A local minimum theorem and nonlinear differential problems

Gabriele Bonanno
University of Messina

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Our aim is to present a local minimum theorem for functionals of the type:

\[ \Phi - \Psi \]
Let $X$ be a real Banach space and $\Phi, \Psi : X \to \mathbb{R}$ two continuously Gâteaux differentiable functions. Put

$$I = \Phi - \Psi$$

and assume that there are $x_0 \in X$ and $r_1, r_2 \in \mathbb{R}$, with $r_1 < \Phi(x_0) < r_2$, such that

$$\sup_{u \in \Phi^{-1}(r_1, r_2]} \Psi(u) \leq r_2 - \Phi(x_0) + \Psi(x_0), \quad (1)$$

$$\sup_{u \in \Phi^{-1}(-\infty, r_1)} \Psi(u) \leq r_1 - \Phi(x_0) + \Psi(x_0). \quad (2)$$

Moreover, assume that $I$ satisfies $(PS)^{[r_1]}$-condition.

Then, there is $u_0 \in \Phi^{-1}(r_1, r_2]$ such that $I(u_0) \leq I(u)$ for all $u \in \Phi^{-1}(r_1, r_2]$ and $I'(u_0) = 0$. 
It is an existence theorem of a critical point for continuously \( \text{Gâteaux differentiable} \) functions, possibly unbounded from below.

The approach is based on Ekeland’s Variational Principle applied to a non-smooth variational framework by using also a novel type of Palais-Smale condition which is more general than the classical one.
Let $X$ be a real Banach space, we say that a Gâteaux differentiable functional

$$I : X \rightarrow \mathbb{R}$$

verifies the Palais-Smale condition (in short (PS)-condition) if any sequence

$\{u_n\}$ such that
(α) \( \{I(u_n)\} \) is bounded,

(β) \( \lim_{n \to +\infty} \|I'(u_n)\|_{X^*} = 0 \),

has a convergent subsequence.

Let \( X \) be a real Banach space and let \( \Phi : X \to \mathbb{R}, \Psi : X \to \mathbb{R} \) two Gâteaux differentiable functions. Put \( I = \Phi - \Psi \).
Fix $r_1, r_2 \in [-\infty; +\infty]$, with $r_1 < r_2$, we say that the function $I$ verifies the Palais-Smale condition cut off lower at $r_1$ and upper at $r_2$ (in short $[r_1](PS)[r_2]$ condition) if any sequence $\{u_n\}$ such that

$(\alpha)$ \( \{I(u_n)\} \) is bounded,

$(\beta)$ \( \lim_{n \to +\infty} \|I'(u_n)\|_{X^*} = 0 \),

$(\gamma)$ $r_1 < \Phi(u_n) < r_2 \ \forall n \in \mathbb{N}$,
has a convergent subsequence.

Clearly, if \( r_1 = -\infty \) and \( r_2 = +\infty \) it coincides with the classical \((PS)\)-condition.

Moreover, if \( r_1 = -\infty \) and \( r_2 \in \mathbb{R} \) we denote it by \((PS)^{[r_2]}\), while if \( r_1 \in \mathbb{R} \) and \( r_2 = +\infty \) we denote it by \([r_1](PS)\).
In particular,

If $I = \Phi - \Psi$ satisfies $(PS)$-condition, then it satisfies $^{[r_1]}(PS)^{[r_2]}$-condition for all $r_1, r_2 \in [-\infty, +\infty]$ with $r_1 < r_2$. 
Proposition. Let $X$ be a reflexive real Banach space; $\Phi : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive and continuously Gâteaux differentiable function whose Gâteaux derivative admits a continuous inverse on $X^*$, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable function whose Gâteaux derivative is compact.

Then, for all $r_1, r_2 \in [-\infty, +\infty[$, with $r_1 < r_2$, the functional $\Phi - \Psi$ satisfies the $^{[r_1]} (PS)^{[r_2]}$-condition.
To prove the local minimum theorem we use the theory for locally Lipschitz functionals investigated by K.C. Chang, which is based on the *Nonsmooth Analysis* by F.H. Clarke, and generalizes the study on the variational inequalities as given by A. Szulkin.
This theory is applied to study variational and variational-hemivariational inequalities. In particular, for instance, differential inclusions and equations with discontinuous nonlinearities are investigated.
Here, by using the nonsmooth theory we obtain results for smooth functions.

THE EKELAND VARIATIONAL PRINCIPLE

Arguing in a classical way of the smooth analysis (as, for instance, Ghossoub), but using the definitions and properties of the non-smooth analysis (as, for instance, Motreanu-Radulescu, the following consequence of the Ekeland variational Principle can be obtained.
Lemma. Let $X$ be a real Banach space and $I : X \to \mathbb{R}$ a locally Lipschitz function bounded from below. Then, for all minimizing sequence of $I$, \( \{u_n\}_{n \in \mathbb{N}} \subseteq X \), there exists a minimizing sequence of $I$, \( \{v_n\}_{n \in \mathbb{N}} \subseteq X \), such that

\[
I(v_n) \leq I(u_n) \quad \forall n \in \mathbb{N},
\]

\[
I^*(v_n; h) \geq -\varepsilon_n \|h\| \quad \forall h \in X, \quad \forall n \in \mathbb{N}, \quad \text{where } \varepsilon_n \to 0^+.
\]
Sketch of Proof. Put

\[ M = r_2 - \Phi(x_0) + \Psi(x_0), \]

\[ \Psi_M(u) = \begin{cases} 
\Psi(u) & \text{if } \Psi(u) < M \\
M & \text{if } \Psi(u) \geq M,
\end{cases} \]

\[ \Phi^{r_1}(u) = \begin{cases} 
\Phi(u) & \text{if } \Phi(u) > r_1 \\
r_1 & \text{if } \Phi(u) \leq r_1,
\end{cases} \]

\[ J = \Phi^{r_1} - \Psi_M. \]

Clearly, \( J \) is locally Lipschitz and bounded from below. Hence, Lemma and a suitable computation ensure the conclusion.
Let $X$ be a real Banach space and $\Phi, \Psi : X \to \mathbb{R}$ two continuously Gâteaux differentiable functions. Put

$$I = \Phi - \Psi$$

and assume that there are $x_0 \in X$ and $r_1, r_2 \in \mathbb{R}$, with $r_1 < \Phi(x_0) < r_2$, such that

$$\sup_{u \in \Phi^{-1}([r_1, r_2])} \Psi(u) \leq r_2 - \Phi(x_0) + \Psi(x_0), \quad (1)$$

$$\sup_{u \in \Phi^{-1}([-\infty, r_1))} \Psi(u) \leq r_1 - \Phi(x_0) + \Psi(x_0). \quad (2)$$

Moreover, assume that $I$ satisfies \text{[(PS)$^{[r_1]}$]$^{[r_2]}$}-condition.

Then, there is $u_0 \in \Phi^{-1}([r_1, r_2])$ such that $I(u_0) \leq I(u)$ for all $u \in \Phi^{-1}([r_1, r_2])$ and $I'(u_0) = 0$. 
Let $X$ be a real Banach space and $\Phi, \Psi : X \to \mathbb{R}$ two continuously Gâteaux differentiable functions with $\Phi$ bounded from below and $\Phi(0) = \Psi(0) = 0$. Fix $r > 0$ such that $\sup_{u \in \Phi^{-1}(-\infty, r]} \Psi < +\infty$ and assume that, for each $\lambda \in \left] 0, \frac{r}{\sup_{u \in \Phi^{-1}(-\infty, r]} \Psi(u)} \right]$, the functional $I_{\lambda} = \Phi - \lambda \Psi$ satisfies $(PS)^{[r]}$-condition.

Then, for each $\lambda \in \left] 0, \frac{r}{\sup_{u \in \Phi^{-1}(-\infty, r]} \Psi(u)} \right]$, there is $u_1 \in \Phi^{-1}(-\infty, r]$ such that $I_{\lambda}(u_1) \leq I_{\lambda}(u)$ for all $u \in \Phi^{-1}(-\infty, r]$ and $I'_{\lambda}(u_1) = 0$. 

First

Three consequences
Let $X$ be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functions. Fix $\inf_X \Phi < r < \sup_X \Phi$ and assume that

$$\rho(r) > 0,$$

where $\rho(r) = \sup_{v \in \Phi^{-1}(]r, \infty[)} \frac{\Psi(v) - \sup_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)}{\Phi(v) - r}$, and

for each $\lambda > \frac{1}{\rho(r)}$ the function $I_\lambda = \Phi - \lambda \Psi$ is bounded from below and satisfies $^{[r]}(PS)$-condition.

Then, for each $\lambda > \frac{1}{\rho(r)}$ there is $u_2 \in \Phi^{-1}(]r, +\infty[)$ such that $I_\lambda(u_2) \leq I_\lambda(u)$ for all $u \in \Phi^{-1}(]r, +\infty[)$ and $I'_{\lambda}(u_2) = 0.$
Let $X$ be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$. Assume that there are $r \in \mathbb{R}$ and $\tilde{u} \in X$, with $0 < \Phi(\tilde{u}) < r$, such that

$$\sup_{u \in \Phi^{-1}([-\infty, r[)} \frac{\Psi(u)}{r} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})}$$

and, for each $\lambda \in \left[ \frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup_{u \in \Phi^{-1}([-\infty, r[)} \Psi(u)} \right]$, the functional $I_{\lambda} = \Phi - \lambda \Psi$ satisfies $(PS)^{[r]}$-condition.

Then, for each $\lambda \in \left[ \frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup_{u \in \Phi^{-1}([-\infty, r[)} \Psi(u)} \right]$, there is $u_0 \in \Phi^{-1}(]0, r[)$ (hence, $u_0 \neq 0$) such that $I_{\lambda}(u_0) \leq I_{\lambda}(u)$ for all $u \in \Phi^{-1}(]0, r[)$ and $I_{\lambda}'(u_0) = 0$.


Some examples on third consequence of LOCAL MINIMUM THEOREM

Consider

\[ (P_\lambda) \quad \begin{cases} \quad -u'' = \lambda \alpha(x) f(u) & x \in ]0, 1[ \\ u(0) = u(1) = 0. \end{cases} \]

\( \alpha \in L^1([0, 1]), \ \alpha(x) \geq 0 \ \text{a.e.} \ x \in [0, 1], \ \alpha \neq 0, \)

\( f : \mathbb{R} \to \mathbb{R} \) is continuous and nonnegative function,

\( F(\xi) = \int_0^\xi f(t) dt \ \forall \xi \in \mathbb{R}, \)

\( \lambda \) is a positive parameter.
Theorem. Assume that $f$ is nonnegative and there exist two positive constants $c, d$, with $d < c$, such that

$$\frac{F(c)}{c^2} < \left( \frac{\int_{\frac{3}{4}}^{\frac{1}{4}} \alpha(x) dx}{2\|\alpha\|_1} \right) \frac{F(d)}{d^2}.$$ 

Then, for each $\lambda \in \left[ \frac{4}{\int_{\frac{3}{4}}^{\frac{1}{4}} \alpha(x) dx} \frac{d^2}{F(d)}, \frac{2}{\|\alpha\|_1} \frac{c^2}{F(c)} \right]$, the problem $(P_\lambda)$ admits at least one positive weak solution $\bar{u}$ such that

$$|\bar{u}(x)| < c \text{ for all } x \in [0, 1].$$
When $\alpha \equiv 1$ the algebraic inequality becomes

there are two positive constants $c, d$, with $d < c$ such that

$$\frac{F(c)}{c^2} < \frac{1}{4} \frac{F(d)}{d^2}.$$ 

In this case, the interval is

$$\left[ 8 \frac{d^2}{F(d)}, 2 \frac{c^2}{F(c)} \right]$$

and the solution is a classical solution.
Corollary Assume that

$$\lim_{t \to 0^+} \frac{f(t)}{t} = +\infty,$$

and put

$$\lambda^* = \frac{2}{\|\alpha\|_1} \sup_{c \in [0,+\infty[} \frac{c^2}{F(c)}.$$

Then, for each $\lambda \in ]0, \lambda^*[,$ the problem $(P_\lambda)$ admits at least one positive weak solution $\tilde{u}.$
We have the same situation for other ordinary nonlinear differential problems and for elliptic problems involving the p-laplacian with $p > n$.

However, the same type of low can be obtained also for $p \leq n$. 
\[
(\bar{D}_\lambda) \quad \begin{cases} 
- \Delta u = \lambda f(u) & \text{in } \Omega \\
 u = 0 & \text{on } \partial \Omega,
\end{cases}
\]
Theorem. Let $f : \mathbb{R} \to \mathbb{R}$ be a nonnegative continuous function satisfying

(h) there exist two non-negative constants $a_1, a_2$ and $q \in ]1, 2N/(N - 2)[$ such that

$$f(t) \leq a_1 + a_2 |t|^{q-1} \quad \forall t \in \mathbb{R}.$$ 

Assume that

$$\lim_{t \to 0^+} \frac{f(t)}{t} = +\infty,$$

and put

$$\lambda^* = \frac{1}{\left(\sqrt{2c_1 a_1} + \frac{2^{q/2} c_q a_2}{q}\right)}.$$

Then, for each $\lambda \in ]0, \lambda^*[$, the problem $(D_\lambda)$ admits at least one positive weak solution.


PHD thesis of Sciammetta:

PHD thesis of Pizzimenti:
Critical nonlinearities

\[
-\Delta u = u^{\frac{N+2}{N-2}} + g(u) \quad \text{in} \quad \Omega, \\
u > 0 \quad \text{in} \quad \Omega, \\
u|_{\partial \Omega} = 0,
\]

\[g(u) = \mu |u|^s, \quad \mu \geq 0 \quad s > 0.\]

The embedding \( H_0^1(\Omega) \) in \( L^{\frac{2N}{N-2}}(\Omega) \) is not compact.
If $\mu = 0$, the problem has no solution (Pohozaev, 1965).

If $s \geq 1$ the problem has at least one solution for suitable $\mu > 0$ (Brezis-Nirenberg, 1983).

If $s < 1$ the problem has at least two solutions for suitable $\mu > 0$ (Ambrosetti-Brezis-Cerami, 1994).
Fix $0 < s < 1$. Then, there is $\Lambda > 0$ such that for each $\mu \in ]0, \Lambda[$ problem

\[
\begin{cases}
-\Delta u = u^{\frac{N+2}{N-2}} + \mu u^s & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u|_{\partial \Omega} = 0,
\end{cases}
\]

admits at least two weak solutions.
Their proof is a clever combination of topological and variational methods. Precisely, they determine the existence of a first solution by using the method of sub- and super-solutions and then, through a deep reasoning, prove that this solution is the minimum of a suitable functional and apply the mountain pass theorem so ensuring the existence of a second solution. However, in their proof, no numerical estimate of $\Lambda$ is provided.
Owing to the local minimum theorem we obtain the first solution directly as a minimum. Moreover, a precise estimate of parameters is given.

\[
\begin{align*}
-\Delta u &= \lambda (h(u) + \mu g(u)) \quad \text{in} \quad \Omega, \\
u|_{\partial \Omega} &= 0,
\end{align*}
\]

\[h(t) = t^{2^*-1}, \quad g(t) = t^{q-1} \quad f(t) = h(t) + \mu g(t) \quad \Phi(u) = \frac{\|u\|^2}{2}, \quad \Psi(u) = \int_\Omega F(u(x))dx\]

\[
\lambda_r^* = \frac{r}{\left(\frac{\mu}{q} c_q^2 (2r)^{q/2} + \frac{(2r)^{2^*/2}}{2^*} c_2^{2^*} \right)}, \quad \tilde{\lambda}_r = \frac{1}{\frac{2^*}{2} c_2^{2^*} (2r N)^{\frac{N}{2-N}}},
\]

\[
\bar{\lambda}_r = \min \left\{ \lambda_r^*, \tilde{\lambda}_r \right\},
\]

\[
c_2 = \frac{1}{\sqrt{N(N-2)\pi}} \left(\frac{N!}{2\Gamma(1+N/2)}\right)^{1/N},
\]

\[
c_s \leq \frac{\text{meas}(\Omega)^{2^*-q}}{\sqrt{N(N-2)\pi}} \left(\frac{N!}{2\Gamma(N/2+1)}\right)^{1/N},
\]

\[I_\lambda = \Phi - \lambda \Psi\]
Lemma Fix $r > 0$. Then, for each $\lambda \in ]0, \lambda_r[$ the functional $I_\lambda = \Phi - \lambda \Psi$ satisfies the $(PS)^{[r]}$-condition.

Theorem Fix $q \in ]1, 2[$. Then, there exists $\mu^* > 0$, where

$$\mu^* = \left( \frac{q}{c_q} \right) \left( \min \left\{ \left( \frac{2^*}{2^{\frac{2^*+2}{c_{2^*}^2}}} \right)^{\frac{2^*}{2^{\frac{2^*+2}{c_{2^*}^2}}}} ; \frac{1}{3N} \left( \frac{1}{c_{2^*}^2} \right)^{\frac{N-2}{2}} \right\} \right)^{\frac{2-q}{2}}$$

such that for each $\mu \in ]0, \mu^*[$ problem

$$(D_\mu) \quad \begin{cases} -\Delta u = u^{2^*-1} + \mu u^{q-1} & \text{in } \Omega, \\ u|_{\partial \Omega} = 0, \end{cases}$$

admits at least one positive weak solution $u_\mu$ such that

$$\|u_\mu\| < \left( \frac{2^*}{c_{2^*}^2} \right)^{\frac{1}{2^*-2}}.$$

Moreover, the mapping

$$\mu \rightarrow \frac{1}{2} \int_\Omega |\nabla u_\mu|^2 dx - \int_\Omega \frac{1}{2^*} |u_\mu|^{2^*} dx - \mu \int_\Omega \frac{1}{q} |u_\mu|^q dx$$

is negative and strictly decreasing in $]0, \mu^*[$.
Theorem

Fix $q \in ]1, 2[$. Then, there exists $\mu^* > 0$, where

$$
\mu^* = \left( \frac{q}{c_q^q} \frac{1}{2^{q+2}} \right) \left( \min \left\{ \left( \frac{2^*}{2^{2^*+2}} \frac{2^*}{c_{2^*}^2} \right)^{\frac{2}{2^*-2}} ; \frac{1}{3N} \left( \frac{1}{c_{2^*}^2} \right)^{\frac{N-2}{2}} \right\} \right)^{\frac{2-q}{2}}
$$

such that for each $\mu \in ]0, \mu^*[$ problem $(D_\mu)$ admits at least two positive weak solutions $u_\mu$ and $w_\mu$ such that $\|u_\mu\| < \left( \frac{2^*}{c_{2^*}^2} \right)^{\frac{1}{2^*+2}}$ and $w_\mu > u_\mu$. 
Example Fix $N = 3$ and let $\Omega = \{x \in \mathbb{R}^3 : |x| < 1\}$. Then, the problem

$$
\begin{cases}
-\Delta u = u^5 + \frac{3}{8}\sqrt{u} & \text{in} & \Omega, \\
 u|_{\partial\Omega} = 0,
\end{cases}
$$

admits at least two positive weak solutions $u_\mu$ and $w_\mu$ such that $\int_{\Omega} |\nabla u_\mu(x)|^2 \, dx < \frac{9\pi^2}{2^{5/2}}$, $w_\mu > u_\mu$.


Mean curvature operator

\[(P_\lambda) \quad - \left( \frac{u'}{\sqrt{1 + |u'|^2}} \right)' = \lambda f(u) \quad \text{in } ]0, 1[,\]

\[u(0) = u(1) = 0,\]
Theorem. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous and nonnegative function and assume that there exist two positive constants $c, d$, with $d < c$, such that

$$
\left[ \left( \frac{F(c)}{c^2} \right)^2 + \left( 2 \max_{[0,c]} f \right)^2 \right]^{1/2} < \frac{1}{4} \frac{F(d)}{d^2}.
$$

Then, for every

$$
\lambda \in \left[ \frac{8d^2}{F(d)}, 2 \left[ \left( \frac{F(c)}{c^2} \right)^2 + \left( 2 \max_{[0,c]} f \right)^2 \right]^{-1/2} \right]
$$

problem $(P_\lambda)$ admits at least one non trivial, nonnegative solution $u_\lambda \in C^{1,\tau}([0,1])$ for some $\tau \in [0,1]$, such that

$$
\|u_\lambda\|_{C^0} < c, \quad \|u'_\lambda\|_{C^0} \leq \frac{2 \max_{[0,c]} f}{F(c)} \frac{c}{c^2}.
$$
Theorem \[ \begin{align*} & \text{Let } f : \mathbb{R} \to \mathbb{R} \text{ be a continuous, nonnegative function such that} \\
& \quad \limsup_{s \to 0^+} \frac{F(s)}{s^2} = +\infty. \\
& \text{Then, for each } \lambda \in ]0, \lambda^*[ \text{, where} \\
& \quad \lambda^* = 2 \left\{ [F(1)]^2 + \left[ \max_{[0,1]} f \right]^2 \right\}^{-1/2}, \\
& \text{problem } (P_\lambda) \text{ admits at least one nontrivial, nonnegative solution } u_\lambda \in C^{1,\tau}([0,1]) \\
& \text{for some } \tau \in ]0,1[, \text{ such that} \\
& \quad \|u_\lambda\|_{C^0} < 1, \quad \|u'_\lambda\|_{C^0} \leq \frac{\max_{[0,1]} f}{F(1)}. \end{align*} \]

BONANNO G. - LIVREA R. - MAWHIN J., Existence results for parametric boundary value problems involving the mean curvature operator. preprint.
Boundary value problems in the real line

Find $u \in W^{1,p}(\mathbb{R})$ satisfying

$$(P_\lambda) \quad -(|u'(x)|^{p-2}u'(x))' + B|u(x)|^{p-2}u(x) = \lambda \alpha(x)g(u(x)) \text{ a.e. in } \mathbb{R},$$

**Theorem:** Assume that there exist two positive constants $\gamma, \kappa$, with $\kappa < \gamma$, such that

$$\frac{G(\gamma)}{\gamma^p} < R \frac{G(\kappa)}{\kappa^p}.$$ 

Then, for each $\lambda \in \left[ \frac{1}{pc_B^p |\alpha|_1} \frac{1}{RG(\kappa)} \frac{1}{G(\gamma)} \right]$, problem $(P_\lambda)$ admits at least one nontrivial and nonnegative solution $u_{0,\lambda}$ such that $|u_{0,\lambda}|_\infty < \gamma.$
Corollary

Assume that

\[ \lim_{t \to 0^+} \frac{g(t)}{t^{p-1}} = +\infty. \]

Then, for each \( \gamma > 0 \) and for each \( \lambda \in \left[ 0, \frac{1}{pc_B |\alpha|_1} \frac{\gamma^p}{G(\gamma)} \right] \) problem \((P_\lambda)\) admits at least one nontrivial and nonnegative solution \( u_{0,\lambda} \) such that \( |u_{0,\lambda}|_\infty < \gamma \).
WALKING IN THE MOUNTAINS
Some remarks on the classical Ambrosetti-Rabinowitz theorem are presented. In particular, it is observed that the geometry of the mountain pass, if the function is bounded from below, is equivalent to the existence of at least two local minima, while, when the function is unbounded from below, it is equivalent to the existence of at least one local minimum.
So, the Ambrosetti-Rabinowitz theorem actually ensures three or two distinct critical points, according to the function is bounded from below or not.
Let $X$ be a real Banach space, $I : X \to \mathbb{R}$ a continuously Gâteaux differentiable function which verifies (PS).
Assume that

\[(G) \text{ there are } u_0, u_1 \in X \text{ and } r \in \mathbb{R}, \text{ with } 0 < r < \|u_1 - u_0\|, \text{ such that}\]

\[
\inf_{\|u-u_0\|=r} I(u) > \max\{I(u_0), I(u_1)\}.
\]

Then, \(I\) admits a critical value \(c\) characterized by

\[
c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t))
\]

where

\[
\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = u_0; \gamma(1) = u_1\}.
\]
(G') there are \( u_0, u_1 \in X \) and \( r, R \in \mathbb{R} \), with \( 0 < r < R < \| u_1 - u_0 \| \), such that

\[
\inf_{r < \| u - u_0 \| < R} I(u) \geq \max\{I(u_0), I(u_1)\}.
\]

**Corollary.** If \( I \) admits two local minima, then \( I \) admits a third critical point.
(MG) there are $u_0, u_1 \in X$ and $r \in \mathbb{R}$, with $0 < r < \|u_1 - u_0\|$, such that

$$\inf_{\|u-u_0\|=r} I(u) \geq \max\{I(u_0), I(u_1)\}.$$
Theorem. Let $X$ be a real Banach space, $I : X \to \mathbb{R}$ a continuously Gâteaux differentiable function which verifies (PS) and it is bounded from below. Then, the following assertions are equivalent:

$(MG)$ there are $u_0, u_1 \in X$ and $r \in \mathbb{R}$, with $0 < r < \|u_1 - u_0\|$, such that

$$
\inf_{\|u - u_0\| = r} I(u) \geq \max\{I(u_0), I(u_1)\};
$$

$(L)$ $I$ admits at least two distinct local minima.
So, the Ambrosetti-Rabinowitz theorem, when the function is bounded from below actually ensures three distinct critical points.

In fact, in this case the mountain pass geometry implies the existence of two local minima and the Pucci-Serrin theorem ensures the third critical point.
In a similar way it is possible to see that, when the function is unbounded from below, the mountain pass geometry is equivalent to the existence of at least one local minimum.

In this case, the following condition is requested:

The function $I$ is bounded from below on every bounded set of $X$. 
Let $X$ be a real Banach space and $I : X \to \mathbb{R}$ be a functional of class $C^1$ satisfying the (PS)–condition and the mountain pass geometry (MG). Assume that $I$ is bounded from below on every bounded set of $X$. Then, $I$ admits two or three distinct critical points according to whether it is unbounded from below or not.


Let $X$ be a real Banach space and $\Phi, \Psi : X \to \mathbb{R}$ two continuously Gâteaux differentiable functions with $\Phi$ bounded from below and $\Phi(0) = \Psi(0) = 0$. Fix $r > 0$ such that $\sup_{u \in \Phi^{-1}([-\infty, r[)} \Psi < +\infty$ and assume that,

for each $\lambda \in \left] 0, \frac{r}{\sup_{u \in \Phi^{-1}([-\infty, r[)} \Psi(u)} \right]$, the functional

$I_\lambda = \Phi - \lambda \Psi$ satisfies $(PS)^{[r]}$-condition.

Then, for each $\lambda \in \left] 0, \frac{r}{\sup_{u \in \Phi^{-1}([-\infty, r[)} \Psi(u)} \right]$, there is $u_1 \in \Phi^{-1}([-\infty, r[)$ such that $I_\lambda(u_1) \leq I_\lambda(u)$ for all $u \in \Phi^{-1}([-\infty, r[)$ and $I'_\lambda(u_1) = 0$. 

Three consequences
Let \( X \) be a real Banach space and let \( \Phi, \Psi : X \to \mathbb{R} \) be two continuously Gâteaux differentiable functions. Fix \( \inf_X \Phi < r < \sup_X \Phi \) and assume that
\[
\rho(r) > 0,
\]
where
\[
\rho(r) = \sup_{v \in \Phi^{-1}([r, \infty[)} \frac{\Psi(v) - \sup_{u \in \Phi^{-1}([-\infty, r])} \Psi(u)}{\Phi(v) - r}, \quad \text{and}
\]
for each \( \lambda > \frac{1}{\rho(r)} \) the function \( I_\lambda = \Phi - \lambda \Psi \) is bounded from below and satisfies \([r](PS)\)-condition.

Then, for each \( \lambda > \frac{1}{\rho(r)} \) there is \( u_2 \in \Phi^{-1}([r, +\infty[) \) such that \( I_\lambda(u_2) \leq I_\lambda(u) \) for all \( u \in \Phi^{-1}([r, +\infty[) \) and \( I'_\lambda(u_2) = 0 \).
From the preceding two variants of the local minimum theorem, a three critical points theorem is obtained. Here a special case is pointed out.
A THREE CRITICAL POINTS THEOREM

Let $X$ be a real Banach space and $\Phi, \Psi : X \to \mathbb{R}$ two continuously Gâteaux differentiable functionals with $\Phi$ bounded from below. Assume that $\Phi(0) = \Psi(0) = 0$ and there are $r > 0$ and $\bar{x} \in X$, with $r < \Phi(\bar{x})$, such that

$$\sup_{u \in \Phi^{-1}([-\infty,r])} \frac{\Psi(u)}{r} < \frac{\Psi(\bar{x})}{\Phi(\bar{x})}. \tag{3}$$

Further assume that, for each

$$\lambda \in \Lambda := \left[ \frac{\Phi(\bar{x})}{\Psi(\bar{x})}, \sup_{u \in \Phi^{-1}([-\infty,r])} \frac{r}{\Psi(u)} \right],$$

the functional $I_\lambda = \Phi - \lambda \Psi$ is bounded from below and satisfies (PS)-condition.

Then, for each $\lambda \in \Lambda$ the functional $I_\lambda$ admits at least three critical points.
Consider the following two point boundary value problem

\[
(D_{\lambda}) \quad \begin{cases} 
-u'' = \lambda f(u) \text{ in } ]0, 1[ \\
u(0) = u(1) = 0,
\end{cases}
\]

where \( f : \mathbb{R} \rightarrow \mathbb{R} \) is a continuous function and \( \lambda \) is a positive real parameter.
Moreover, put

\[ F(\xi) = \int_0^\xi f(t)\,dt \]

for all \( \xi \in \mathbb{R} \) and assume, for clarity, that \( f \) is nonnegative.
Theorem. Assume that there are two positive constants \( c \) and \( d \), with \( c < d \), such that

\[
\frac{F(c)}{c^2} < \frac{1}{4} \frac{F(d)}{d^2}
\]

and there are two positive constants \( a \) and \( s \), with \( s < 2 \), such that

\[
F(\xi) \leq a(1 + |\xi|^s) \quad \forall \xi \in \mathbb{R}.
\]

Then, for each \( \lambda \in \left[ 8 \frac{d^2}{F(d)}, 2 \frac{c^2}{F(c)} \right] \), problem \((D_\lambda)\) admits at least three (nonnegative) classical solutions.
TWO-POINT BOUNDARY VALUE PROBLEMS

NEUMANN BOUNDARY VALUE PROBLEMS

MIXED BOUNDARY VALUE PROBLEMS

STURM-LIOUVILLE BOUNDARY VALUE PROBLEMS

HAMILTONIAN SYSTEMS

FOURTH-ORDER ELASTIC BEAM EQUATIONS
ELLiptic Dirichlet Problems involving the p-Laplacian with p > n

Elliptic Neumann Problems involving the p-Laplacian with p > n

Elliptic Systems

Boundary Value Problems on the Half Line

Nonlinear Difference Problems
Nonlinear eigenvalue problems in Orlics-Sobolev spaces

Nonlinear elliptic problems on the Sierpiński Gasket

Generalized Yamabe equations on Riemannian manifolds

Elliptic problems involving the p(x)-Laplacian
{\begin{align*}
-\Delta u &= \lambda f(x, u) \quad \text{in } \Omega, \\
u|_{\partial \Omega} &= 0,
\end{align*}}

(h) There exist two non-negative constants $a_1, a_2$ and $q \in ]1, 2N/(N - 2)[\ such \ that

\[ |f(x, t)| \leq a_1 + a_2 |t|^{q-1}, \]

for every $(x, t) \in \Omega \times \mathbb{R}$.

(j) There exist two positive constants $c$ and $d$, with $d > c\kappa$ such that

\[ \inf_{x \in \Omega} \frac{F(x, d)}{d^2} > a_1 \frac{K_1}{c} + a_2 K_2 c^{q-2}, \tag{1} \]

where $a_1, a_2$ are given in (h) and $\kappa, K_1, K_2$ are given by

\[ \kappa := D\sqrt{2} \frac{\Gamma(1 + N/2)}{2\pi^{N/4}} \left( \frac{D}{D^N - (D/2)^N} \right)^{1/2}, \quad K_1 := \frac{2\sqrt{2}c_1(2^N - 1)}{D^2}, \quad K_2 := \frac{2^{q+2} c_q(2^N - 1)}{qD^2}. \]


…and others
Theorem 1. Let $X$ be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that $\Phi$ is bounded from below and $\Phi(0) = \Psi(0) = 0$. Fix $r > 0$ such that $\sup_{u \in \Phi^{-1}([-\infty, r])} \Psi(u) < +\infty$ and assume that, for each

$$\lambda \in \left]0, \frac{r}{\sup_{u \in \Phi^{-1}([-\infty, r])} \Psi(u)} \right[,$$

the functional $I_\lambda = \Phi - \lambda \Psi$ satisfies the (PS)-condition and it is unbounded from below. Then, for each

$$\lambda \in \left]0, \frac{r}{\sup_{u \in \Phi^{-1}([-\infty, r])} \Psi(u)} \right[,$$

the functional $I_\lambda$ admits two distinct critical points.
Theorem 2. Let $X$ be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$. Assume that there are $r \in \mathbb{R}$ and $\tilde{u} \in X$, with $0 < \Phi(\tilde{u}) < r$, such that
\[
\frac{\sup_{u \in \Phi^{-1}(]0, r]} \Psi(u)}{r} > \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})}
\]
and, for each
\[
\lambda \in \left[ \frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup_{u \in \Phi^{-1}(]0, r]} \Psi(u)} \right],
\]
the functional $I_\lambda = \Phi - \lambda \Psi$ satisfies the (PS)-condition and it is unbounded from below. Then, for each
\[
\lambda \in \left[ \frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup_{u \in \Phi^{-1}(]0, r]} \Psi(u)} \right],
\]
the functional $I_\lambda$ admits two distinct critical points $u_1$ and $u_2$ such that
\[
\Phi(u_1) < r, \quad u_1 \neq 0 \quad \text{and} \quad I_\lambda(u_2) = \inf_{\gamma \in \Gamma_{\tilde{x}}} \max_{t \in [0, 1]} I_\lambda(\gamma(t)),
\]
where $\tilde{x}$ is such that $I_\lambda(\tilde{x}) \leq I_\lambda(u_1)$. 
A TWO CRITICAL POINTS THEOREM

**Theorem 3.** Let $X$ be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$. Assume that there are $r \in \mathbb{R}$ and $\tilde{u} \in X$, with $0 < \Phi(\tilde{u}) < r$, such that

$$\frac{\sup_{u \in \Phi^{-1}([-\infty, r])} \Psi(u)}{r} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})}$$

and, for each

$$\lambda \in \left[ \frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup_{u \in \Phi^{-1}([-\infty, r])} \Psi(u)} \right],$$

the functional $I_\lambda = \Phi - \lambda \Psi$ satisfies the (PS)-condition and it is unbounded from below. Then, for each

$$\lambda \in \left[ \frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup_{u \in \Phi^{-1}([-\infty, r])} \Psi(u)} \right],$$

the functional $I_\lambda$ admits two distinct critical points $u_1$ and $u_2$ such that

$$\Phi(u_1) < r, \quad u_1 \neq 0 \quad \text{and} \quad u_2 \neq 0.$$
\[ (P_\lambda) \quad \begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial \Omega, \end{cases} \]
Theorem. Let $f : \mathbb{R} \to \mathbb{R}$ be a nonnegative continuous function satisfying

(h) there exist two non-negative constants $a_1, a_2$ and $q \in ]1, 2N/(N - 2)[$ such that

$$f(t) \leq a_1 + a_2 |t|^{q-1} \quad \forall t \in \mathbb{R}.$$ 

Put

$$\lambda^* = \frac{1}{\left( \sqrt{2c_1 a_1} + \frac{2^{q/2} c_q^q a_2}{q} \right)}.$$ 

Assume that

$$0 < \mu F(t) \leq tf(t) \quad \text{(A-R)}$$

for all $|t| \geq r$, for some $r > 0$ and for some $\mu > 2$. Then, for each $\lambda \in ]0, \lambda^*[,$ the problem $(P_\lambda)$ admits at least two weak solutions, whose at least one is positive.
Further, if in addition, assume
\[
\lim_{t \to 0^+} \frac{f(t)}{t} = +\infty,
\]
(1)

Then, for each \( \lambda \in ]0, \lambda^*[, \) the problem \((P_\lambda)\) admits at least two positive weak solutions.


WALKING IN THE MOUNTAINS
1. Mountain pass geometry
2. Palais-Smale condition

Then, there is a critical point.
1. **Strong mountain pass geometry**

2. **Weak Palais-Smith condition**

Then, there is a critical point near to local minimum.

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If we apply two times the first special case of the local minimum theorem and owing to a novel version of the mountain pass theorem where the (PS) cut off upper at $r$ is assumed we can give a variant of the three critical theorem. In the applications it became
Theorem. Assume that there are three positive constants \( c_1, d \) and \( c_2 \), with \( c_1 < d < \frac{\sqrt{2}}{2} c_2 \), such that

\[
\frac{F(c_1)}{c_1^2} < \frac{1}{6} \frac{F(d)}{d^2} \tag{1}
\]

and

\[
\frac{F(c_2)}{c_2^2} < \frac{1}{12} \frac{F(d)}{d^2}. \tag{2}
\]

Then, for each \( \lambda \in \left[ 12 \frac{d^2}{F(d)}, \min \left\{ 2 \frac{c_1^2}{F(c_1)}, \frac{c_2^2}{F(c_2)} \right\} \right] \), problem \( (D_\lambda) \) admits at least three (nonnegative) classical solutions \( u_i, i = 1, 2, 3 \), such that

\[
\max_{x \in [0,1]} |u_i(x)| < c_2, \quad i = 1, 2, 3.
\]
FURTHER APPLICATIONS OF THE LOCAL MINIMUM THEOREM
If we apply iteratively the first special case of the local minimum theorem in a suitable way, we obtain an infinitely many critical points theorem. As an example of application, here, we present the following result.
Theorem. Assume that

$$(1) \quad \liminf_{\xi \to +\infty} \frac{F(\xi)}{\xi^2} < \frac{1}{4} \limsup_{\xi \to +\infty} \frac{F(\xi)}{\xi^2}.$$ 

Then, for each $\lambda \in \left[ \frac{8}{\limsup_{\xi \to +\infty} \frac{F(\xi)}{\xi^2}}, \frac{2}{\liminf_{\xi \to +\infty} \frac{F(\xi)}{\xi^2}} \right]$, the problem $(D_\lambda)$ admits a sequence of pairwise distinct positive classical solutions.
Previous results can be applied to perturbed problems, as, for instance, the following

\[
\begin{align*}
-u'' &= \lambda f(u) + \mu g(x, u) \quad \text{in } (0, 1] \\
\quad u(0) &= u(1) = 0,
\end{align*}
\]

or in the framework of the non-smooth Analysis. As example, here, the following problem is considered.
Let $\Omega$ be a non-empty, bounded, open subset of the Euclidean space $\mathbb{R}^N$, $N \geq 1$, with $C^1$-boundary $\partial \Omega$, let $p \in ]N, +\infty[$, and let $q \in L^\infty(\Omega)$ satisfy $\text{ess inf}_{x \in \Omega} q(x) > 0$.

**Problem:** Find $u \in K$ such that, for all $v \in K$,

\[
\int_\Omega |\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla (v(x) - u(x)) \, dx + \int_\Omega q(x)|u(x)|^{p-2} u(x)(v(x) - u(x)) \, dx \\
+ \int_\Omega \lambda \alpha(x) F^\circ(u(x); v(x) - u(x)) \, dx + \int_{\partial \Omega} \mu \beta(x) G^\circ(\gamma u(x); \gamma v(x) - \gamma u(x)) \, d\sigma \geq 0,
\]

where $K$ is a closed convex subset of $W^{1,p}(\Omega)$ containing the constant functions, and $\alpha \in L^1(\Omega)$, $\beta \in L^1(\partial \Omega)$, with $\alpha(x) \geq 0$ for a.a. $x \in \Omega$, $\alpha \not= 0$, $\beta(x) \geq 0$ for a.a. $x \in \partial \Omega$, and $\lambda, \mu$ are real parameters, with $\lambda > 0$ and $\mu \geq 0$. Here, $F^\circ$ and $G^\circ$ stand for Clarke’s generalized directional derivatives of locally Lipschitz
functions $F, G : \mathbb{R} \to \mathbb{R}$ given by $F(\xi) = \int_0^\xi f(t)dt$, $G(\xi) = \int_0^\xi g(t)dt$, $\xi \in \mathbb{R}$, with $f, g : \mathbb{R} \to \mathbb{R}$ locally essentially bounded functions, and $\gamma : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ denotes the trace operator.

A prototype of the previous problem for $K = W^{1,p}(\Omega)$ is the following boundary value problem with nonsmooth potential and nonhomogeneous, nonsmooth Neumann boundary condition

\[
\begin{cases}
\Delta_p u - q(x)|u|^{p-2}u \in \lambda\alpha(x)\partial F(u) & \text{in } \Omega, \\
|\nabla u|^{p-2} \frac{\partial u}{\partial \nu} \in -\mu\beta(x)\partial G(\gamma u) & \text{on } \partial\Omega.
\end{cases}
\]


Arguing as ARINO-GAUTIER-PENOT and by starting from Ky FAN theorem, the following fixed point theorem for multifunctions with weakly sequential closed graph has been established:

**Theorem** Let $X$ be a real Banach space and let $K$ be a weakly compact convex subset of $X$. Suppose $\Phi$ is a multifunction from $K$ into itself with nonempty convex values and weakly sequentially closed graph. Then, there exists $x_0 \in K$ such that $x_0 \in \Phi(x_0)$. 
Recently, as a consequence, the following coincidence point theorem has been established:

**Theorem 1** \( \text{Let } X, Y \text{ be two real Banach spaces, let } K \text{ be a weakly compact convex subset of } X, \text{ and let } F, G \text{ be two weakly sequentially continuous functions from } K \text{ into } Y. \text{ Assume that } F \text{ is invertible and} \)
\[
G(K) \subseteq F(K).
\]
Then, there exists \( x_0 \in K \) such that \( F(x_0) = G(x_0) \).

Taking into account that a critical point of \( \Phi - \Psi \) is a coincidence point for \( \Phi' \) and \( \Psi' \), one has the following critical point theorem

**Theorem 2** \( \text{Let } X \text{ be a real reflexive Banach spaces and let } \Phi, \Psi : X \to \mathbb{R} \text{ be two Gateaux differentiable functionals such that } \Phi' : X \to X^* \text{ is invertible and } \Phi'^{-1}, \Psi' \text{ are weakly sequentially continuous. Assume that there is } r > 0 \text{ such that for all } u \in X \text{ such that } ||u|| \leq r \text{ there is } v \in X \text{ with } ||v|| \leq r \text{ such that } \Psi'(u) = \Phi'(v). \text{ Then, the functional } \Phi - \Psi \text{ admits at least a critical point } x_0 \text{ such that } ||x_0|| \leq r. \)
\[ \begin{align*}
\begin{cases}
-u'' = g(t, u, u') & \text{in } (0, 1), \\
u(0) = u(1) = 0.
\end{cases}
\end{align*} \]

**Theorem:** Assume that there is \( r > 0 \) such that

\[
\max_{(t,s,\xi)\in[0,1] \times [-r/2,r/2] \times [-r,r]} |g(t, s, \xi)| \leq r,
\]

Then, problem (1) has a classical solution \( u \) satisfying \((u(t), u'(t)) \in [-r/2, r/2] \times [-r, r] \) for all \( t \in [0, 1] \).

**Example**  The following problem

\[ \begin{align*}
\begin{cases}
-u'' = \frac{1}{4} (1 + u'^2) & \text{in } (0, 1), \\
u(0) = u(1) = 0
\end{cases}
\end{align*} \]

admits at least one nonzero classical solution.

BONANNO G. - CANDITO P. - MOTREANU D., A coincidence point theorem for sequentially continuous mapping, preprint.

Thank you very much for your kind attention