On the generalized Emden-Fowler differential equations

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Joint research with Mauro Marini

Topological and Variational Methods for ODEs
Dedicated to Massimo Furi Professor Emeritus at the University of Florence, June 4-5, 2014
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Historical survey

Emden-Fowler differential equation

\[ x'' + b(t)|x|^\beta \text{sgn } x = 0, \quad \beta \neq 1, \beta > 0. \]  \hspace{1cm} (1)

where \( b \) is a positive continuous function for \( t \geq 1 \).

\( \beta > 1 \): super-linear equation, \( \beta < 1 \): sub-linear equation

- Atkinson (1955), Moore and Nehari (1959) … \( \beta > 1 \)
- Belohorec (1961) … \( \beta < 1 \)
A solution $x$ is *nonoscillatory* if it has no zero for large $t$.
In view of the sign of $b$, all nonoscillatory sol’s of (1) satisfy

$$x(t)x'(t) > 0 \quad \text{for large } t.$$  

If $x$ is a sol. of (1), then $-x$ is a sol. too. So, we will consider only nonoscillatory solutions which are eventually positive.

Positive solutions can be classified as:

- **subdominant** $\iff x(\infty) = c_x, \ x'(\infty) = 0$,
- **intermediate** $\iff x(\infty) = \infty, \ x'(\infty) = 0$,
- **dominant** $\iff x(\infty) = \infty, \ x'(\infty) = d_x$,

$c_x, \ d_x$ are positive constants.

If $x, \ y$ and $z$ are subdominant, intermediate and dominant sols, then

$$0 < x(t) < y(t) < z(t) \quad \text{for large } t.$$
Moore and Nehari (Trans. Amer. Math. J. 1959): $\beta > 1$

- Necessary/sufficient conditions for the existence of dominant solutions of (1)
- Necessary/sufficient conditions for the existence of subdominant solutions of (1)
- The above three types of nonoscillatory sols cannot coexist simultaneously!
- No conditions for the existence of intermediate solutions are given.

Two years later Belohorec proved the same results in the sublinear case, i.e. $\beta < 1$. 
Due to the interest for radially symmetric solutions of PDE with p-Laplacian, Kusano and Elbert (1990), Kusano et al (1998) and others considered the above problems for equation

\[
(a(t)|x'|^{\alpha} \text{sgn } x')' + b(t)|x|^\beta \text{sgn } x = 0 \quad (E)
\]

Assumptions: \(\alpha > 0, \beta > 0, a, b \in C[0, \infty), a(t) > 0 \) and \(b(t) > 0\) for \(t \geq 0\) and

\[
\int_{0}^{\infty} a^{-1/\alpha}(s)ds = \infty, \quad \int_{0}^{\infty} b(s)ds < \infty.
\]

For (E) the above classification as subdominant, intermediate and dominant solutions continues to hold by replacing in the Moore and Nehari classification the derivative with the quasiderivative

\[
x' \leadsto x^{[1]} = a(t)|x'(t)|^{\alpha} \text{sgn } x'(t).
\]
Existence results (Kusano et all) – characteristic integrals:

\[ J_\alpha = \int_0^\infty \frac{1}{a^{1/\alpha}(t)} \left( \int_t^\infty b(s) \, ds \right)^{1/\alpha} \, dt, \]

\[ K_\beta = \int_0^\infty b(t) \left( \int_0^t \frac{1}{a^{1/\alpha}(s)} \, ds \right)^\beta \, dt. \]

- (E) has subdominant solutions \( \iff J_\alpha < \infty \).
- (E) has dominant solutions \( \iff K_\beta < \infty \).
- When (1) or (E) has intermediate solutions ???

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Two open problems:

1. Coexistence problem: Is it possible for (E) the coexistence of subdominant, intermediate and dominant solutions?

2. Sufficient conditions for the existence of intermediate solutions.

- \( \alpha = \beta \) (half-linear case)
- \( \alpha > \beta \) (sub-linear case)
- \( \alpha < \beta \) (super-linear case)
• $\alpha = \beta$ (half-linear case)

Ad 1 (coexistence problem): These three types of nonoscillatory sols cannot coexist simultaneously!


Method: the extension of the wronskian identity.

Ad 2: The existence of intermediate solutions
– Sturm-theory for half-linear equation
– the notion of principal and nonprincipal solutions.
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Coexistence problem

Existence of intermediate solutions

- $\alpha > \beta$ (sub-linear case)

Ad 1 (coexistence problem): These three types of nonoscillatory sols cannot coexist simultaneously!


- $\alpha < \beta$ (super-linear case)

Partial answer when $0 < \alpha < 1$: These three types of nonoscillatory sols cannot coexist simultaneously!


Our subject: intermediate solutions when $\alpha < \beta$
Change of integration for $J_\alpha$, $K_\beta$

$\leadsto$ Compatibility of conditions in case $\alpha < \beta$

- $J_\alpha = \infty$, $K_\beta = \infty$ (all sols oscillatory)
- $J_\alpha < \infty$, $K_\beta < \infty$
- $J_\alpha < \infty$, $K_\beta = \infty$.

- **Coexistence problem:** Can subdominant, intermediate and dominant solutions coexist simultaneously?
- **Do exist intermediate solution if $\alpha < \beta$?**
Coexistence problem

\((a(t)|x'|^\alpha \text{sgn } x')' + b(t)|x|^{\beta} \text{sgn } x = 0 \quad (t \geq 0) \quad (E)\)

where \(b(t) \geq 0\) and \(\alpha < \beta\).

**Theorem 1**

Let \(J_\alpha < \infty\) and \(K_\beta < \infty\). Then (E) does not have intermediate solutions.

Consequently,

\(E\) never has simultaneously subdominant, intermediate and dominant solutions!

This is an extension of Moore-Nehari result for (1).
Idea of the proof. 

Step 1. New Holder-type inequality:

Let $\lambda, \mu$ be such that $\mu > 1$, $\lambda \mu > 1$ and let $f, g$ be nonnegative continuous functions for $t \geq T$. Then

$$
\left( \int_T^t g(s) \left( \int_s^t f(\tau) d\tau \right)^\lambda d\tau \right)^\mu \leq K \left( \int_T^t f(\tau) \left( \int_T^\tau g(s) ds \right)^\mu d\tau \right) \left( \int_T^t f(\tau) d\tau \right)^{\lambda \mu - 1}
$$

where

$$
K = \lambda^\mu \left( \frac{\mu - 1}{\lambda \mu - 1} \right)^{\mu - 1}
$$
Step 2. Asymptotic property of intermediate solutions for equation

\[(|x'|^\alpha \text{ sgn } x')' + b(t)|x|^\beta \text{ sgn } x = 0.\] (E1)

Lemma

Let $1 < \alpha < \beta$ and assume

\[\int_0^\infty s^\beta b(s)ds < \infty.\]

Then for any intermediate solution $x$ of (E1) we have

\[\liminf_{t \to \infty} \frac{tx'(t)}{x(t)} > 0.\]
Step 3. Extension to the general weight $a$:

Set

$$A(t) = \int_{0}^{t} a^{-1/\alpha}(\sigma)d\sigma.$$ 

The change of variable

$$s = A(t), \quad X(s) = x(t), \quad t \in [0, \infty), \quad s \in [0, \infty) \quad (2)$$

transforms (E), $t \in [0, \infty)$, into

$$\frac{d}{ds} \left( |\dot{X}(s)|^{\alpha} \operatorname{sgn} \dot{X}(s) \right) + c(s)X^{\beta}(s) = 0, \quad s \in [0, \infty),$$

$t(s)$ is the inverse function of $s(t)$, the function $c$ is given by

$$c(s) = a^{1/\alpha}(t(s)))b(t(s)).$$
Existence of intermediate solutions

Consider

\[
\left( a(t)|x'|^\alpha \text{sgn } x' \right)' + b(t)|x|^\beta \text{sgn } x = 0 \quad (\alpha < \beta). \quad (E)
\]

If it has intermediate solutions, then

\[
J_\alpha < \infty, \quad K_\beta = \infty. \quad (3)
\]

For Emden-Fowler equation

\[
x'' + b(t)|x|^\beta \text{sgn } x = 0, \quad \beta > 1
\]

(3) reads

\[
\int_1^\infty t b(t)dt < \infty, \quad \int_1^\infty t^\beta b(t)dt = \infty.
\]
Consider Emden-Fowler equation

\[ x'' + b(t)|x|^\beta \text{sgn} \, x = 0, \quad \beta > 1 \quad \text{(EF)} \]

where \( t \geq 1 \).

Define the function

\[ F(t) = t^{(\beta+3)/2} b(t). \]

- \( F \) is nondecreasing on \([T, \infty)\). Then (1) has an oscillatory solution (Kurzweil 1956).
- \( F \) is nonincreasing on \([T, \infty)\)

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Theorem 2

Let \( F(t) = t^{(\beta+3)/2} b(t) \) be nonincreasing for \( t \geq T \) and

\[
\int_1^\infty t b(t) dt < \infty, \quad \int_1^\infty t^\beta b(t) dt = \infty.
\]

Then (1) has infinitely many intermediate solutions which are positive increasing on \([T, \infty)\).
Example 1. (Moore-Nehari) Consider

\[ x'' + \frac{1}{4t^{(\beta+3)/2}} |x|^\beta \text{sgn } x = 0 \quad \beta > 1 \quad (4) \]

where \( t \geq 1 \). We have

\[ F(t) = t^{(\beta+3)/2} b(t) = 1/4. \]

By Theorem 2 this equation has intermediate solutions such that

\[ x(t) > 0, \quad x'(t) > 0 \quad t \geq 1. \quad (5) \]

One of them is

\[ x(t) = \sqrt{t}. \]

Moreover, this equation has also oscillatory solutions, and subdominant solutions satisfying (5).
Example 2. Consider the equation

\[ x'' + \frac{3}{16} \left(\frac{1}{t}\right)^{7/2} \ x^3(t) = 0 \quad (t \geq 1). \]  

(6)

The function \( F \) is nonincreasing for \( t \geq 1 \). By Theorem 2, equation (6) has intermediate solutions which are positive increasing on \([1, \infty)\). One of them is

\[ x(t) = t^{3/4}. \]
Case when (1) has an oscillatory solution

**Theorem 3**

Let the function $F(t) = t^{(\beta+3)/2} b(t)$ is nondecreasing on $[T, \infty)$,

$$
\int_1^\infty t \, b(t) \, dt < \infty, \quad \int_1^\infty t^\beta \, b(t) \, dt = \infty
$$

and

$$
\int_1^\infty (b(t))^{-1/(\beta-1)} t^{-2\beta/(\beta-1)} \, dt < \infty.
$$

Then (1) has intermediate solutions

$$
0 < x(t) \leq ct^{1/2} \quad (t \geq T).
$$

Idea of the proof – topological limit process:

*Step 1.* We construct the sequence of subdominant solutions: for any $n > 0$ equation (1) has a subdominant solution $x_n$ such that

$$\lim_{t\to\infty} x_n(t) = n.$$  

*Step 2.* We prove that

$$0 < x_n(t) \leq (F(t))^{1/(\beta-1)}t^{1/2} \quad \text{for } t \in [T, \infty).$$

*Step 3.* The sequences $\{x_n\}$, $\{x'_n\}$ are equibounded and equicontinuous on every finite subinterval of $[T, \infty)$. Hence there exists a converging subsequence $\{x_n^{(i)}\}$, $i = 0, 1$, which uniformly converges to a function $x$ on every finite subinterval of $[T, \infty)$. Then $x$ is an unbounded (intermediate) solution.
Example 3. Consider

\[ x'' + \frac{1}{t^2 \ln^2 t} |x|^2 \text{sgn} x = 0 \quad (t \geq 2). \]

We have \( \beta = 2 \) and

\[ F(t) = \frac{\sqrt{t}}{\ln^2 t}, \quad \int_2^\infty \frac{\ln^4 t}{t^2 \ln^2 t} dt < \infty. \]

Hence, (7) has following solutions:

- subdominant solution which are positive increasing on \((2, \infty)\),
- oscillatory solution (every solution with zero is oscillatory),
- intermediate solution. One of them is

\[ x(t) = \ln t. \]
References


References II


Thank you for your attention!