On the generalized Emden-Fowler differential equations

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Topological and Variational Methods for ODEs
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Historical survey

Emden-Fowler differential equation

$$x'' + b(t)|x|^{\beta} \operatorname{sgn} x = 0, \quad \beta \neq 1, \beta > 0. \tag{1}$$

where b is a positive continuous function for $t \geq 1$.

$\beta > 1$: super-linear equation, $\beta < 1$: sub-linear equation

- Emden (1907): Gaskugeln, Anwendungen der mechanischen Warmentheorie auf Kosmologie und metheorologische Probleme, Leipzig.
- Fowler (1930): The solutions of Emden's and similar differential equations, Monthly Notices Roy. Astronom. Soc.
- Atkinson (1955), Moore and Nehari (1959) . . . $\beta > 1$
- Belohorec (1961) . . . $\beta < 1$

A solution x is *nonoscillatory* if it has no zero for large t.
In view of the sign of b, all nonoscillatory sol's of (1) satisfy

$$x(t)x'(t) > 0 \quad \text{for large } t.$$

If x is a sol. of (1), then $-x$ is a sol. too. So, we will consider only nonoscillatory solutions which are eventually positive.

Positive solutions can be classified as:

subdominant $\iff$ $x(\infty) = c_x,\ x'(\infty) = 0,$

intermediate $\iff$ $x(\infty) = \infty,\ x'(\infty) = 0,$

dominant $\iff$ $x(\infty) = \infty,\ x'(\infty) = d_x,$

c_x, d_x are positive constants.

If x, y and z are subdominant, intermediate and dominant sols, then

$$0 < x(t) < y(t) < z(t) \quad \text{for large } t.$$

Moore and Nehari (Trans. Amer. Math. J. 1959): $\beta > 1$

- Necessary/sufficient conditions for the existence of dominant solutions of (1)
- Necessary/sufficient conditions for the existence of subdominant solutions of (1)
- The above three types of nonoscillatory sols cannot coexist simultaneously!
- No conditions for the existence of intermediate solutions are given.

Two years later Belohorec proved the same results in the sublinear case, i.e. $\beta < 1$.

Due to the interest for radially symmetric solutions of PDE with p-Laplacian, Kusano and Elbert (1990), Kusano et all (1998) and others considered the above problems for equation

$$\left(a(t)|x'|^{\alpha}\operatorname{sgn} x'\right)' + b(t)|x|^{\beta}\operatorname{sgn} x = 0 \qquad \text{(E)}$$

Assumptions: $\alpha > 0$, $\beta > 0$, $a, b \in C[0,\infty)$, $a(t) > 0$ and $b(t) > 0$ for $t \geq 0$ and

$$\int_0^{\infty} a^{-1/\alpha}(s)ds = \infty, \quad \int_0^{\infty} b(s)ds < \infty.$$

For (E) the above classification as subdominant, intermediate and dominant solutions continues to hold by replacing in the Moore and Nehari classification the derivative with the quasiderivative

$$x' \qquad \rightsquigarrow \qquad x^{[1]} = a(t)|x'(t)|^{\alpha}\operatorname{sgn} x'(t).$$

Existence results (Kusano et all) – characteristic integrals:

$$J_{\alpha} = \int_0^{\infty} \frac{1}{a^{1/\alpha}(t)} \left(\int_t^{\infty} b(s)\, ds \right)^{1/\alpha} dt,$$

$$K_{\beta} = \int_0^{\infty} b(t) \left(\int_0^{t} \frac{1}{a^{1/\alpha}(s)}\, ds \right)^{\beta} dt.$$

- (E) has subdominant solutions $\iff J_{\alpha} < \infty$.
- (E) has dominant solutions $\iff K_{\beta} < \infty$.
- When (1) or (E) has intermediate solutions ???

Two open problems:

1. Coexistence problem: Is it possible for (E) the coexistence of subdominant, intermediate and dominant solutions?
2. Sufficient conditions for the existence of intermediate solutions.

- $\alpha = \beta$ (half-linear case)
- $\alpha > \beta$ (sub-linear case)
- $\alpha < \beta$ (super-linear case)

- $\alpha = \beta$ (half-linear case)

Ad 1 (coexistence problem): These three types of nonoscillatory sols cannot coexist simultaneously!

M. Cecchi, M. Marini, Z.D., *On intermediate solutions and the Wronskian for half-linear differential equations,*
J. Math. Anal. Appl. 336 (2007).

Method: the extension of the wronskian identity.

Ad 2: The existence of intermediate solutions
– Sturm-theory for half-linear equation
– the notion of principal and nonprincipal solutions.

- $\alpha > \beta$ (sub-linear case)

Ad 1 (coexistence problem): These three types of nonoscillatory sols cannot coexist simultaneously!

M. Naito, *On the asymptotic behavior of nonoscillatory solutions of second order quasilinear ordinary differential equations*, J. Math. Anal. Appl. (2011).

- $\alpha < \beta$ (super-linear case)

Partial answer when $0 < \alpha < 1$: These three types of nonoscillatory sols cannot coexist simultaneously!

M. Cecchi, M. Marini, Z.D., *Intermediate solutions for Emden-Fowler type equations: continuous versus discrete*, Advances Dynam. Systems Appl. (2008).

Our subject: intermediate solutions when $\alpha < \beta$

Change of integration for J_α, K_β
Cecchi-Z.D.-Marini-Vrkoč (2006)

$\rightsquigarrow$ Compatibility of conditions in case $\alpha < \beta$

$$J_\alpha = \infty, \quad K_\beta = \infty \quad \text{(all sols oscillatory)}$$
$$J_\alpha < \infty, \quad K_\beta < \infty$$
$$J_\alpha < \infty, \quad K_\beta = \infty.$$

- Coexistence problem: Can subdominant, intermediate and dominant solutions coexist simultaneously?
- Do exist intermediate solution if $\alpha < \beta$?

Coexistence problem

$$\left(a(t)|x'|^{\alpha}\operatorname{sgn} x'\right)' + b(t)|x|^{\beta}\operatorname{sgn} x = 0 \quad (t \geq 0) \qquad \text{(E)}$$

where $b(t) \geq 0$ and $\alpha < \beta$.

Theorem 1

Let $J_\alpha < \infty$ and $K_\beta < \infty$. Then (E) *does not have intermediate solutions.*

Consequently,

(E) never has simultaneously subdominant, intermediate and dominant solutions!

This is an extension of Moore-Nehari result for (1).

Idea of the proof.

Step 1. New Holder-type inequality:

Lemma

Let λ, μ be such that $\mu > 1, \lambda\mu > 1$ and let f, g be nonnegative continuous functions for $t \geq T$. Then

$$\left(\int_T^t g(s) \left(\int_s^t f(\tau) d\tau \right)^\lambda ds \right)^\mu$$

$$\leq K \left(\int_T^t f(\tau) \left(\int_T^\tau g(s) ds \right)^\mu d\tau \right) \left(\int_T^t f(\tau) d\tau \right)^{\lambda\mu - 1}$$

$$K = \lambda^\mu \left(\frac{\mu - 1}{\lambda\mu - 1} \right)^{\mu - 1}$$

Step 2. Asymptotic property of intermediate solutions for equation

$$\left(|x'|^{\alpha} \operatorname{sgn} x'\right)' + b(t)|x|^{\beta} \operatorname{sgn} x = 0. \tag{E1}$$

Lemma

Let $1 < \alpha < \beta$ *and assume*

$$\int_0^{\infty} s^{\beta} b(s) ds < \infty.$$

Then for any intermediate solution x *of (E1) we have*

$$\liminf_{t\to\infty} \frac{tx'(t)}{x(t)} > 0.$$

Step 3. Extension to the general weight a:

Set

$$A(t) = \int_0^t a^{-1/\alpha}(\sigma) d\sigma.$$

The change of variable

$$s = A(t), \quad X(s) = x(t), \ t \in [0, \infty), \ s \in [0, \infty) \tag{2}$$

transforms (E), $t \in [0, \infty)$, into

$$\frac{d}{ds}\left(|\dot{X}(s)|^{\alpha} \operatorname{sgn} \dot{X}(s) \right) + c(s) X^{\beta}(s) = 0, \quad s \in [0, \infty),$$

$t(s)$ is the inverse function of $s(t)$, the function c is given by

$$c(s) = a^{1/\alpha}(t(s)) b(t(s)).$$

Existence of intermediate solutions

Consider

$$\left(a(t)|x'|^{\alpha}\operatorname{sgn} x'\right)' + b(t)|x|^{\beta}\operatorname{sgn} x = 0 \quad (\alpha < \beta). \tag{E}$$

If it has intermediate solutions, then

$$J_{\alpha} < \infty, \quad K_{\beta} = \infty. \tag{3}$$

For Emden-Fowler equation

$$x'' + b(t)|x|^{\beta}\operatorname{sgn} x = 0, \quad \beta > 1$$

(3) reads

$$\int_1^{\infty} t\, b(t)dt < \infty, \quad \int_1^{\infty} t^{\beta}\, b(t)dt = \infty.$$

Consider Emden-Fowler equation

$$x'' + b(t)|x|^{\beta} \operatorname{sgn} x = 0, \quad \beta > 1 \tag{EF}$$

where $t \geq 1$.

Define the function

$$F(t) = t^{(\beta+3)/2} b(t).$$

- F is nondecreasing on $[T, \infty)$. Then (1) has an oscillatory solution (Kurzweil 1956).
- F is nonincreasing on $[T, \infty)$

Theorem 2

Let $F(t) = t^{(\beta+3)/2} b(t)$ be nonincreasing for $t \geq T$ and

$$\int_1^\infty t\, b(t) dt < \infty, \quad \int_1^\infty t^\beta\, b(t) dt = \infty.$$

Then (1) has infinitely many intermediate solutions which are positive increasing on $[T, \infty)$.

Example 1. (Moore-Nehari) Consider

$$x'' + \frac{1}{4t^{(\beta+3)/2}}|x|^{\beta}\operatorname{sgn} x = 0 \qquad \beta > 1 \tag{4}$$

where $t \geq 1$. We have

$$F(t) = t^{(\beta+3)/2} b(t) = 1/4.$$

By Theorem 2 this equation has intermediate solutions such that

$$x(t) > 0, \quad x'(t) > 0 \quad t \geq 1. \tag{5}$$

One of them is

$$x(t) = \sqrt{t}.$$

Moreover, this equation has also oscillatory solutions, and subdominant solutions satisfying (5).

Example 2. Consider the equation

$$x'' + \frac{3}{16}\left(\frac{1}{t}\right)^{7/2} x^3(t) = 0 \quad (t \geq 1). \tag{6}$$

The function F is nonincreasing for $t \geq 1$.
By Theorem 2, equation (6) has intermediate solutions which are positive increasing on $[1, \infty)$.
One of them is

$$x(t) = t^{3/4}.$$

Case when (1) has an oscillatory solution

Theorem 3

Let the function $F(t) = t^{(\beta+3)/2} b(t)$ is nondecreasing on $[T, \infty)$,

$$\int_1^\infty t\, b(t) dt < \infty, \quad \int_1^\infty t^\beta\, b(t) dt = \infty$$

and

$$\int_1^\infty (b(t))^{-1/(\beta-1)} t^{-2\beta/(\beta-1)}\, dt < \infty.$$

Then (1) has intermediate solutions

$$0 < x(t) \le ct^{1/2} \quad (t \ge T).$$

M. Marini, Z.D., J. Math. Anal. Appl. (2014)

Idea of the proof – topological limit process:

Step 1. We construct the sequence of subdominant solutions: for any $n > 0$ equation (1) has a subdominant solution x_n such that

$$\lim_{t\to\infty} x_n(t) = n.$$

Step 2. We prove that

$$0 < x_n(t) \le (F(t))^{1/(\beta-1)} t^{1/2} \quad \text{for } t \in [T,\infty).$$

Step 3. The sequences $\{x_n\}$, $\{x_n'\}$ are equibounded and equicontinuous on every finite subinterval of $[T,\infty)$. Hence there exists a converging subsequence $\{x_{n_j}^{(i)}\}$, $i = 0, 1$, which uniformly converges to a function x on every finite subinterval of $[T,\infty)$. Then x is an unbounded (intermediate) solution.

Example 3. Consider

$$x'' + \frac{1}{t^2 \ln^2 t} |x|^2 \operatorname{sgn} x = 0 \quad (t \geq 2). \tag{7}$$

We have $\beta = 2$ and

$$F(t) = \frac{\sqrt{t}}{\ln^2 t}, \quad \int_2^\infty \frac{\ln^4 t}{t^2 \ln^2 t} dt < \infty.$$

Hence, (7) has following solutions:

- subdominant solution which are positive increasing on $(2, \infty)$,
- oscillatory solution (every solution with zero is oscillatory),
- intermediate solution. One of them is

$$x(t) = \ln t.$$

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Thank you for your attention!