d)}, \min\left\{ 2 \frac{c_1^2}{F(c_1)}, \frac{c_2^2}{F(c_2)} \right\} \right[$, *problem* (D_λ) *admits at least three (nonnegative) classical solutions* u_i, $i = 1, 2, 3$, *such that*

$$\max_{x \in [0,1]} |u_i(x)| < c_2, \qquad i = 1, 2, 3.$$

FURTHER APPLICATIONS OF THE LOCAL MINIMUM THEOREM

INFINITELY MANY CRITICAL POINTS THEOREM

If we apply iteratively the first special case of the local minimum theorem in a suitable way, we obtain an infinitely many critical points theorem. As an example of application, here, we present the following result.

Theorem. *Assume that*

(1) $$\liminf_{\xi\to+\infty}\frac{F(\xi)}{\xi^2}<\frac{1}{4}\limsup_{\xi\to+\infty}\frac{F(\xi)}{\xi^2}.$$

Then, for each $\lambda\in\left]\dfrac{8}{\limsup\limits_{\xi\to+\infty}\dfrac{F(\xi)}{\xi^2}},\dfrac{2}{\liminf\limits_{\xi\to+\infty}\dfrac{F(\xi)}{\xi^2}}\right[$,

the problem (D_λ) *admits a sequence of pairwise distinct positive classical solutions.*

PERTURBED PROBLEMS

Previous results can be applied to perturbed problems, as, for instance, the following

$$\begin{cases} -u'' = \lambda f(u) + \mu g(x,u) \quad \text{in }]0,1[\\ u(0) = u(1) = 0, \end{cases}$$

or in the framework of the non-smooth Analysis. As example, here, the following problem is considered.

VARIATIONAL-HEMIVARIATIONAL INEQUALITIES

Let Ω be a non-empty, bounded, open subset of the Euclidian space $\mathbb{R}^N$, $N \geq 1$, with C^1-boundary $\partial\Omega$, let $p \in]N, +\infty[$, and let $q \in L^\infty(\Omega)$ satisfy $\operatorname{ess\,inf}_{x\in\Omega} q(x) > 0$.

Problem: Find $u \in K$ such that, for all $v \in K$,

$$\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x) \cdot \nabla(v(x) - u(x))dx + \int_\Omega q(x)|u(x)|^{p-2}u(x)(v(x) - u(x))dx$$
$$+ \int_\Omega \lambda\alpha(x)F^\circ(u(x); v(x) - u(x))dx + \int_{\partial\Omega} \mu\beta(x)G^\circ(\gamma u(x); \gamma v(x) - \gamma u(x))d\sigma \geq 0,$$

where K is a closed convex subset of $W^{1,p}(\Omega)$ containing the constant functions, and $\alpha \in L^1(\Omega)$, $\beta \in L^1(\partial\Omega)$, with $\alpha(x) \geq 0$ for a.a. $x \in \Omega$, $\alpha \not\equiv 0$, $\beta(x) \geq 0$ for a.a. $x \in \partial\Omega$, and λ, μ are real parameters, with $\lambda > 0$ and $\mu \geq 0$. Here, F° and G° stand for Clarke's generalized directional derivatives of locally Lipschitz functions $F, G : \mathbb{R} \to \mathbb{R}$ given by $F(\xi) = \int_0^\xi f(t)dt$, $G(\xi) = \int_0^\xi g(t)dt$, $\xi \in \mathbb{R}$, with $f, g : \mathbb{R} \to \mathbb{R}$ locally essentially bounded functions, and $\gamma : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ denotes the trace operator.

A prototype of the previous problem for $K = W^{1,p}(\Omega)$ is the following boundary value problem with nonsmooth potential and nonhomogeneous, nonsmooth Neumann boundary condition

$$\boxed{\begin{cases} \Delta_p u - q(x)|u|^{p-2}u \in \lambda\alpha(x)\partial F(u) & \text{in } \Omega, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} \in -\mu\beta(x)\partial G(\gamma u) & \text{on } \partial\Omega. \end{cases}}$$

BONANNO G. – MOTREANU D. – WINKERT P., *Variational-hemivariational inequalities with small perturbations of nonhomogeneous Neumann boundary conditions*, Journal of Mathematical Analysis and Application, **381** (2011), 627-637.

BONANNO G. – WINKERT P., *Multiplicity results to a class of variational-hemivariational inequalities,* Topological Methods in Nonlinear Analysis, in corso di stampa.

BONANNO G. – MOTREANU D. – WINKERT P., *Boundary value problems with nonsmooth potential, constraints and parameters,* Dynamic Systems and Applications **22** (2013), 385-396.

A COINCIDENCE POINT THEOREM

Arguing as ARINO-GAUTIER-PENOT and by starting from Ky FAN theorem, the following fixed point theorem for multifunctions with weakly sequentialy closed graph has been established:

1996

Theorem *Let X be a real Banach space and let K be a weakly compact convex subset of X. Suppose Φ is a multifunction from K into itself with nonempty convex values and weakly sequentially closed graph. Then, there exists $x_0 \in K$ such that $x_0 \in \Phi(x_0)$.*

Recently, as a consequence, the following coincidence point theorem has been established:

Theorem 1 *Let X, Y be two real Banach spaces, let K be a weakly compact convex subset of X, and let F, G be two weakly sequentially continuous functions from K into Y. Assume that F is invertible and*

$$G(K) \subseteq F(K).$$

Then, there exists $x_0 \in K$ such that $F(x_0) = G(x_0)$.

Taking into account that a critical point of ***Φ-Ψ*** *is a coincidence point for* ***Φ'*** *and* ***Ψ'****, one has the following critical point theorem*

Theorem 2 *Let X be a real reflexive Banach spaces and let $\Phi, \Psi : X \to \mathbb{R}$ be two Gateaux differentiable functionals such that $\Phi' : X \to X^*$ is invertible and Φ'^{-1}, Ψ' are weakly sequentially continuous. Assume that there is $r > 0$ such that for all $u \in X$ such that $\|u\| \leq r$ there is $v \in X$ with $\|v\| \leq r$ such that $\Psi'(u) = \Phi'(v)$. Then, the functional $\Phi - \Psi$ admits at least a critical point x_0 such that $\|x_0\| \leq r$.*

(1) $$\begin{cases} -u'' = g(t,u,u') & \text{in } (0,1), \\ u(0) = u(1) = 0. \end{cases}$$

Theorem *Assume that there is $r > 0$ such that*

$$\max_{(t,s,\xi)\in[0,1]\times[-r/2,r/2]\times[-r,r]} |g(t,s,\xi)| \le r,$$

Then, problem (1) has a classical solution u satisfying $(u(t),u'(t)) \in [-r/2,r/2] \times [-r,r]$ for all $t \in [0,1]$.

Example The following problem

$$\begin{cases} -u'' = \frac{1}{4}\left(1 + u'^2\right) & \text{in } (0,1), \\ u(0) = u(1) = 0 \end{cases}$$

admits at least one nonzero classical solution.

BONANNO G. - CANDITO P. - MOTREANU D., *A coincidence point theorem for sequentially continuous mapping*, preprint.

BONANNO G. - MARANO S.A., *Positive solutions of elliptic equations with discontinuous nonlinearities,* Topological Methods in Nonlinear Analysis **8** (1996), 263-273.

Thank you very much for your kind attention