Nontrivial solutions of local and nonlocal Neumann boundary value problems

Gennaro Infante

Dipartimento di Matematica ed Informatica
Università della Calabria, Cosenza, Italy
www.mat.unical.it/∼infante
gennaro.infante@unical.it

Joint work with Paolamaria Pietramala and F. Adrián F. Tojo; arXiv:1404.1390

Topological and Variational Methods for ODEs

Dedicated to Massimo Furi Professor Emeritus at the University of Florence

Firenze, June 3-4, 2014

Introduction

We discuss the existence, localization, multiplicity and non-existence of nontrivial solutions of the second order differential equation

$$u''(t) + h(t, u(t)) = 0, \ \ t \in (0, 1), \tag{1}$$

subject to (local) Neumann boundary conditions (BCs)

$$u'(0) = u'(1) = 0, \tag{2}$$

or to non-local BCs of Neumann type

$$u'(0) = \alpha[u], \quad u'(1) = \beta[u], \tag{3}$$

where $\alpha[\cdot]$, $\beta[\cdot]$ are linear functionals given by Stieltjes integrals, namely

$$\alpha[u] = \int_0^1 u(s)\, dA(s), \quad \beta[u] = \int_0^1 u(s)\, dB(s).$$

The existence of positive solutions of the local BVP (1)-(2) has been studied by Miciano and Shivaji in [33] and by Li and co-authors [31, 32]. Note that, since $\lambda = 0$ is an eigenvalue of the associated linear problem

$$u''(t) + \lambda u(t) = 0, \quad u'(0) = u'(1) = 0,$$

the correspondent Green's function does not exist. Therefore we use a shift argument similar to the ones used by Han [14], Torres [41] and Webb and Zima [51] for different BCs and we study two related BVPs for which the Green's function can be constructed, namely

$$-u'' - \omega^2 u = f(t,u) := h(t,u) - \omega^2 u, \; u'(0) = u'(1) = 0, \quad (4)$$

and (with an abuse of notation)

$$-u'' + \omega^2 u = f(t,u) := h(t,u) + \omega^2 u, \; u'(0) = u'(1) = 0. \quad (5)$$

The BVPs (4) and (5) have been recently object of interest by a number of authors [3, 7, 10, 38, 40, 42, 43, 44, 53, 54, 55, 56, 57]; We study the properties of the associated Green's functions and improve some estimates that occur in earlier papers.

The formulation of the nonlocal BCs in terms of linear functionals includes multi-point and integral conditions, namely

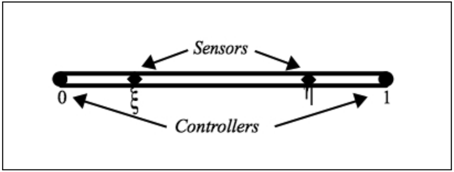
$$\alpha[u] = \sum_{j=1}^{m} \alpha_j u(\eta_j) \quad \text{or} \quad \alpha[u] = \int_0^1 \phi(s)u(s)ds.$$

The study of nonlocal multi-point BCs goes back to Picone (1908) [37] and has been developed in the years; we mention the work of Whyburn (1942) [52] on integral BCs, the more recent reviews by Ma [30] and Ntouyas [35] and the papers by Karakostas and Tsamatos [25, 26] and by Webb and GI [48].

One motivation for studying nonlocal problems is that they occur when modelling heat-flow problems. For example the BVP

$$u''(t) + h(t, u(t)) = 0, \ u'(0) = \alpha u(\xi), \ u'(1) = \beta u(\eta), \ \xi, \eta \in [0, 1],$$

models a thermostat where two controllers at $t = 0$ and $t = 1$ add or remove heat according to the temperatures detected by two sensors at $t = \xi$ and $t = \eta$.



For some references this type of thermostat models see Cabada et al. [6], GI [16, 17], GI and Webb [23], Palamides et al. [27, 36] and Webb [45, 46, 47].

Our methodology is to build a general theory the existence of nontrivial solutions of the *perturbed* Hammerstein integral equation of the form

$$u(t) = \gamma(t)\alpha[u] + \delta(t)\beta[u] + \int_0^1 k(t,s)g(s)f(s,u(s))\,ds := Tu(t), \tag{6}$$

by working in a cone of functions that are allowed to *change sign*.

This setting covers, as *special cases*, the BVP (1)-(3) and the BVP (1)-(2).

The approach that we use relies on classical fixed point index theory and we make use of ideas from the papers [6, 21, 48, 50].

The fixed point index

- What is the fixed point index of a compact map T?
- Roughly speaking, is the algebraic count of the fixed points of T in a certain set.

  The definition is rather technical and involves the knowledge of the *Leray-Schauder degree*.
- Usually the best candidate for a set on which to compute the fixed point index is a *cone*.

  A cone K in a Banach space X, is a closed, convex set such that $\lambda x \in K$ for $x \in K$ and $\lambda \geq 0$ and $K \cap (-K) = \{0\}$.

More details on the fixed point index can be found in the review of Amann [1] and in the book of Guo and Lakshmikantham [13].

Properties of the fixed point index

- Let D be an open bounded set of X with $D_K \neq \emptyset$ and $\overline{D}_K \neq K$, where $D_K = D \cap K$.
  Assume that $T : \overline{D}_K \to K$ is a compact map such that $x \neq Tx$ for $x \in \partial D_K$.
  Then the fixed point index $i_K(T, D_K)$ has the following properties:

(1) If there exists $e \in K \setminus \{0\}$ such that $x \neq Tx + \lambda e$ for all $x \in \partial D_K$ and all $\lambda > 0$, then $i_K(T, D_K) = 0$.

(2) If $Tx \neq \lambda x$ for all $x \in \partial D_K$ and all $\lambda > 1$, then $i_K(T, D_K) = 1$.

The Leray-Schauder condition in (2) holds, for example, if $\|Tx\| \leq \|x\|$ for $x \in \partial D_K$.

Main assumptions I

- $k:[0,1]\times[0,1]\to\mathbb{R}$ is measurable, and for every $\tau\in[0,1]$ we have

$$\lim_{t\to\tau}|k(t,s)-k(\tau,s)|=0 \text{ for almost every } s\in[0,1].$$

- There exist $[a,b]\subseteq[0,1]$, $\Phi\in L^{\infty}[0,1]$, and $c_1\in(0,1]$ such that

$$|k(t,s)|\le\Phi(s) \text{ for } t\in[0,1] \text{ and almost every } s\in[0,1],$$
$$k(t,s)\ge c_1\Phi(s) \text{ for } t\in[a,b] \text{ and almost every } s\in[0,1].$$

- $g\,\Phi\in L^1[0,1]$, $g(s)\ge 0$ for almost every $s\in[0,1]$, and $\int_a^b\Phi(s)g(s)\,ds>0$.

Main assumptions II

- $f : [0,1] \times (-\infty,\infty) \to [0,\infty)$ satisfies Carathéodory conditions, that is, $f(\cdot,u)$ is measurable for each fixed $u \in (-\infty,\infty)$, $f(t,\cdot)$ is continuous for almost every $t \in [0,1]$, and for each $r > 0$, there exists $\phi_r \in L^\infty[0,1]$ such that

$$f(t,u) \le \phi_r(t) \;\text{ for all }\; u \in [-r,r], \;\text{ and almost every }\; t \in [0,1].$$

- A, B are of bounded variation functions and $\mathcal{K}_A(s),\ \mathcal{K}_B(s) \ge 0$ for almost every $s \in [0,1]$, where

$$\mathcal{K}_A(s) := \int_0^1 k(t,s)\, dA(t) \text{ and } \mathcal{K}_B(s) := \int_0^1 k(t,s)\, dB(t).$$

Main assumptions III

- $\gamma \in C[0,1],\ 0 \leq \alpha[\gamma] < 1,\ \ \beta[\gamma] \geq 0.$
  There exists $c_2 \in (0,1]$ such that $\gamma(t) \geq c_2\|\gamma\|$ for $t \in [a,b]$.
- $\delta \in C[0,1],\ 0 \leq \beta[\delta] < 1,\ \ \alpha[\delta] \geq 0.$
  There exists $c_3 \in (0,1]$ such that $\delta(t) \geq c_3\|\delta\|$ for $t \in [a,b]$.
- $D := (1-\alpha[\gamma])(1-\beta[\delta]) - \alpha[\delta]\beta[\gamma] > 0.$

The assumptions above allow us to work in the cone

$$K := \{u \in C[0,1] : \min_{t\in[a,b]} u(t) \geq c\|u\|,\ \alpha[u], \beta[u] \geq 0\},$$

where $c = \min\{c_1, c_2, c_3\}$.

Cone invariance and bounded sets

Under the assumptions above the the operator T maps K into K and is compact.

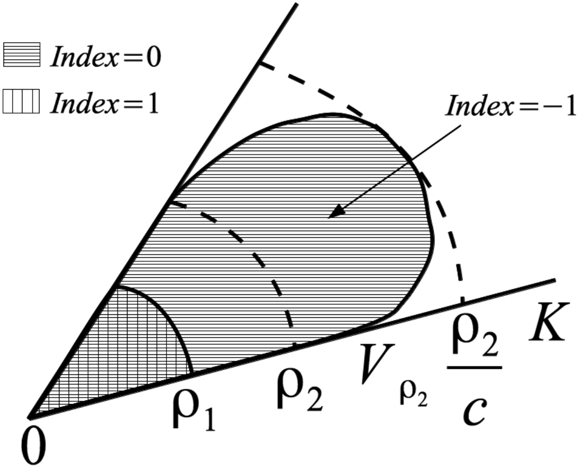
We use the following open bounded sets (relative to K):

$$K_\rho := \{u \in K : \|u\| < \rho\},$$

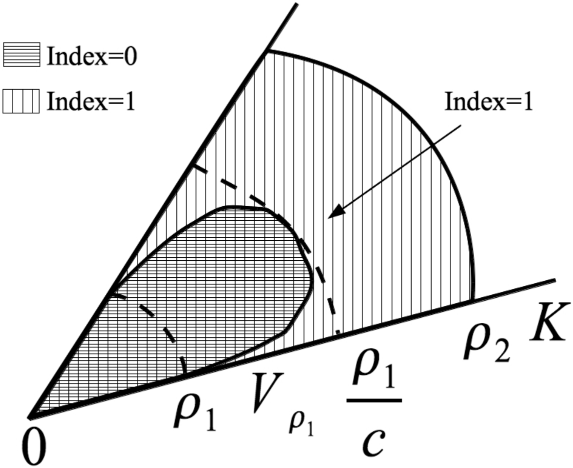
$$V_\rho := \{u \in K : \min_{t\in[a,b]} u(t) < \rho\}.$$

Note that $K_\rho \subset V_\rho \subset K_{\rho/c}$.

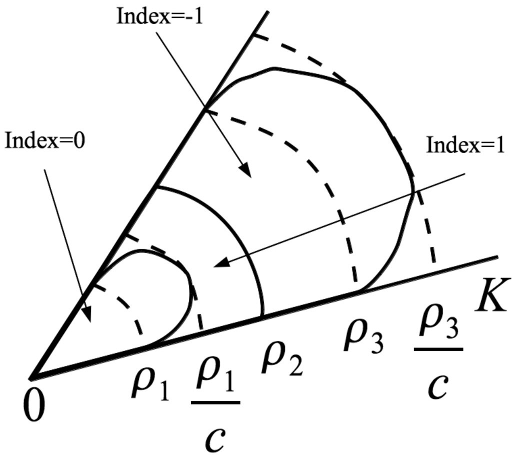
One nontrivial solution



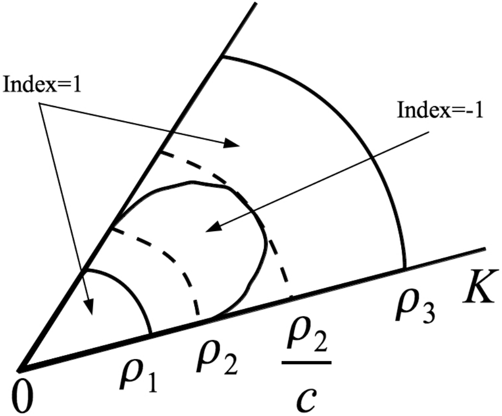
One nontrivial solution



Two nontrivial solutions



Two nontrivial solutions



Index calculations I

Assume that there exists $\rho > 0$ such that

$$f^{-\rho,\rho}\left[\left(\frac{\|\gamma\|}{D}(1-\beta[\delta])+\frac{\|\delta\|}{D}\beta[\gamma]\right)\int_0^1\mathcal{K}_A(s)g(s)\,ds\right.$$
$$\left.+\left(\frac{\|\gamma\|}{D}\alpha[\delta]+\frac{\|\delta\|}{D}(1-\alpha[\gamma])\right)\int_0^1\mathcal{K}_B(s)g(s)\,ds+\frac{1}{m}\right]<1.$$

where

$$f^{-\rho,\rho}:=\sup\left\{\frac{f(t,u)}{\rho}:\ (t,u)\in[0,1]\times[-\rho,\rho]\right\},$$

$$\frac{1}{m}:=\sup_{t\in[0,1]}\left\{\max\left\{\int_0^1 k^+(t,s)g(s)\,ds,\int_0^1 k^-(t,s)g(s)\,ds\right\}\right\}.$$

Then $i_K(T,K_\rho)=1$.

Index calculations II

Assume that there exist $\rho > 0$ such that

$$f_{\rho,\rho/c}\left(\left(\frac{c_2\|\gamma\|}{D}(1-\beta[\delta])+\frac{c_3\|\delta\|}{D}\beta[\gamma]\right)\int_a^b \mathcal{K}_A(s)g(s)\,ds\right.$$
$$\left.+\left(\frac{c_2\|\gamma\|}{D}\alpha[\delta]+\frac{c_3\|\delta\|}{D}(1-\alpha[\gamma])\right)\int_a^b \mathcal{K}_B(s)g(s)\,ds+\frac{1}{M(a,b)}\right)>1, \tag{7}$$

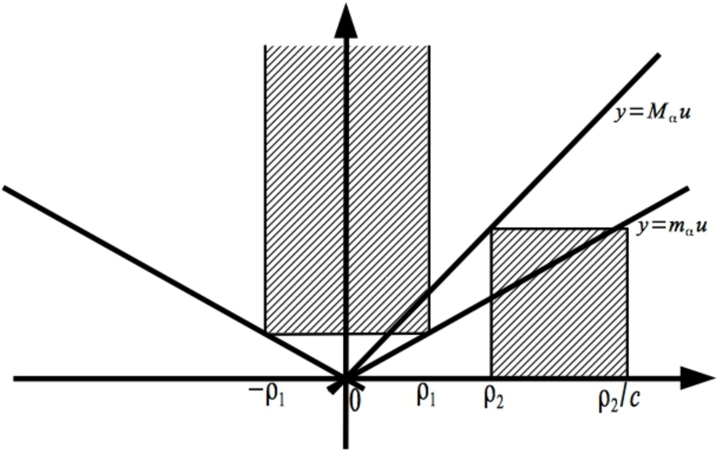
where

$$f_{\rho,\rho/c} := \inf\left\{\frac{f(t,u)}{\rho}:\ (t,u)\in[a,b]\times[\rho,\rho/c]\right\},$$

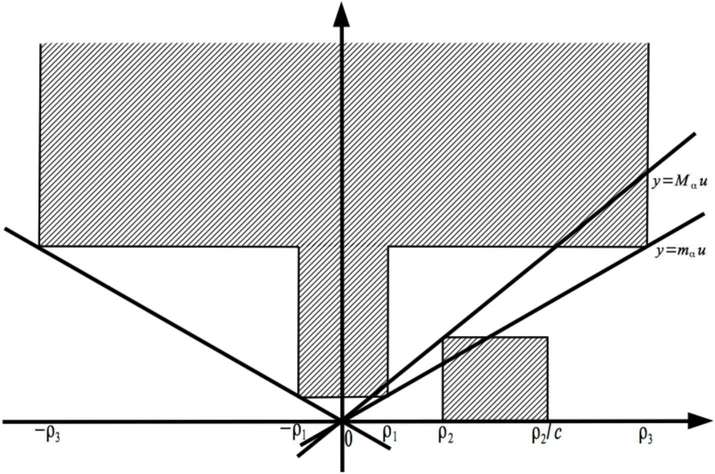
$$\frac{1}{M(a,b)} := \inf_{t\in[a,b]}\int_a^b k(t,s)g(s)\,ds.$$

Then $i_K(T,V_\rho)=0$.

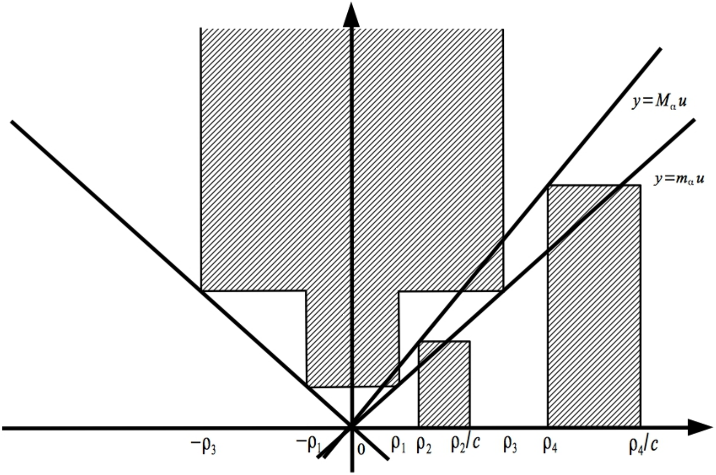
One solution



Two solutions



Three solutions



An auxiliary operator

We now consider the auxiliary Hammerstein integral equation

$$u(t) = \int_0^1 k_S(t,s)g(s)f(s,u(s))ds := Su(t), \qquad (8)$$

where the kernel k_S is given by the formula

$$\begin{aligned} k_S(t,s) = {} & \frac{\gamma(t)}{D}\left[(1-\beta[\delta])\mathcal{K}_A(s) + \alpha[\delta]\mathcal{K}_B(s)\right] \\ & + \frac{\delta(t)}{D}\left[\beta[\gamma]\mathcal{K}_A(s) + (1-\alpha[\gamma])\mathcal{K}_B(s)\right] + k(t,s). \qquad (9) \end{aligned}$$

The operator S shares a number of useful properties with T: the cone invariance, the compactness and the same fixed points in K.

A non-existence result

Assume that one of the following conditions holds:

(1) $f(t,u) < m_S|u|$ for every $t \in [0,1]$ and $u \in \mathbb{R}\backslash\{0\}$, where

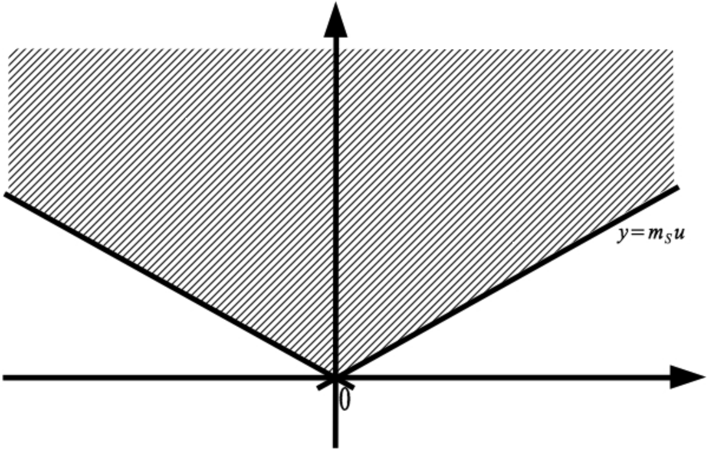
$$\frac{1}{m_S} := \sup_{t\in[0,1]} \left\{ \max\left\{ \int_0^1 k_S^+(t,s)g(s)\,ds, \int_0^1 k_S^-(t,s)g(s)\,ds \right\} \right\},$$

(2) $f(t,u) > M_S u$ for every $t \in [a,b]$ and $u \in \mathbb{R}^+$, where

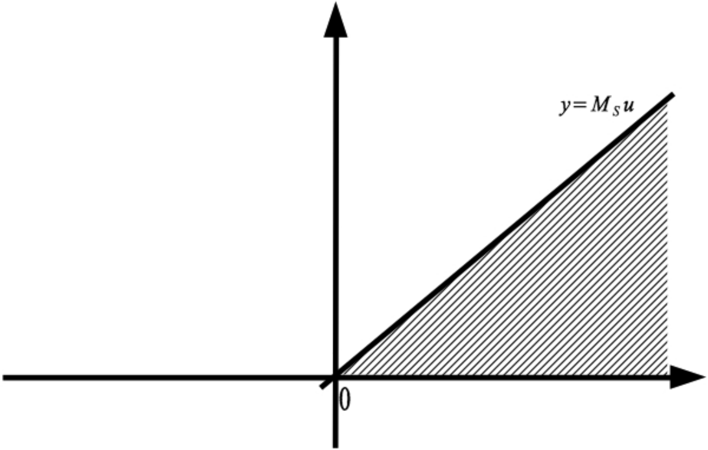
$$\frac{1}{M_S(a,b)} = \frac{1}{M_S} := \inf_{t\in[a,b]} \int_a^b k_S(t,s)g(s)\,ds.$$

Then the equations (6) and (8) have no non-trivial solution in K.

No solutions in K, condition (1)



No solutions in K, condition (2)



Consider the BVP

$$-u''(t)+u(t)=\lambda t e^{u(t)}, t\in[0,1], \quad u'(0)=u'(1)=0. \quad (10)$$

In [3] Bonanno and Pizzimenti establish the existence of at least one positive solution such that $\|u\|<2$ for $\lambda\in(0,2e^{-2})$.
The BVP (10) is equivalent to the integral equation

$$u(t)=\int_0^1 k(t,s)g(s)f(u(s))ds,$$

where

$$g(s)=s,\ f(u)=\lambda e^u$$

and

$$k(t,s):=\frac{1}{\sinh(1)}\begin{cases}\cosh(1-t)\cosh s, & 0\le s\le t\le 1,\\ \cosh(1-s)\cosh t, & 0\le t\le s\le 1.\end{cases}$$

The kernel k is positive, we can take $[a,b]=[0,1]$ and work in the cone

$$K=\{u\in C[0,1]:\min_{t\in[0,1]}u(t)\geq c\|u\|\},$$

where

$$c=c(0,1)=1/\cosh 1=0.648.$$

In this case

$$m=\frac{e+1}{2}=1.859,\quad M(0,1)=\frac{e+1}{e-1}=2.163,$$
$$f^{-\rho,\rho}=f_{\rho,\rho/c}=\lambda e^{\rho}/\rho.$$

Taking $\rho_2=2$ we have that the index is 1 on K_{ρ_2} for $\lambda<(e+1)e^{-2}$, and taking $0<\rho_1<c/2$ we have that the index is 0 on V_{ρ_1} for $\lambda>[(e+1)/(e-1)]\rho_1 e^{-\rho_1}$.

Hence there exists a positive solution of norm less than 2 whenever

$$\lambda \in \left(0, \frac{e+1}{e^2}\right) \supset (0, 2e^{-2}).$$

Reasoning as in [24], when $\lambda = 1/4$ the choice of $\rho_2 = 0.16$ and $\rho_1 = 0.1$ gives the following localization for the solution

$$0.064 \le u(t) \le 0.16, \;\; t \in [0,1].$$

Furthermore, for

$$\lambda > \frac{e+1}{e(e-1)},$$

there are no solutions in K (the trivial solution does not satisfy the differential equation). Note that $T : P \to K$; this shows that there are no positive solutions for the BVP (10).

H. Amann, Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, *SIAM. Rev.*, **18** (1976), 620–709.

G. Bonanno, A critical point theorem via the Ekeland variational principle, *Nonlinear Anal.*, **75** (2012), 2992–3007.

G. Bonanno and P. F. Pizzimenti, Neumann boundary value problems with not coercive potential, *Mediterr. J. Math.*, **9** (2012), 601–609.

A. Cabada and F. A. F. Tojo, Comparison results for first order linear operators with reflection and periodic boundary value conditions, *Nonlinear Anal.*, **78** (2013), 32–46.

A. Cabada, G. Infante and F. A. F. Tojo, Nontrivial solutions of perturbed Hammerstein integral equations with reflections, *Bound. Value Probl.*, **2013:86** (2013).

A. Cabada, G. Infante and F. A. F. Tojo, Nonzero solutions of perturbed Hammerstein integral equations with deviated arguments and applications, *arXiv: 1306.6560v1 [math.CA] 27 Jun 2013*.

Y. Dongming, Z. Qiang and P. Zhigang, Existence of positive solutions for Neumann boundary value problem with a variable coefficient, *Int. J. Differ. Equ.*, (2011), doi:10.1155/2011/376753.

H. Fan and R. Ma, Loss of positivity in a nonlinear second order ordinary differential equations, *Nonlinear Anal.*, **71** (2009), 437–444.

F. Faraci and V. Moroz, Solutions of Hammerstein integral equations via a variational principle, *J. Integral Equations Appl.*, **15** (2003), 385–402.

Y. Feng and G. Li, Exact three positive solutions to a second-order Neumann boundary value problem with singular nonlinearity, *Arab. J. Sci. Eng.*, **35, 2D** (2010).

D. Franco, G. Infante and D. O'Regan, Positive and nontrivial solutions for the Urysohn integral equation, *Acta Math. Sin.*, **22** (2006), 1745–1750.

D. Franco, G. Infante and D. O'Regan, Nontrivial solutions in abstract cones for Hammerstein integral systems, *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.*, **14** (2007), 837–850.

D. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, 1988.

X. Han, Positive solutions for a three-point boundary value problems at resonance, *J. Math. Anal. Appl.*, **336** (2007), 556–568.

G. Infante, Eigenvalues of some non-local boundary-value problems, *Proc. Edinb. Math. Soc.*, **46** (2003), 75–86.

G. Infante, Positive solutions of some nonlinear BVPs involving singularities and integral BCs, *Discrete Contin. Dyn. Syst. Series S*, **1** (2008), 99–106.

G. Infante, Nonlocal boundary value problems with two nonlinear boundary conditions, *Commun. Appl. Anal.*, **12** (2008), 279–288.

G. Infante and P. Pietramala, Nonlocal impulsive boundary value problems with solutions that change sign, *CP1124, Mathematical Models in Engineering, Biology, and Medicine, Proceedings of the International Conference on Boundary Value Problems, edited by A. Cabada, E. Liz, and J.J. Nieto*, (2009), 205–213.

G. Infante and P. Pietramala, Perturbed Hammerstein integral inclusions with solutions that change sign, *Comment. Math. Univ. Carolin.*, **50** (2009), 591–605.

G. Infante and J. R. L. Webb, Nonzero solutions of Hammerstein integral equations with discontinuous kernels, *J. Math. Anal. Appl.*, **272** (2002), 30–42.

G. Infante and J. R. L. Webb, Three point boundary value problems with solutions that change sign, *J. Integral Equations Appl.*, **15** (2003), 37–57.

G. Infante and J. R. L. Webb, Loss of positivity in a nonlinear scalar heat equation, *NoDEA Nonlinear Differential Equations Appl.*, **13** (2006), 249–261.

G. Infante and J. R. L. Webb, Nonlinear nonlocal boundary value problems and perturbed Hammerstein integral equations, *Proc. Edinb. Math. Soc.*, **49** (2006), 637–656.

G. Infante, P. Pietramala and M. Tenuta, Existence and localization of positive solutions for a nonlocal BVP arising in chemical reactor theory, *Commun. Nonlinear Sci. Numer. Simulat.*, **19** (2014), 2245–2251.

G. L. Karakostas and P. Ch. Tsamatos, Existence of multiple positive solutions for a nonlocal boundary value problem, *Topol. Methods Nonlinear Anal.*, **19** (2002), 109–121.

G. L. Karakostas and P. Ch. Tsamatos, Multiple positive solutions of some Fredholm integral equations arisen from nonlocal boundary-value problems, *Electron. J. Differential Equations*, **2002**, No. 30, 17 pp.

I. Karatsompanis and P. K. Palamides, Polynomial approximation to a non-local boundary value problem, *Comput. Math. Appl.*, **60** (2010), 3058–3071.

M. A. Krasnosel'skiĭ and P. P. Zabreĭko, *Geometrical methods of nonlinear analysis*, Springer-Verlag, Berlin, (1984).

K. Q. Lan, Multiple positive solutions of Hammerstein integral equations with singularities, *Differential Equations and Dynamical Systems*, **8** (2000), 175–195.

R. Ma, A survey on nonlocal boundary value problems, *Appl. Math. E-Notes*, **7** (2001), 257–279.

K. Li, Multiple solutions for an asymptotically linear Duffing equation with Neumann boundary value conditions, *Nonlinear Anal.*, **74** (2011), 2819–2830.

K. Li, J. Li and W. Mao, Multiple solutions for asymptotically linear Duffing equations with Neumann boundary value conditions (II), *J. Math. Anal. Appl.*, **401** (2013), 548–553.

A. R. Miciano and R. Shivaji, Multiple positive solutions for a class of semipositone Neumann two-point boundary value problems, *J. Math. Anal. Appl.*, **178** (1993), 102–115.

J. J. Nieto and J. Pimentel, Positive solutions of a fractional thermostat model, *Bound. Value Probl.*, **2013:5** (2013).

S. K. Ntouyas, Nonlocal initial and boundary value problems: a survey, *Handbook of differential equations: ordinary differential equations. Vol. II*, 461–557, Elsevier B. V., Amsterdam, 2005.

P. K. Palamides, G. Infante and P. Pietramala, Nontrivial solutions of a nonlinear heat flow problem via Sperner's lemma, *Appl. Math. Lett.*, **22** (2009), 1444–1450.

M. Picone, Su un problema al contorno nelle equazioni differenziali lineari ordinarie del secondo ordine, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, **10** (1908), 1–95.

J. P. Sun and W. T. Li, Multiple positive solutions to second-order Neumann boundary value problems, *Appl. Math. Comput.*, **146** (2003), 187–194.

Y. Sun, Y. J. Cho and D. O'Regan, Positive solutions for singular second order Neumann boundary value problems via a cone fixed point theorem, *Appl. Math. Comput.*, **210** (2009), 80–86.

J. P. Sun, W. T. Li and S. S. Cheng, Three positive solutions for second-order Neumann boundary value problems, *Appl. Math. Lett.*, **17** (2004), 1079–1084.

P. J. Torres, Existence of one-signed periodic solutions of some second-order differential equations via a Krasnoselskii fixed point theorem, *J. Differential Equations*, **190** (2003), 643–662.

F. Wang and F. Zhang, Existence of positive solutions of Neumann boundary value problem via a cone compression-expansion fixed point theorem of functional type, *J. Appl. Math. Comput.*, **35** (2011), 341–349.

F. Wang, Y. Cui and F. Zhang, A singular nonlinear second-order Neumann boundary value problem with positive solutions, *Thai J. Math.*, **7** (2009), 243–257.

F. Wang, F. Zhang and Y. Yu, Existence of positive solutions of Neumann boundary value problem via a convex functional compression-expansion fixed point theorem, *Fixed Point Theory*, **11** (2010), 395–400.

J. R. L. Webb, Multiple positive solutions of some nonlinear heat flow problems, *Discrete Contin. Dyn. Syst.*, **suppl.** (2005), 895–903.

J. R. L. Webb, Optimal constants in a nonlocal boundary value problem, *Nonlinear Anal.*, **63** (2005), 672–685.

J. R. L. Webb, Existence of positive solutions for a thermostat model, *Nonlinear Anal. Real World Appl.*, **13** (2012), 923–938

J. R. L. Webb and G. Infante, Positive solutions of nonlocal boundary value problems: a unified approach, *J. London Math. Soc.*, **74** (2006), 673–693.

J. R. L. Webb and G. Infante, Positive solutions of nonlocal boundary value problems involving integral conditions, *NoDEA Nonlinear Differential Equations Appl.*, **15** (2008), 45–67.

J. R. L. Webb and K. Q. Lan, Eigenvalue criteria for existence of multiple positive solutions of nonlinear boundary value problems of local and nonlocal type, *Topol. Methods Nonlinear Anal.*, **27** (2006), 91–115.

J. R. L. Webb and M. Zima, Multiple positive solutions of resonant and non-resonant nonlocal boundary value problems, *Nonlinear Anal.*, **71** (2009), 1369–1378.

W. M. Whyburn, Differential equations with general boundary conditions, *Bull. Amer. Math. Soc.*, **48** (1942), 692–704.

Q. Yao, Successively iterative method of nonlinear Neumann boundary value problems, *Appl. Math. Comput.*, **217** (2010), 2301–2306.

Q. Yao, Multiple positive solutions of nonlinear Neumann problems with time and space singularities, *Applied Mathematical Letters*, **25** (2012), 93–98.

J. Zhang and C. Zhai, Existence and uniqueness results for perturbed Neumann boundary value problems, *Bound. Value Probl.*, **2010:494210** (2010), doi:10.1155/2010/494210.

Y. W. Zhang and H. X. Li, Positive solutions of a second-order Neumann boundary value problem with a parameter, *Bull. Aust. Math. Soc.*, **86** (2012), 244–253.

L. Zhilong, Existence of positive solutions of superlinear second-order Neumann boundary value problem, *Nonlinear Anal.*, **72** (2010), 3216–3221.

Best wishes Massimo!