A global bifurcation result for a second order singular equation

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Topological and Variational Methods for ODEs
Dedicated to Massimo Furi Professor Emeritus at the University of Florence

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We develop a global bifurcation approach.

Literature

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A global bifurcation result for the a second order BVP in $[0, \pi]$
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THEOREM.

Assume $b \in C$ and non-negative and $f(t,\xi,\eta,\lambda) = \lambda a(t)\xi + h(t,\xi,\eta,\lambda)$ (being a continuous and positive and $h(t,\xi,\eta,\lambda) = o(\sqrt{\xi^2+\eta^2})$, $(\xi,\eta) \to (0,0)$).

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The bifurcating branches $C_k \subset \mathbb{R} \times C^1([0, \pi], \mathbb{R})$ are unbounded in $\mathbb{R} \times C^1([0, \pi], \mathbb{R})$; moreover, if $(\lambda, u) \in C_k$ and $u \neq 0$, then u has $(k - 1)$ simple zeros in $(0, \pi)$.

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$$w_1(x) = x, \qquad x \sim 0,$$

$$w_2(x) = \int_0^x \int_t^1 q(s)dsdt - 1, \qquad x \sim 0.$$

From the abstract spectral theory, we learn that a self-adjoint extension is obtained by means of the boundary condition

$$\lim_{x\to 0}(w_\alpha(x)u'(x) - w'_\alpha(x)u(x)) = 0,$$

where $w_\alpha := \cos\alpha w_1 + \sin\alpha w_2, \alpha \in [0, 2\pi)$, must be such that the function

$$W_\alpha(x) := \frac{1}{|x|}w_\alpha(|x|) =$$

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Notice that the condition $\alpha < 5/4$ guarantees that $\tau w_2 \in L^2((0,1])$.

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$$M(c,d,\lambda) := \text{number of zeros in } (c,d) \text{ of the solution to}$$

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and

$$m(\mu,\lambda) = \liminf_{c \to 0^+, d \to 1^-} (M(c,d,\lambda) - M(c,d,\mu)).$$

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(a) A is bounded below if and only if there exists a real number μ such that $(\tau - \mu)u = 0$ is non-oscillatory.
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$$\sigma_{ess}(A) = \{\lambda \in \mathbb{R} : m(\lambda - \epsilon, \lambda + \epsilon) = \infty \ \forall \epsilon > 0\}$$

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where $Y = (w, w')^T$ and

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$$u_f(x) = \int_0^x G(x,t) f(t)\, dt, \quad \forall\, x \in (0,1),$$

$$G(x,t) = u_{1,0}(t) u_{2,0}(x) - u_{2,0}(t) u_{1,0}(x), \quad \forall\, x \in (0,1), \; t \in (0,1)$$

PROPOSITION 1. For every $\lambda \in \mathbb{R}$ the equation has two linearly independent solutions $u_{1,\lambda}$, $u_{2,\lambda}$ such that

$$u_{1,\lambda}(x) = 1 + o(1),\ u'_{1,\lambda}(x) = o\left(\frac{1}{x}\right) \quad x \to 0^+,$$

$$u_{2,\lambda}(x) = x + o(x),\ u'_{2,\lambda}(x) = 1 + o(1), \quad x \to 0^+,$$

and $u_{2,\lambda} \in H^2(0,1)$.
For every $f \in L^2(0,1)$ the solutions of $\tau u = f$ are given by

$$u(x) = c_1 u_{1,0}(x) + c_2 u_{2,0}(x) + u_f(x), \quad \forall\, x \in (0,1),\ c_1, c_2 \in \mathbb{R},$$

where

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fulfill $G \in L^\infty((0,1)^2)$, $u_f(0) = 0 = u'_f(0)$ and $u_f \in H^2(0,1)$.

From the spectral theory for singular differential operators, it follows that the differential operator A defined by $Au = \tau u$, being

$$D(A) = \{u \in L^2(0,1) : \; u, u' \in AC(0,1), \; \tau u \in L^2(0,1),$$

$$\lim_{x \to 0^+} (xu'(x) - u(x)) = 0 = u(1)\},$$

is a self-adjoint realization of τ.

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the condition $\alpha < 5/4$ guarantees again that $qu \in L^2(0,1)$ and therefore $\tau u = -u'' + qu \in L^2(0,1)$. Finally, the regularity of u and u' imply that

$$\lim_{x \to 0^+} (xu'(x) - u(x)) = 0$$

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Hence $u \in D(A)$ if and only if $c_1 = 0$; the last statement of Proposition 1 implies then that $u \in H^2(0,1)$.

As in the first part of the proof, the regularity of u allows to conclude that the boundary condition in $x = 0$ given in $D(A)$ reduces to $u(0) = 0$.

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Hence $u = c_2 u_2 + u_f$; from Proposition 1 this function belongs to $H^2(0,1)$ and satisfies the boundary condition $u(0) = 0$.
In order to prove that the missing condition $u(1) = 0$ is fulfilled for every $f \in L^2(0,1)$, let us observe that $u_2(1) \neq 0$, otherwise u_2 would be an eigenfunction of A associated to the zero eigenvalue.
Therefore, $u(1) = 0$ is satisfied if $c_2 = -\dfrac{u_f(1)}{u_2(1)}$, for every $f \in L^2(0,1)$. □

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An application of the Sturm comparison theorem proves that every solution has at most one zero in $(0, c(\lambda))$; as a consequence, we obtain the following result:

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and for every $n \in \mathbb{N}$ the eigenfunction u_n of A associated to the eigenvalue λ_n has $(n-1)$ simple zeros in $(0, 1)$.

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We will look for solutions u such that $u \in H_0^2(0,1)$.
Let Σ denote the set of nontrivial solutions in $H_0^2(0,1) \times \mathbb{R}$ and let $\Sigma' = \Sigma \cup \{(0,\lambda) \in H_0^2(0,1) \times \mathbb{R} : \ \lambda \text{ is an eigenvalue of } A\}$.

Let M denote the Nemitskii operator associated to g, given by

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for every $u \in H_0^2(0, 1)$.
The search of solutions $u \in H_0^2(0, 1)$ is equivalent to the search of solutions of the abstract equation

$$Au = \lambda u + M(u), \quad (u, \lambda) \in H_0^2(0, 1) \times \mathbb{R};$$

which can be written in the form

$$w = \lambda Rw + M(Rw),\ (w, \lambda) \in L^2(0, 1) \times \mathbb{R},$$

where $R : L^2(0, 1) \to H_0^2(0, 1)$ is the inverse of A.

It is easy to see that $M : H_0^2(0,1) \longrightarrow L^2(0,1)$ is a continuous map and satisfies

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In this framework, Rabinowitz global bifurcation theorem is applicable.

In order to obtain a more precise description of the bifurcating branch, first observe that for every nontrivial solution $u \in H_0^2(0,1)$ the function u is a nontrivial solution of the linearized equation

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For the proof, we use the fact that for every $\lambda \in \mathbb{R}$ and for every nontrivial solution $u \in H_0^2(0,1)$ there exist a neighbourhood $U \subset H_0^2(0,1) \times \mathbb{R}$ of (u,λ) and $x_{u,\lambda} \in (0,1)$ such that
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We are then allowed to define the functional $j : \Sigma' \to \mathbb{N}$ by setting

$$j(u,\lambda) = \begin{cases} n(u) & \text{if } u \not\equiv 0 \\ \\ n-1 & \text{if } u \equiv 0 \text{ and } \lambda = \lambda_n, \end{cases}$$

for every $(u,\lambda) \in \Sigma'$.

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PROPOSITION 6 The function $j : \Sigma' \to \mathbb{N}$ is continuous.

MAIN RESULT For every eigenvalue λ_n of A there exists a continuum C_n of nontrivial solutions in $H_0^2(0,1) \times \mathbb{R}$ bifurcating from $(0, \lambda_n)$ and such that C_n is unbounded in $H_0^2(0,1) \times \mathbb{R}$ and

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The continuity of j enables to exclude the second alternative.

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a 27 years uninterrupted friendship...thanks Massimo and best wishes !!!!