A global bifurcation result for a second order singular equation

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Topological and Variational Methods for ODEs
Dedicated to Massimo Furi Professor Emeritus at the University of Florence
Firenze, Dipartimento di Matematica e Informatica ”U. Dini”, June 3 - 4, 2014
We are concerned with a second order ODE of the form

\[-u'' + q(x)u = \lambda u + g(x, u)u, \quad \lambda \in \mathbb{R}, \ x \in (0, 1),\]

where \(q \in C((0, 1])\) satisfies

\[\lim_{x \to 0^+} q(x) = 1\]

for some \(l > 0\) and \(\alpha \in (0, 5/4)\), and \(g \in C([0, 1] \times \mathbb{R})\) is such that

\[\lim_{u \to 0} g(x, u) = 0, \quad \text{uniformly in } x \in (0, 1).\]

We will look for solutions \(u\) such that \(u \in H^2_0(0, 1)\).

In what follows, we set

\[\tau u = -u'' + q(\cdot)u.\]

The constant \(5/4\) arises in the study of the differential operator \(\tau\).

We develop a global bifurcation approach.
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Literature

For the linear problem

Weidmann, "Linear Operators in Hilbert Spaces", 1980

For the spectral properties of the Schrödinger operator

Pearson, "Quantum scattering and spectral theory", 1988
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For singular problems

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Sim-Kajikiya-Lee, "On a criterion for discrete or continuous spectrum of \( p \)-Laplace eigenvalue problems with singular sign-changing weights", Nonlinear Anal., 2010
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**THEOREM.**

Assume \(b \in C\) and non-negative and 
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f(t, \xi, \eta, \lambda) = \lambda a(t)\xi + h(t, \xi, \eta, \lambda)\] (being a continuous and positive and \(h(t, \xi, \eta, \lambda) = o(\sqrt{\xi^2 + \eta^2}), (\xi, \eta) \to (0, 0))\).
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Then for every $k$, $(\lambda_k, 0)$ is a bifurcation point for the nonlinear BVP

$$\begin{cases} 
-u'' + b(t)u = f(t, u, u', \lambda), & t \in [0, \pi], \\
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The bifurcating branches $C_k \subset \mathbb{R} \times C^1([0, \pi], \mathbb{R})$ are unbounded in $\mathbb{R} \times C^1([0, \pi], \mathbb{R})$;
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The bifurcating branches \(C_k \subset \mathbb{R} \times C^1([0, \pi], \mathbb{R})\) are unbounded in \(\mathbb{R} \times C^1([0, \pi], \mathbb{R})\); moreover, if \((\lambda, u) \in C_k\) and \(u \neq 0\), then \(u\) has \((k - 1)\) simple zeros in \((0, \pi)\).
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On the other hand, in Weidmann’s book the existence of a self-adjoint realization of the formal differential expression $\tau u = -u'' + q(\cdot)u$ is tackled from an abstract point of view.
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\[ w_1(x) = x, \quad x \sim 0, \]
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\[
\begin{align*}
    w_1(x) &= x, \quad x \sim 0, \\
    w_2(x) &= \int_0^x \int_t^1 q(s)dsdt - 1, \quad x \sim 0.
\end{align*}
\]
From the abstract spectral theory, we learn that a self-adjoint extension is obtained by means of the boundary condition

$$\lim_{x \to 0} \left( w_\alpha(x)u'(x) - w'_\alpha(x)u(x) \right) = 0,$$

where $w_\alpha := \cos \alpha w_1 + \sin \alpha w_2$, $\alpha \in [0, 2\pi)$, must be such that the function

$$W_\alpha(x) := \frac{1}{|x|} w_\alpha(|x|) =$$

$$\cos \alpha + \frac{1}{|x|} \left( -\sin \alpha + \sin \alpha \int_0^{|x|} \int_t^1 q(s) ds dt \right), \quad x \sim 0$$

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Notice that the condition $\alpha < 5/4$ guarantees that $\tau w_2 \in L^2((0, 1])$. 
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$$M(c, d, \lambda) := \text{number of zeros in } (c, d) \text{ of the solution to } (\tau - \lambda)u = 0 \text{ satisfying } u(c) = 0 \text{ or } u(d) = 0$$
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$$m(\mu, \lambda) = \liminf_{c \to 0^+, d \to 1^-} (M(c, d, \lambda) - M(c, d, \mu)).$$
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(a) \( A \) is bounded below if and only if there exists a real number \( \mu \) such that \((\tau - \mu)u = 0\) is non-oscillatory.
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Theorem (Weidmann, LNM)
(a) $A$ is bounded below if and only if there exists a real number $\mu$ such that $(\tau - \mu)u = 0$ is non-oscillatory.
(b) $$\sigma_{ess}(A) = \{ \lambda \in \mathbb{R} : m(\lambda - \epsilon, \lambda + \epsilon) = \infty \ \forall \epsilon > 0 \}$$
The linear theory

Consider the linear equation

$$-u'' + q(x)u = \lambda u, \quad x \in (0, 1], \quad \lambda \in \mathbb{R}.$$  

Recall that $q \in C((0, 1])$ and that $q(x) \sim lx^\alpha, \quad x \to 0^+,$

for some $l > 0$ and $\alpha \in (0, 5/4)$.

Without loss of generality we may suppose that $q(x) > 0, \forall x \in (0, 1]$. 

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where $Y = (w, w')^T$ and

$$C = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}, \quad R(t) = \begin{pmatrix} 0 & 0 \\ e^{-2t}q(e^{-t}) - \lambda e^{-2t} & 0 \end{pmatrix}, \quad \forall \ t > 0.$$
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As an application of Levinson theorem, we get
PROPOSITION 1. For every $\lambda \in \mathbb{R}$ the equation has two linearly independent solutions $u_{1,\lambda}, u_{2,\lambda}$ such that

$$u_{1,\lambda}(x) = 1 + o(1), \quad u'_{1,\lambda}(x) = o\left(\frac{1}{x}\right) \quad x \to 0^+,$$

and $u_{2,\lambda} \in H^2(0,1)$.
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\end{align*}
\]

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For every \( f \in L^2(0, 1) \) the solutions of \( \tau u = f \) are given by

\[
  u(x) = c_1 u_{1,0}(x) + c_2 u_{2,0}(x) + u_f(x), \quad \forall \ x \in (0, 1), \ c_1, c_2 \in \mathbb{R},
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fulfill $G \in L^\infty((0, 1)^2)$, $u_f(0) = 0 = u'_f(0)$ and $u_f \in H^2(0, 1)$. 
From the spectral theory for singular differential operators, it follows that the differential operator $A$ defined by $Au = \tau u$, being

$$D(A) = \{ u \in L^2(0,1) : u, u' \in AC(0,1), \tau u \in L^2(0,1), \lim_{x \to 0^+} (xu'(x) - u(x)) = 0 = u(1) \},$$

is a self-adjoint realization of $\tau$. 
PROPOSITION 2. The relation $D(A) = H^2_0(0,1)$ holds true.

Moreover, $A$ has a bounded inverse $A^{-1}$: $\mathbb{L}^2(0,1) \rightarrow H^2_0(0,1)$.

Sketch of the proof. Let us start proving that $H^2_0(0,1) \subset D(A)$.

It is well known that $H^2_0(0,1) \subset \mathbb{C}^1(0,1)$; hence, for every $u \in H^2_0(0,1)$ we have $u, u' \in AC(0,1)$.

Moreover, using the fact that $u(0) = 0$ we deduce that $u(x) = u'(0)x + o(x), x \to 0$ and $q(u(x)) = u'(0)x^2 - \alpha + o(x^2), x \to 0$; the condition $\alpha < 5/4$ guarantees again that $qu \in \mathbb{L}^2(0,1)$ and therefore $\tau u = -u'' + qu \in \mathbb{L}^2(0,1)$.

Finally, the regularity of $u$ and $u'$ imply that $\lim_{x \to 0^+} (xu'(x) - u(x)) = 0$ and so also the boundary condition in the definition of $D(A)$ is satisfied.
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q(x)u(x) = u'(0)x^{1-\alpha} + o(x^{1-\alpha}), \quad x \to 0^+;
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Hence $u \in D(A)$ if and only if $c_1 = 0$; the last statement of Proposition 1 implies then that $u \in H^2(0, 1)$. 
Now, let us prove that \( D(A) \subset H^2_0(0,1) \). For every \( u \in D(A) \) let \( f = \tau u \in L^2(0,1) \). According to Proposition 1, \( u \) can be written as
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u = c_1 u_1 + c_2 u_2 + u_f,\
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for some \( c_1, c_2 \in \mathbb{R} \); it is easy to see that the function \( u_1 \) does not satisfy the boundary condition given in \( x = 0 \) in the definition of \( D(A) \), while \( u_2 \) and \( u_f \) do.

Hence \( u \in D(A) \) if and only if \( c_1 = 0 \); the last statement of Proposition 1 implies then that \( u \in H^2(0,1) \).

As in the first part of the proof, the regularity of \( u \) allows to conclude that the boundary condition in \( x = 0 \) given in \( D(A) \) reduces to \( u(0) = 0 \).
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Moreover, since $A$ is self-adjoint on $H$, this follows from the surjectivity of $A$; hence, it is sufficient to prove that $A$ is surjective. To this aim, let us first observe that $0$ cannot be an eigenvalue of $A$. Now, let us fix $f \in L^2(0,1)$ and let us prove that there exists $u \in H^2_0(0,1)$ such that $Au = f$, i.e. $\tau u = f$. The same argument of the first part of the proof implies that $c_1 = 0$. Hence $u = c_2 u_2 + u f(1)$. From Proposition 1 this function belongs to $H^2_0(0,1)$ and satisfies the boundary condition $u(0) = 0$. In order to prove that the missing condition $u(1) = 0$ is fulfilled for every $f \in L^2(0,1)$, let us observe that $u_2(1) \neq 0$, otherwise $u_2$ would be an eigenfunction of $A$ associated to the zero eigenvalue. Therefore, $u(1) = 0$ is satisfied if $c_2 = -u f(1) u_2(1)$, for every $f \in L^2(0,1)$. 
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Spectral properties of $A$

The regularity assumptions on $q$ imply that solutions to 
$$-u'' + q(x)u = \lambda u,$$
have a finite number of zeros in any interval of the form $[a, 1)$, for every $0 < a < 1$.

Moreover, for every $\lambda \in \mathbb{R}$ there exists $c(\lambda) \in (0, 1]$ such that $\lambda - q(x) < 0$, $\forall x \in (0, c(\lambda))$.

An application of the Sturm comparison theorem proves that every solution has at most one zero in $(0, c(\lambda))$; as a consequence, we obtain the following result:
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and for every $n \in \mathbb{N}$ the eigenfunction $u_n$ of $A$ associated to the eigenvalue $\lambda_n$ has $(n - 1)$ simple zeros in $(0, 1)$. 
The nonlinear problem

\[-u'' + q(x)u = \lambda u + g(x, u), \ \lambda \in \mathbb{R}, \ x \in (0, 1),\]

where \(q \in C((0, 1])\) satisfies

\[\lim_{x \to 0^+} q(x) = 1\]

for some \(l > 0\) and \(\alpha \in (0, 5/4)\), and \(g \in C([0, 1] \times \mathbb{R})\) is such that

\[\lim_{u \to 0} g(x, u) = 0, \ \text{uniformly in} \ x.\]

We will look for solutions \(u\) such that \(u \in H^2_0(0, 1)\).

Let \(\Sigma\) denote the set of nontrivial solutions in \(H^2_0(0, 1) \times \mathbb{R}\) and let \(\Sigma' = \Sigma \cup \{\lambda \in H^2_0(0, 1) \times \mathbb{R}: \lambda \text{ is an eigenvalue of } A\}\).
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The nonlinear problem

Consider

\[-u'' + q(x)u = \lambda u + g(x, u)u, \quad \lambda \in \mathbb{R}, \quad x \in (0, 1),\]

where \(q \in C((0, 1])\) satisfies

\[\lim_{x \to 0^+} \frac{q(x)}{x^\alpha} = 1\]

for some \(l > 0\) and \(\alpha \in (0, 5/4)\) and \(g \in C([0, 1] \times \mathbb{R})\) is such that

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We will look for solutions \( u \) such that \( u \in H^2_0(0, 1) \).

Let \( \Sigma \) denote the set of nontrivial solutions in \( H^2_0(0, 1) \times \mathbb{R} \) and let

\( \Sigma' = \Sigma \cup \{(0, \lambda) \in H^2_0(0, 1) \times \mathbb{R} : \lambda \) is an eigenvalue of \( A \}\).
Let $M$ denote the Nemitskii operator associated to $g$, given by

$$M(u)(x) = g(x, u(x))u(x), \quad \forall \, x \in [0, 1],$$

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The search of solutions $u \in H^2_0(0, 1)$ is equivalent to the search of solutions of the abstract equation

$$Au = \lambda u + M(u), \quad (u, \lambda) \in H^2_0(0, 1) \times \mathbb{R};$$

which can be written in the form

$$w = \lambda Rw + M(Rw), \quad (w, \lambda) \in L^2(0, 1) \times \mathbb{R},$$

where $R : L^2(0, 1) \to H^2_0(0, 1)$ is the inverse of $A$. 
It is easy to see that \( M : H_0^2(0, 1) \rightarrow L^2(0, 1) \) is a continuous map and satisfies
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M(u) = o(\|u\|), \quad u \rightarrow 0.
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In this framework, Rabinowitz global bifurcation theorem is applicable.
In order to obtain a more precise description of the bifurcating branch, first observe that for every nontrivial solution \( u \in H^2_0(0,1) \) the function \( u \) is a nontrivial solution of the linearized equation

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-w'' + (q(x) - g(x, u(x)) - \lambda)w = 0.
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**PROPOSITION 5** All the nontrivial solutions of the linearized equation (in particular \( u \)) have a finite number of zeros in \((0, 1)\).
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**Proposition 5** All the nontrivial solutions of the linearized equation (in particular \( u \)) have a finite number of zeros in \((0, 1)\). Denote by \( n(u) \) this number.
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**PROPOSITION 5** All the nontrivial solutions of the linearized equation (in particular \( u \)) have a finite number of zeros in \((0, 1)\). Denote by \( n(u) \) this number.

For the proof, we use the fact that for every \( \lambda \in \mathbb{R} \) and for every nontrivial solution \( u \in H^2_0(0, 1) \) there exist a neighbourhood \( U \subset H^2_0(0, 1) \times \mathbb{R} \) of \((u, \lambda)\) and \( x_{u,\lambda} \in (0, 1) \) such that

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q(x) - g(x, v(x)) - \lambda > 0, \quad \forall (v, \mu) \in U, \ x \in (0, x_{u,\lambda}].
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$$q(x) - g(x, u(x)) - \lambda > 0, \quad \forall (v, \mu) \in U, \ x \in (0, x_{u,\lambda}].$$
We are then allowed to define the functional $j : \Sigma' \rightarrow \mathbb{N}$ by setting

$$j(u, \lambda) = \begin{cases} n(u) & \text{if } u \not\equiv 0 \\ n - 1 & \text{if } u \equiv 0 \text{ and } \lambda = \lambda_n, \end{cases}$$

for every $(u, \lambda) \in \Sigma'$. 

Let us observe that the definition $j(0, \lambda_n) = n - 1$ is suggested by Proposition 4.

**Proposition 6**
The function $j : \Sigma' \rightarrow \mathbb{N}$ is continuous.
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**PROPOSITION 6** The function $j : \Sigma' \to \mathbb{N}$ is continuous.
**MAIN RESULT**  For every eigenvalue $\lambda_n$ of $A$ there exists a continuum $C_n$ of nontrivial solutions in $H^2_0(0, 1) \times \mathbb{R}$ bifurcating from $(0, \lambda_n)$ and such that $C_n$ is unbounded in $H^2_0(0, 1) \times \mathbb{R}$ and

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1. \( C_n \) is unbounded in \( H_0^2(0, 1) \times \mathbb{R} \);
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1. $C_n$ is unbounded in $H^2_0(0,1) \times \mathbb{R}$;
2. $C_n$ contains $(0, \lambda_{n'}) \in \Sigma'$, with $n' \neq n$.

The continuity of $j$ enables to exclude the second alternative.
1987-2014 and much more ....... :
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a 27 years uninterrupted friendship...thanks Massimo and best wishes !!!!