Topological and variational methods for ODEs
Dedicated to Massimo Furi Professor Emeritus at the University of Florence

GLOBAL CONTINUATION OF PERIODIC SOLUTIONS FOR RFDE’S ON MANIFOLDS

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Setting of the problem

We study retarded functional differential equations (RFDE) on $M$ of the type:

$$x'(t) = \lambda F(t, x_t)$$

(1)

where:

- $M \subseteq \mathbb{R}^k$ is a smooth manifold (possibly noncompact),
- $\lambda \geq 0$ is a parameter,
- $F$ is a functional vector field on $M$.

Notation: $x_t(\theta) = x(t + \theta), \theta \in (-\infty, 0]$. 
**Functional vector fields**

The map $F : \mathbb{R} \times BU((−\infty, 0], M) \to \mathbb{R}^k$ is continuous, $T$-periodic in the first variable and such that

$$F(t, \varphi) \in T_{\varphi(0)}M, \quad \forall (t, \varphi) \in \mathbb{R} \times BU((−\infty, 0], M)$$

where $T_pM \subseteq \mathbb{R}^k$ denotes the tangent space of $M$ at $p$. 
Remark: We work in the space \( BU((-\infty, 0], M) \) of the bounded, uniformly continuous maps

\[
\varphi : (-\infty, 0] \to M.
\]

- \( BU((-\infty, 0], M) \) is a subset of the Banach space \( BU((-\infty, 0], \mathbb{R}^k) \) with the supremum norm;
- the topology in the space \( BU((-\infty, 0], M) \) is stronger than the compact-open topology of \( C((-\infty, 0], M) \);
- if \( x : J \to M \) is a solution of (1),
then the curve \( t \mapsto x_t \in BU((-\infty, 0], M) \), \( t \in J \), is continuous.
Goal: to prove global continuation results for $T$-periodic solutions of equation (1).

Tools:
- Fixed Point Index theory for locally compact maps on ANRs (ANRs = absolute neighborhood retracts)

References: Granas, Nussbaum, Eells–Fournier.
- Degree of a tangent vector field (Euler characteristic, rotation number).
Application: Retarded spherical pendulum

Consider the following second order equation on a boundaryless manifold $N \subseteq \mathbb{R}^s$:

$$x''_\pi(t) = G(t, x_t),$$

where (regarding (2) as a motion equation)

- $x''_\pi(t)$ is the tangential part of the acceleration $x''(t)$,
- the applied force $G$ is a $T$-periodic functional vector field.
Equivalently (2) can be written as

\[ x''(t) = r(x(t), x'(t)) + G(t, x_t), \]

where \( r(q, v) \) is the reactive force.

A **forced oscillation** of (2) is a solution which is \( T \)-periodic and globally defined on \( \mathbb{R} \).

**Problem:** to prove the existence of forced oscillations of (2).
Continuation results for ODEs on manifolds

Consider the parametrized ODE on $M \subseteq \mathbb{R}^k$

$$x'(t) = \lambda f(t, x(t))$$ (3)

where $f : \mathbb{R} \times M \rightarrow \mathbb{R}^k$ is a $T$-periodic tangent vector field on $M$.

Furi and Pera (1986) have obtained global continuation results for equation (3) by means of topological methods.
Applications to the spherical pendulum

Consider the following second order ODE on a boundaryless manifold \( N \subseteq \mathbb{R}^s \):

\[
x''_{\pi}(t) = g(t, x(t))
\]

Furi and Pera (1990) proved that equation (4) has forced oscillations in the case \( N = S^2 \) (the spherical pendulum) and \( N = S^{2n} \).

**Conjecture:** Equation (4) has forced oscillations if \( \chi(N) \neq 0 \) (Euler–Poincaré characteristic).
– Difficulty: they use in a crucial way the geometry of the sphere. The case of the ellipsoid is still open!

**Related works:**
- Capietto, Mawhin and Zanolin (1990);
Delay differential equations: some references

**General reference:** Hale and Verduyn Lunel (1993).

- **in Euclidean spaces:** Gaines and Mawhin (1977); Nussbaum and Mallet-Paret (1994); Krisztin and Walter (1999).

- **equations on manifolds:** Oliva (1976).
Equations with infinite delay, or Retarded Functional Differential Equations (RFDEs)


- **equations on manifolds:** *no general results were available!* Benevieri, C., Furi, Pera (2013) Discrete Contin. Dyn. Syst.
Delay differential equations on manifolds (finite delay)

We study the parametrized delay differential equation on $M$

$$x'(t) = \lambda f(t, x(t), x(t - \tau))$$ (5)

where $\tau > 0$ is the delay, and $f : \mathbb{R} \times M \times M \to \mathbb{R}^k$ is continuous, $T$-periodic in the first variable and tangent to $M$ in the second one; i.e.,

$$f(t + T, p, q) = f(t, p, q) \in T_pM, \quad \forall (t, p, q) \in \mathbb{R} \times M \times M.$$

We call $f$ a (generalized) vector field on $M$. 

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Remark. When $\partial M \neq \emptyset$ we require $f$ to be inward along $\partial M$; i.e.,

$$f(t, p, q) \in C_pM, \quad \forall (t, p, q) \in \mathbb{R} \times \partial M \times M.$$  

($C_pM \subseteq \mathbb{R}^k$ is the tangent cone of $M$ at $p$)

**Goal:** to obtain global continuation results for $T$-periodic solutions.

Main difficulty: we work in an infinite-dimensional setting.
Let $C_T(M)$ be the metric space of the continuous, $T$-periodic $M$-valued maps.

**Definition.** $(\lambda, x)$ in $[0, +\infty) \times C_T(M)$ is a $T$-periodic pair if $x : \mathbb{R} \to M$ is a $T$-periodic solution of (5) corresponding to $\lambda$.

A $T$-periodic pair of the type $(0, p_0)$, with $p_0 \in M$, is said to be trivial.

**Remark.** $C([-\tau, 0], M)$ and $C_T(M)$ are ANRs.

(when $M$ is boundaryless $\Rightarrow$ Banach manifolds)
**Fixed Point Index on ANRs** (Granas, 1972)

Let $X$ be a metric ANR (Borsuk, 1930), $k : \mathcal{D}(k) \subseteq X \rightarrow X$ locally compact, $U \subseteq X$ open, contained in $\mathcal{D}(k)$.

If $\text{Fix}(k, U) = \{x \in U : x = k(x)\}$ is compact, the pair $(k, U)$ is called *admissible*.

→ *fixed point index* of $k$ in $U$:

\[ \text{ind}_X(k, U) \in \mathbb{Z}. \]
Properties:

analogous to those of the classical **Leray–Schauder degree**

(\textit{Normalization, Additivity, Homotopy invariance}...)

- **Existence Property:**
  \[
  \text{ind}_X(k,U) \neq 0 \Rightarrow \text{Fix}(k,U) \text{ nonempty.}
  \]

- **Strong Normalization Property:**
  \( M \) a compact manifold \( \Rightarrow \text{ind}_M(I,M) = \chi(M) \)
  (the **Euler–Poincaré characteristic** of \( M \)).
Bifurcation points: necessary condition.

**Definition.** $p_0 \in M$ is a **bifurcation point** (of equation (5)) if every neighborhood of $(0, p_0)$ in $[0, +\infty) \times C_T(M)$ contains a nontrivial $T$-periodic pair (i.e., with $\lambda > 0$).

**Proposition.** $p_0 \in M$ bifurcation point $\Rightarrow$ the **mean value tangent vector field** $w : M \to \mathbb{R}^k$, defined by

$$w(p) = \frac{1}{T} \int_0^T f(t, p, p) \, dt,$$

vanishes at $p_0$. 
Global continuation result


- $M$ is closed in $\mathbb{R}^k$ (possibly noncompact)
- $U \subseteq M$ open such that $\deg(w, U)$ is defined and nonzero

$\Rightarrow$ there exists in $[0, +\infty) \times C_T(M)$ a **connected branch**

of nontrivial $T$-periodic pairs of (5) whose closure meets the set

$\{(0, p) : p \in U, w(p) = 0\}$ and satisfies at least one of the following properties:

(i) it is unbounded;

(ii) it contains a pair $(0, p_0)$, where $p_0 \in M \setminus U$ is a bifurcation point.
Theorem 2.

- $M$ is compact, possibly with boundary, with $\chi(M) \neq 0$,
- $f$ inward along $\partial M$

$\Rightarrow$ there exists in $[0, +\infty) \times C_T(M)$ an unbounded (w.r.t. $\lambda$) connected branch of nontrivial $T$-periodic pairs of (5), whose closure intersects the set of the trivial $T$-periodic pairs.
Sketch of the proof (finite delay, $M$ compact)

- First we assume $f$ of class $C^1$ and consider the delayed IVP
  \[
  \begin{cases}
  x'(t) = \lambda f(t, x(t), x(t - \tau)), & t > 0, \\
  x(t) = \phi(t), & t \in [-\tau, 0].
  \end{cases}
  \]
  \hspace{1cm} (6)

- $x(\lambda, \phi) : [-\tau, \infty) \to M$ the unique solution of (6).

Given $\lambda \in [0, +\infty)$, we define the Poincaré-type operator

$$P_\lambda : C([-\tau, 0], M) \to C([-\tau, 0], M)$$

$$P_\lambda(\phi)(s) = x(\lambda, \phi)(s + T) \quad s \in [-\tau, 0].$$
Poincaré-type operator

- The fixed points of $P_\lambda$ correspond to the $T$-periodic solutions of the equation (5); i.e., $\varphi$ is a fixed point of $P_\lambda$ if and only if it is the restriction to $[-\tau, 0]$ of a $T$-periodic solution.

- The map

$$P : [0, +\infty) \times C([-\tau, 0], M) \to C([-\tau, 0], M)$$

$$(\lambda, \varphi) \mapsto P_\lambda(\varphi)$$

is continuous and “locally compact”.
Proposition ($M$ noncompact)
Let $U$ be a relatively compact open subset of $M$ such that there are no zeros of $w$ on $\partial U$.
⇒ there exists $\bar{\lambda} > 0$ such that, for any $0 < \lambda < \bar{\lambda}$

$$\text{ind}_{\tilde{M}} (P(\lambda, \cdot), \tilde{U}) = \deg(-w, U).$$

Notation: $\tilde{U} = C([-\tau, 0], U)$. 
RFDE on manifolds (infinite delay)

We study the RFDE (1) on $M$:

$$x'(t) = \lambda F(t, x_t)$$

Assumptions on the functional vector field $F$:

(H1) $F$ is locally Lipschitz in the second variable;
(H2) $F$ sends bounded subsets of $\mathbb{R} \times BU((−\infty, 0], M) \to \mathbb{R}^k$ into bounded subsets of $\mathbb{R}^k$. 
Examples.

1) The case of ODEs is obtained with

\[ F(t, \varphi) := f(t, \varphi(0)). \]

2) The previous case (finite delay) is obtained with

\[ F(t, \varphi) := f(t, \varphi(0), \varphi(-\tau)). \]

3) Given \( h : \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}^k \), define

\[ F(t, \varphi) := h(t, \varphi(0)) + \int_{-\infty}^{0} e^{\theta} \varphi(\theta) \, d\theta. \]
Goals:  

i) to extend to equation (1) the global continuation results for $T$-periodic solutions,  

ii) to give applications to second order equations.

Main difficulty: to study RFDEs requires much more effort than delay equations.
Initial value problem (general properties)

Consider the initial value problem

\[
\begin{cases}
x'(t) = \lambda F(t, x_t), & t > 0 \\
x(t) = \eta(t), & t \leq 0.
\end{cases}
\]

where \( \eta : (-\infty, 0] \to M \) is a continuous map.

**Proposition.** If \( F \) is locally Lipschitz in the second variable
\( \Rightarrow \) existence, uniqueness and continuous dependence.
Global continuation result


- $M$ is closed in $\mathbb{R}^k$ (possibly noncompact)
- $F$ verifies (H1)–(H2)
- $U \subseteq M$ open such that $\text{deg}(w, U)$ is defined and nonzero

$\Rightarrow$ there exists in $[0, +\infty) \times C_T(M)$ a connected branch of nontrivial $T$-periodic pairs of (1) whose closure meets the set

$$\{(0, p) : p \in U, w(p) = 0\}$$

and

(i) either is unbounded;

(ii) or contains a pair $(0, p_0)$, where $p_0 \in M \setminus U$ is a bifurcation point.
Theorem 4.

- $M$ is compact, possibly with boundary, with $\chi(M) \neq 0$,
- $F$ is inward and verifies (H1)–(H2)

$\Rightarrow$ there exists in $[0, +\infty) \times C_T(M)$ an unbounded (w.r.t. $\lambda$) connected branch of nontrivial $T$-periodic pairs of (1), whose closure intersects the set of the trivial $T$-periodic pairs.
Applications to constrained motion problems with infinite delay

Consider the following **retarded functional motion equation** on a boundaryless manifold $N \subseteq \mathbb{R}^s$:

$$x''(t) = G(t, x_t) - \varepsilon x'(t),$$

where $G$ is a functional vector field on $N$, and $\varepsilon \geq 0$ is the frictional coefficient.

- $N$ is compact, boundaryless, with $\chi(N) \neq 0$,
- $G$ is $T$-periodic and verifies (H1)–(H2).
- Assume $\varepsilon > 0$

Then the equation

$$x''(t) = G(t, x_t) - \varepsilon x'(t)$$

admits a forced oscillation.
“Retarded spherical pendulum”

Assume $N = S^2$. Let $G$ be a $T$-periodic functional vector field
on $S^2$ which verifies (H1)–(H2)
⇒ the equation

$$x''_{\pi}(t) = G(t, x_t)$$

admits a forced oscillation.
Thank you for your attention!