The Poincaré - Birkhoff theorem in the framework of Hamiltonian systems

Alessandro Fonda

(Università degli Studi di Trieste)

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a collaboration with Antonio J. Ureña

Jules Henri Poincaré (1854 – 1912)

SUR UN THÉORÈME DE GÉOMÉTRIE.

Par M. **H. Poincaré** (Paris).

Adunanza del 10 marzo 1912.

§ 1.

Introduction.

Je n'ai jamais présenté au public un travail aussi inachevé; je crois donc nécessaire d'expliquer en quelques mots les raisons qui m'ont déterminé à le publier, et d'abord celles qui m'avaient engagé à l'entreprendre. J'ai démontré, il y a longtemps déjà, l'existence des solutions périodiques du problème des trois corps; le résultat laissait cependant encore à désirer; car, si l'existence de chaque sorte de solution était établie pour les petites valeurs des masses, on ne voyait pas ce qui devait arriver pour des valeurs plus grandes, quelles étaient celles de ces solutions qui subsistaient et dans quel ordre elles disparaissaient. En réfléchissant à cette question, je me suis assuré que la réponse devait dépendre de l'exactitude ou de la fausseté d'un certain théorème de géométrie dont l'énoncé est très simple, du moins dans le cas du problème restreint et des problèmes de Dynamique où il n'y a que deux degrés de liberté.

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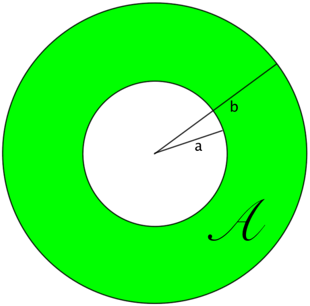

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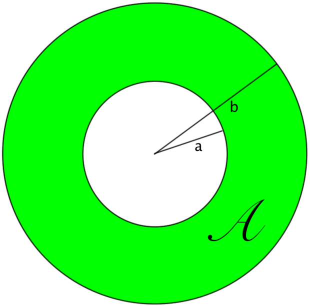

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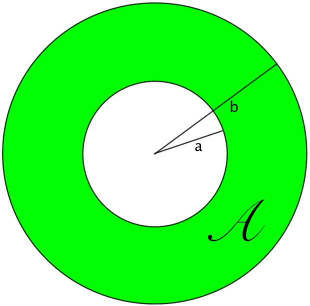

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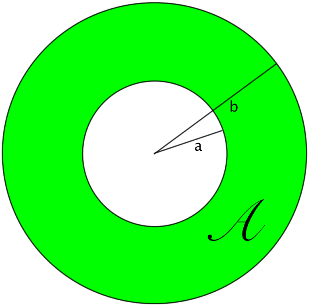

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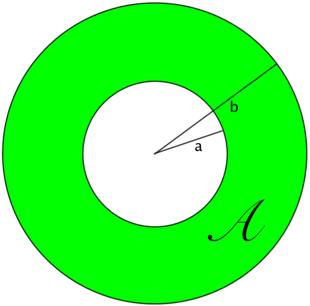

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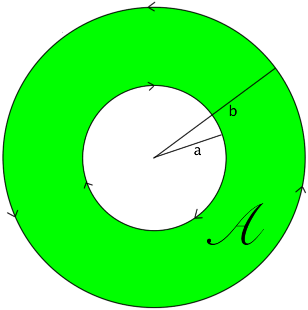

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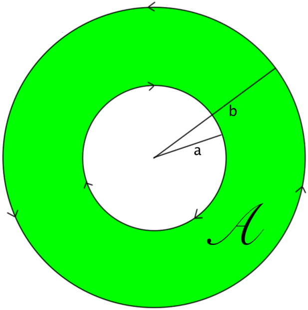

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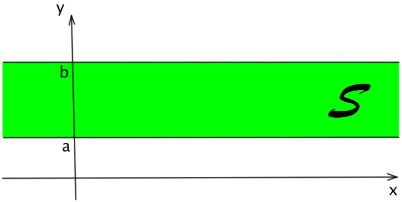

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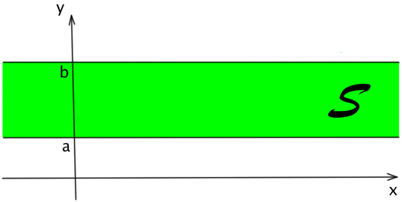

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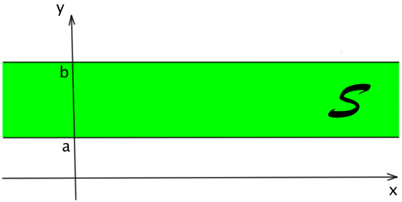

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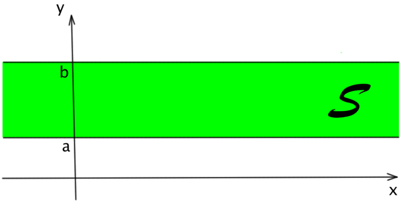

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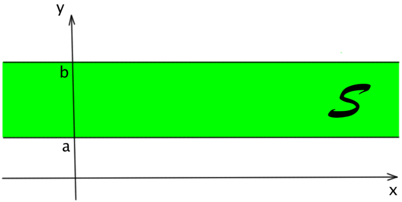

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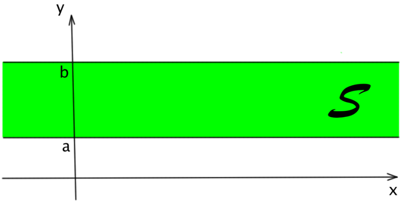

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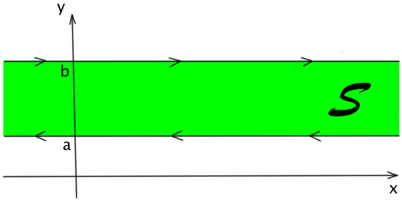

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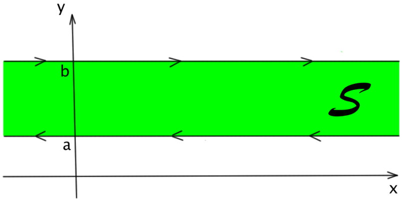

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George David Birkhoff (1884 – 1944)

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Applications to the existence of periodic solutions were provided by:

Bonheure, Boscaggin, Butler, Corsato, Del Pino, T. Ding, Fabry, Garrione, Hartman, Manásevich, Mawhin, Omari, Ortega, Sabatini, Sfecci, Smets, Torres, Zanini, Zanolin, ...

Periodic solutions as fixed points of the Poincaré map

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$$\dot{x} = \frac{\partial H}{\partial y}(t, x, y), \qquad \dot{y} = -\frac{\partial H}{\partial x}(t, x, y),$$

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Bad news:

It is very difficult to find an invariant annulus for $\mathcal{P}$.

Generalizing the Poincaré – Birkhoff theorem (in the framework of Hamiltonian systems)

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Twist condition: the solutions $(x(t),y(t))$ with "starting point" $(x(0),y(0))$ on $\partial\mathcal{S}$ are defined on $[0,T]$ and satisfy

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However,

a genuine generalization of the Poincaré – Birkhoff theorem to higher dimensions has never been given.

[Moser and Zehnder, Notes on Dynamical Systems, 2005].

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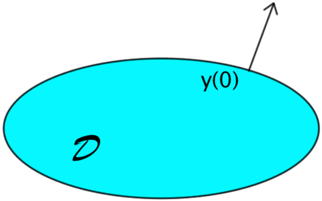

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$$\dot{x} = \frac{\partial H}{\partial y}(t, x, y)\,, \qquad \dot{y} = -\frac{\partial H}{\partial x}(t, x, y)\,,$$

and assume that the Hamiltonian $H(t, x, y)$ is T-periodic in t.

Here, $x = (x_1, \dots, x_N)$ and $y = (y_1, \dots, y_N)$.

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[Mawhin–Willem 1984]

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[Ding–Zanolin 1992, Boscaggin–Ortega 2014]

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[Jacobowitz 1976, Hartman 1977, F.–Sfecci 2014]

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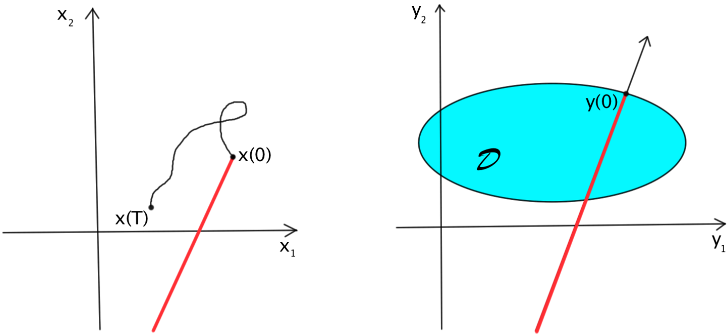

a collaboration with Antonio J. Ureña