Topological and variational methods for ODEs

Dedicated to Massimo Furi Professor Emeritus at the University of Florence

GLOBAL CONTINUATION OF PERIODIC SOLUTIONS FOR RFDE'S ON MANIFOLDS

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Setting of the problem

We study retarded functional differential equations (RFDE) on M of the type:

$$x'(t) = \lambda F(t, x_t) \tag{1}$$

where:

- $M \subseteq \mathbb{R}^k$ is a smooth manifold (possibly noncompact),
- $\lambda \geq 0$ is a parameter,
- F is a **functional vector field** on M.

Notation: $x_t(\theta) = x(t+\theta)$, $\theta \in (-\infty, 0]$.

Functional vector fields

The map $F : \mathbb{R} \times BU((-\infty, 0], M) \to \mathbb{R}^k$ is continuous, T-periodic in the first variable and such that

$$F(t, \varphi) \in T_{\varphi(0)}M \,, \quad \forall\, (t, \varphi) \in \mathbb{R} \times BU((-\infty, 0], M)$$

where $T_pM \subseteq \mathbb{R}^k$ denotes the tangent space of M at p.

Remark: We work in the space $BU((-\infty, 0], M)$ of the bounded, uniformly continuous maps

$$\varphi : (-\infty, 0] \to M.$$

- $BU((-\infty, 0], M)$ is a subset of the Banach space $BU((-\infty, 0], \mathbb{R}^k)$ with the supremum norm;
- the topology in the space $BU((-\infty, 0], M)$ is stronger than the compact-open topology of $C((-\infty, 0], M)$;
- if $x \colon J \to M$ is a solution of (1),
then the curve $t \mapsto x_t \in BU((-\infty, 0], M)$, $t \in J$, is continuous.

Goal: to prove global continuation results for T-periodic solutions of equation (1).

Tools:

- Fixed Point Index theory for locally compact maps on ANRs

(ANRs = absolute neighborhood retracts)

<u>References:</u> Granas, Nussbaum, Eells–Fournier.

- Degree of a tangent vector field

(Euler characteristic, rotation number).

Application: Retarded spherical pendulum

Consider the following second order equation on a boundaryless manifold $N \subseteq \mathbb{R}^s$:

$$x''_\pi(t) = G(t, x_t), \tag{2}$$

where (regarding (2) as a motion equation)

- $x''_\pi(t)$ is the tangential part of the acceleration $x''(t)$,
- the applied force G is a T-periodic functional vector field.

Equivalently (2) can be written as

$$x''(t) = r(x(t), x'(t)) + G(t, x_t),$$

where $r(q, v)$ is the reactive force.

A **forced oscillation** of (2) is a solution which is T-periodic and globally defined on $\mathbb{R}$.

Problem: to prove the existence of forced oscillations of (2).

Continuation results for ODEs on manifolds

Consider the parametrized ODE on $M \subseteq \mathbb{R}^k$

$$x'(t) = \lambda f(t, x(t)) \tag{3}$$

where $f : \mathbb{R} \times M \to \mathbb{R}^k$ is a T-periodic tangent vector field on M.

Furi and Pera (1986) have obtained global continuation results for equation (3) by means of topological methods.

Applications to the spherical pendulum

Consider the following second order ODE on a boundaryless manifold $N \subseteq \mathbb{R}^s$:

$$x''_\pi(t) = g(t, x(t)) \tag{4}$$

Furi and Pera (1990) proved that equation (4) has forced oscillations in the case $N = S^2$ (the spherical pendulum) and $N = S^{2n}$.

Conjecture: Equation (4) has forced oscillations if $\chi(N) \neq 0$ **(Euler–Poincaré characteristic)**.

– Motivation: Poincaré–Hopf Theorem.

– Difficulty: they use in a crucial way the geometry of the sphere.

The case of the ellipsoid is still open!

Related works:

- Capietto, Mawhin and Zanolin (1990);
- Benci and Degiovanni (1990).

Delay differential equations: some references

General reference: Hale and Verduyn Lunel (1993).

- **in Euclidean spaces:** Gaines and Mawhin (1977);

Nussbaum and Mallet-Paret (1994); Krisztin and Walter (1999).

- **equations on manifolds:** Oliva (1976).

Equations with infinite delay, or Retarded Functional Differential Equations (RFDEs)

- **in Euclidean spaces:** Hale and Kato (1978);

Hino, Murakami and Naito (1991); Novo, Obaya and Sanz (2007).

- **equations on manifolds:** *no general results were available!*

Benevieri, C., Furi, Pera (2013) Discrete Contin. Dyn. Syst.

Delay differential equations on manifolds (finite delay)

We study the parametrized delay differential equation on M

$$x'(t) = \lambda f(t, x(t), x(t - \tau)) \tag{5}$$

where $\tau > 0$ is the delay, and $f : \mathbb{R} \times M \times M \to \mathbb{R}^k$ is continuous, T-periodic in the first variable and tangent to M in the second one; i.e.,

$$f(t + T, p, q) = f(t, p, q) \in T_p M\,, \quad \forall\, (t, p, q) \in \mathbb{R} \times M \times M.$$

We call f a **(generalized) vector field** on M.

Remark. When $\partial M \neq \emptyset$ we require f to be *inward* along ∂M; i.e.,

$$f(t,p,q) \in C_pM\,, \quad \forall\,(t,p,q) \in \mathbb{R} \times \partial M \times M.$$

($C_pM \subseteq \mathbb{R}^k$ is the *tangent cone* of M at p)

Goal: to obtain global continuation results for T-periodic solutions.

Main difficulty: we work in an infinite-dimensional setting.

Let $C_T(M)$ be the metric space of the continuous, T-periodic M-valued maps.

Definition. (λ, x) in $[0, +\infty) \times C_T(M)$ is a T-**periodic pair** if $x : \mathbb{R} \to M$ is a T-periodic solution of (5) corresponding to λ.

A T-periodic pair of the type $(0, p_0)$, with $p_0 \in M$, is said to be *trivial*.

Remark. $C([-\tau, 0], M)$ and $C_T(M)$ are ANRs.
(when M is boundaryless $\Rightarrow$ Banach manifolds)

Fixed Point Index on ANRs (Granas, 1972)

X a metric ANR (Borsuk, 1930),

$k : \mathcal{D}(k) \subseteq X \to X$ locally compact,

$U \subseteq X$ open, contained in $\mathcal{D}(k)$.

If $\text{Fix}(k, U) = \{x \in U : x = k(x)\}$ is compact, the pair (k, U) is called *admissible*

$\to$ *fixed point index* of k in U:

$$\text{ind}_X(k, U) \in \mathbb{Z}.$$

Properties:

analogous to those of the classical **Leray–Schauder degree** (*Normalization*, *Additivity*, *Homotopy invariance*...)

- *Existence Property*:

$$\text{ind}_X(k,U) \neq 0 \;\Rightarrow\; \text{Fix}(k,U) \;\text{ nonempty.}$$

- *Strong Normalization Property*:

M a compact manifold $\Rightarrow$ $\text{ind}_M(I,M) = \chi(M)$ (the **Euler–Poincaré characteristic** of M).

Bifurcation points: necessary condition.

Definition. $p_0 \in M$ is a **bifurcation point** (of equation (5)) if every neighborhood of $(0, p_0)$ in $[0, +\infty) \times C_T(M)$ contains a *nontrivial* T-periodic pair (i.e., with $\lambda > 0$).

Proposition. $p_0 \in M$ bifurcation point $\Rightarrow$ the *mean value tangent vector field* $w : M \to \mathbb{R}^k$, defined by

$$w(p) = \frac{1}{T} \int_0^T f(t, p, p)\, dt,$$

vanishes at p_0.

Global continuation result

Theorem 1. Benevieri, C., Furi, Pera (2009) Z. Anal. Anwend.

- M is closed in $\mathbb{R}^k$ (possibly noncompact)
- $U \subseteq M$ open such that $\deg(w, U)$ is defined and nonzero

$\Rightarrow$ there exists in $[0, +\infty) \times C_T(M)$ a <u>connected branch</u> of nontrivial T-periodic pairs of (5) whose closure meets the set $\{(0, p) : p \in U, w(p) = 0\}$ and satisfies at least one of the following properties:

(i) it is unbounded;

(ii) it contains a pair $(0, p_0)$, where $p_0 \in M \setminus U$ is a bifurcation point.

Theorem 2.

- M is compact, possibly with boundary, with $\chi(M) \neq 0$,
- f inward along ∂M

$\Rightarrow$ there exists in $[0, +\infty) \times C_T(M)$ an unbounded (w.r.t. λ) connected branch of nontrivial T-periodic pairs of (5), whose closure intersects the set of the trivial T-periodic pairs.

Sketch of the proof (finite delay, M compact)

- First we assume f of class C^1 and consider the delayed IVP

$$\begin{cases} x'(t) = \lambda f(t, x(t), x(t-\tau)), & t > 0, \\ x(t) = \varphi(t), & t \in [-\tau, 0]. \end{cases} \tag{6}$$

- $x_{(\lambda,\varphi)} : [-\tau, \infty) \to M$ the unique solution of (6).

Given $\lambda \in [0, +\infty)$, we define the Poincaré-type operator

$$P_\lambda : C([-\tau, 0], M) \to C([-\tau, 0], M)$$

$$P_\lambda(\varphi)(s) = x_{(\lambda,\varphi)}(s + T) \quad s \in [-\tau, 0].$$

Poincaré-type operator

- The fixed points of P_λ correspond to the T-periodic solutions of the equation (5); i.e., φ is a fixed point of P_λ if and only if it is the restriction to $[-\tau, 0]$ of a T-periodic solution.

- The map

$$P : [0, +\infty) \times C([-\tau, 0], M) \to C([-\tau, 0], M)$$

$$(\lambda, \varphi) \mapsto P_\lambda(\varphi)$$

is continuous and "locally compact".

Proposition (M noncompact)

Let U be a relatively compact open subset of M such that there are no zeros of w on ∂U.

$\Rightarrow$ there exists $\bar{\lambda} > 0$ such that, for any $0 < \lambda < \bar{\lambda}$

$$\mathrm{ind}_{\widetilde{M}}(P(\lambda, \cdot), \tilde{U}) = \deg(-w, U).$$

Notation: $\tilde{U} = C([-\tau, 0], U)$.

RFDE on manifolds (infinite delay)

We study the RFDE (1) on M:

$$x'(t) = \lambda F(t, x_t)$$

Assumptions on the functional vector field F:

(H1) F is locally Lipschitz in the second variable;

(H2) F sends bounded subsets of $\mathbb{R} \times BU((-\infty, 0], M) \to \mathbb{R}^k$ into bounded subsets of $\mathbb{R}^k$.

Examples.

1) The case of ODEs is obtained with

$$F(t,\varphi) := f(t,\varphi(0)).$$

2) The previous case (finite delay) is obtained with

$$F(t,\varphi) := f(t,\varphi(0),\varphi(-\tau)).$$

3) Given $h : \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}^k$, define

$$F(t,\varphi) := h(t,\varphi(0)) + \int_{-\infty}^{0} \mathrm{e}^{\theta}\varphi(\theta)\,\mathrm{d}\theta.$$

Goals: *i)* to extend to equation (1) the global continuation results for T-periodic solutions,
ii) to give applications to second order equations.

Main difficulty: to study RFDEs requires much more effort than delay equations.

Initial value problem (general properties)

Consider the initial value problem

$$\begin{cases} x'(t) = \lambda F(t, x_t), & t > 0 \\ x(t) = \eta(t), & t \leq 0. \end{cases}$$

where $\eta : (-\infty, 0] \to M$ is a continuous map.

Proposition. If F is locally Lipschitz in the second variable $\Rightarrow$ existence, uniqueness and continuous dependence.

Global continuation result

Theorem 3. Benevieri, C., Furi, Pera (2013) Bound. Value Probl.

- M is closed in $\mathbb{R}^k$ (possibly noncompact)
- F verifies (H1)–(H2)
- $U \subseteq M$ open such that deg(w, U) is defined and nonzero

$\Rightarrow$ there exists in $[0, +\infty) \times C_T(M)$ a connected branch of nontrivial T-periodic pairs of (1) whose closure meets the set $\{(0, p) : p \in U, w(p) = 0\}$ and

(i) either is unbounded;

(ii) or contains a pair $(0, p_0)$, where $p_0 \in M \setminus U$ is a bifurcation point.

Theorem 4.

- M is compact, possibly with boundary, with $\chi(M) \neq 0$,
- F is inward and verifies (H1)–(H2)

$\Rightarrow$ there exists in $[0, +\infty) \times C_T(M)$ an unbounded (w.r.t. λ) connected branch of nontrivial T-periodic pairs of (1), whose closure intersects the set of the trivial T-periodic pairs.

Applications to constrained motion problems with infinite delay

Consider the following **retarded functional motion equation** on a boundaryless manifold $N \subseteq \mathbb{R}^s$:

$$x''_{\pi}(t) = G(t, x_t) - \varepsilon x'(t), \tag{7}$$

where G is a functional vector field on N, and $\varepsilon \geq 0$ is the frictional coefficient.

Theorem 5. Benevieri, C., Furi, Pera (2012) Rend. Trieste

- N is compact, boundaryless, with $\chi(N) \neq 0$,
- G is T-periodic and verifies (H1)–(H2).
- Assume $\varepsilon > 0$

$\Rightarrow$ the equation

$$x''_\pi(t) = G(t, x_t) - \varepsilon x'(t)$$

admits a forced oscillation.

“Retarded spherical pendulum”

Theorem 6. Benevieri, C., Furi, Pera (2011) J. Dynam. Diff. Eq.
Assume $N = S^2$. Let G be a T-periodic functional vector field on S^2 which verifies (H1)–(H2)
$\Rightarrow$ the equation

$$x''_\pi(t) = G(t, x_t)$$

admits a forced oscillation.

Thank you for your attention!