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Dedicated to Massimo Furi
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AN ASYMMETRIC POINCARÉ INEQUALITY AND APPLICATIONS

Joint work with FRANCO OBERSNEL **and** SABRINA RIVETTI **(UNITS)**

Structure of this talk

This talk is divided into two parts:

1. we introduce an asymmetric version of the Poincaré inequality in the space of bounded variation functions

2. based on this result, we study the existence of bounded variation solutions of a class of capillarity problems with possibly asymmetric perturbations.

POINCARÉ INEQUALITIES

The classical Poincaré-Wirtinger inequality

Let Ω be a bounded domain in $\mathbb{R}^N$, with a Lipschitz boundary $\partial\Omega$.

The classical Poincaré-Wirtinger inequality in $BV(\Omega)$ asserts that there exists a constant $c > 0$ such that every $u \in BV(\Omega)$, with

$$\int_\Omega u\,dx = 0 \qquad \qquad \left(i.e. \quad r = \frac{\int_\Omega u^+\,dx}{\int_\Omega u^-\,dx} = 1, \text{ if } u \neq 0\right),$$

satisfies

$$c\int_\Omega |u|\,dx \leq \int_\Omega |Du|.$$

Recall that $u \in BV(\Omega)$ if $u \in L^1(\Omega)$ and its distributional gradient is a vector valued Radon measure with finite total variation

$$\int_\Omega |Dv| := \sup\left\{\int_\Omega v\,\mathrm{div}\,w\,dx : w \in C_0^1(\Omega, \mathbb{R}^N) \text{ and } \|w\|_{L^\infty(\Omega)} \leq 1\right\}.$$

The Poincaré constant

The largest constant $c = c(\Omega)$ for which the inequality

$$c \int_\Omega |u|\, dx \leq \int_\Omega |Du|$$

holds is called the Poincaré constant and is variationally characterized by

$$c = \inf \left\{ \int_\Omega |Dv| : v \in BV(\Omega), \int_\Omega v\, dx = 0, \int_\Omega |v|\, dx = 1 \right\}.$$

Clearly, all minimizers, if any, yield the equality in the Poincaré inequality.

Why $BV(\Omega)$ instead of $W^{1,1}(\Omega)$?

Elementary examples show that

$$\inf\left\{\int_\Omega |\nabla v|\,dx : v \in W^{1,1}(\Omega), \int_\Omega v\,dx = 0, \int_\Omega |v|\,dx = 1\right\}$$

is not attained in $W^{1,1}(\Omega)$;

whereas, we have

$$\begin{aligned}
&\inf\left\{\int_\Omega |Dv| : v \in BV(\Omega), \int_\Omega v\,dx = 0, \int_\Omega |v|\,dx = 1\right\}\\
&= \min\left\{\int_\Omega |Dv| : v \in BV(\Omega), \int_\Omega v\,dx = 0, \int_\Omega |v|\,dx = 1\right\}\\
&= \inf\left\{\int_\Omega |\nabla v|\,dx : v \in W^{1,1}(\Omega), \int_\Omega v\,dx = 0, \int_\Omega |v|\,dx = 1\right\}.
\end{aligned}$$

An asymmetric variant of the Poincaré inequality

Our aim is to discuss the validity of an asymmetric counterpart of the Poincaré inequality, where u^+ and u^- weigh differently, i.e.

$$r = \frac{\int_\Omega u^+\,dx}{\int_\Omega u^-\,dx} \neq 1.$$

Namely, we show that for each $r > 0$ there exist constants $\mu > 0$ and $\nu > 0$, with $\nu/\mu = r$,

such that every $u \in BV(\Omega)$, with

$$\mu \int_\Omega u^+\,dx - \nu \int_\Omega u^-\,dx = 0 \qquad \left(i.e. \quad \frac{\int_\Omega u^+\,dx}{\int_\Omega u^-\,dx} = r \right),$$

satisfies

$$\mu \int_\Omega u^+\,dx + \nu \int_\Omega u^-\,dx \leq \int_\Omega |Du|.$$

Variational characterization

For each $r > 0$ we define μ and ν through the variational formulas

$$\mu = \mu(r, \Omega) = \inf \Big\{ \int_\Omega |Dv| : v \in BV(\Omega), \int_\Omega v^+ dx - r \int_\Omega v^- dx = 0, \\ \int_\Omega v^+ dx + r \int_\Omega v^- dx = 1 \Big\}$$

and

$$\nu = \nu(r, \Omega) = \inf \Big\{ \int_\Omega |Dv| : v \in BV(\Omega), r^{-1} \int_\Omega v^+ dx - \int_\Omega v^- dx = 0, \\ r^{-1} \int_\Omega v^+ dx + \int_\Omega v^- dx = 1 \Big\}.$$

Needless to say that in this way we find the best constants for which the inequality holds.

Minimum properties

For each $r > 0$, we have

$$\mu(r) = \min\left\{ \int_\Omega |Dv| : v \in BV(\Omega), \int_\Omega v^+ \, dx = \tfrac{1}{2}, \int_\Omega v^- \, dx = \tfrac{1}{2r} \right\}$$

and

$$\nu(r) = \min\left\{ \int_\Omega |Dv| : v \in BV(\Omega), \int_\Omega v^+ \, dx = \tfrac{r}{2}, \int_\Omega v^- \, dx = \tfrac{1}{2} \right\}.$$

Moreover,

$$\mu(r), \nu(r) > 0$$

and

$$\nu(r) = r\mu(r).$$

The curve $\mathcal{C}$ and its properties.

We study the functions

$$r \mapsto \mu(r) \quad \text{and} \quad r \mapsto \nu(r),$$

and the plane curve

$$\mathcal{C} = \{(\mu(r), \nu(r)) : r \in \mathbb{R}_0^+\}.$$

Of course, by construction of $\mathcal{C}$, the following holds: if $(\mu, \nu) \in \mathcal{C}$, then every $v \in BV(\Omega)$, with

$$\mu \int_\Omega v^+ \, dx - \nu \int_\Omega v^- \, dx = 0,$$

satisfies

$$\mu \int_\Omega v^+ \, dx + \nu \int_\Omega v^- \, dx \le \int_\Omega |Dv|.$$

Symmetry

For each $r > 0$, we have $\mu(r^{-1}) = \nu(r)$;
hence $\mathcal{C}$ is symmetric with respect to the diagonal.

Continuity

The function $r \mapsto \mu(r)$ (and hence $r \mapsto \nu(r)$)
is both lower and upper semicontinuous (and thus continuous).

Monotonicity

The function $r \mapsto \mu(r)$ is strictly decreasing
(and the function $r \mapsto \nu(r)$ is strictly increasing).

Recall:

$$\mu(r) = \min\left\{ \int_\Omega |Dv| : v \in BV(\Omega), \int_\Omega v^+ \, dx = \tfrac{1}{2}, \int_\Omega v^- \, dx = \tfrac{1}{2r} \right\}.$$

Asymptotic behaviour as $r \to 0^+$

We have

$$\lim_{r \to 0^+} \mu(r) = +\infty \quad \text{(and hence} \quad \lim_{r \to +\infty} \nu(r) = +\infty).$$

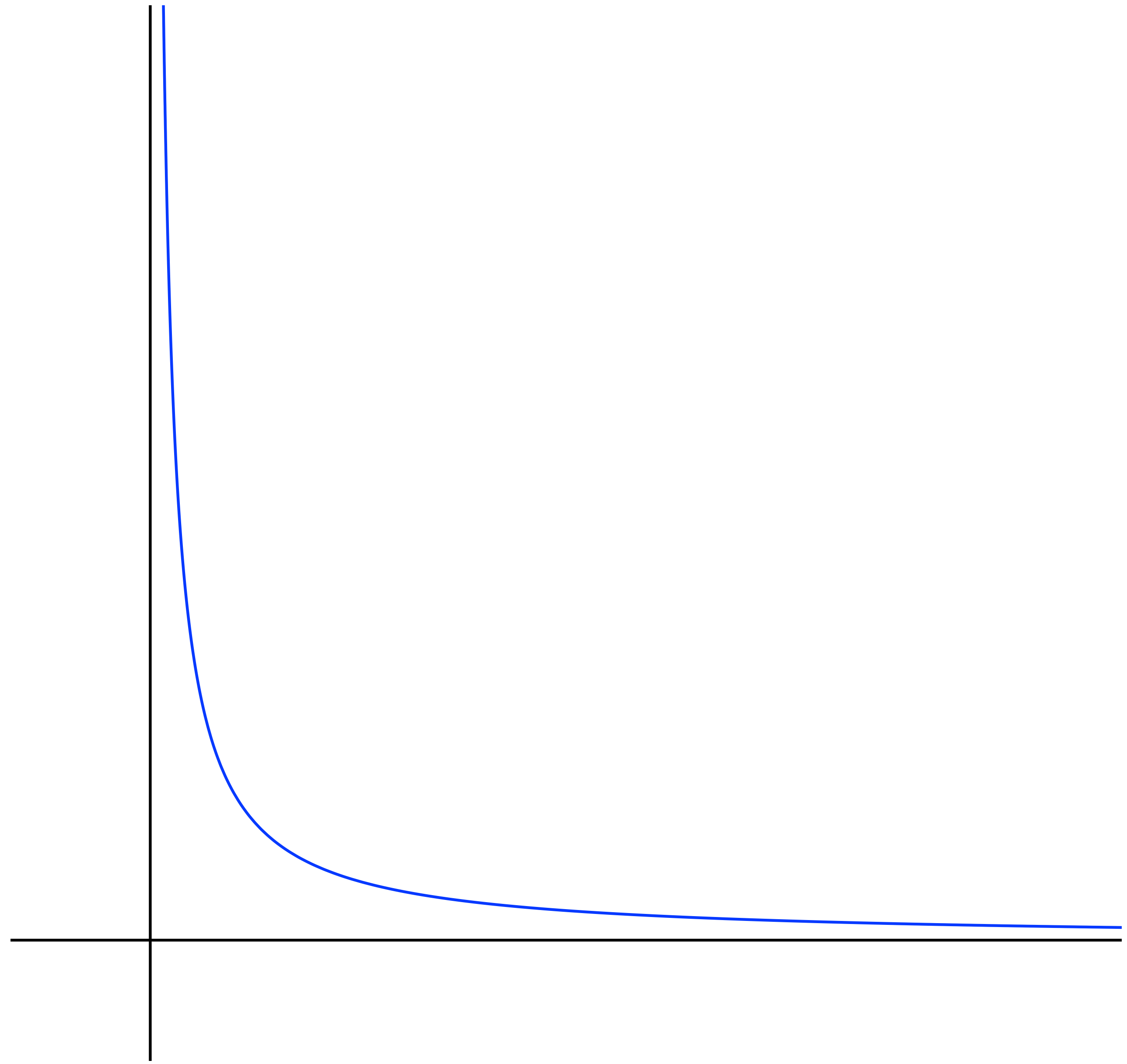
Asymptotic behaviour as $r \to +\infty$ in dimension $N \geq 2$

Assume $N \geq 2$. Then, we have

$$\lim_{r \to +\infty} \mu(r) = 0 \quad \text{(and hence} \quad \lim_{r \to 0^+} \nu(r) = 0).$$

Recall:

$$\mu(r) = \min \left\{ \int_\Omega |Dv| : v \in BV(\Omega), \int_\Omega v^+ \, dx = \tfrac{1}{2}, \int_\Omega v^- \, dx = \tfrac{1}{2r} \right\}.$$

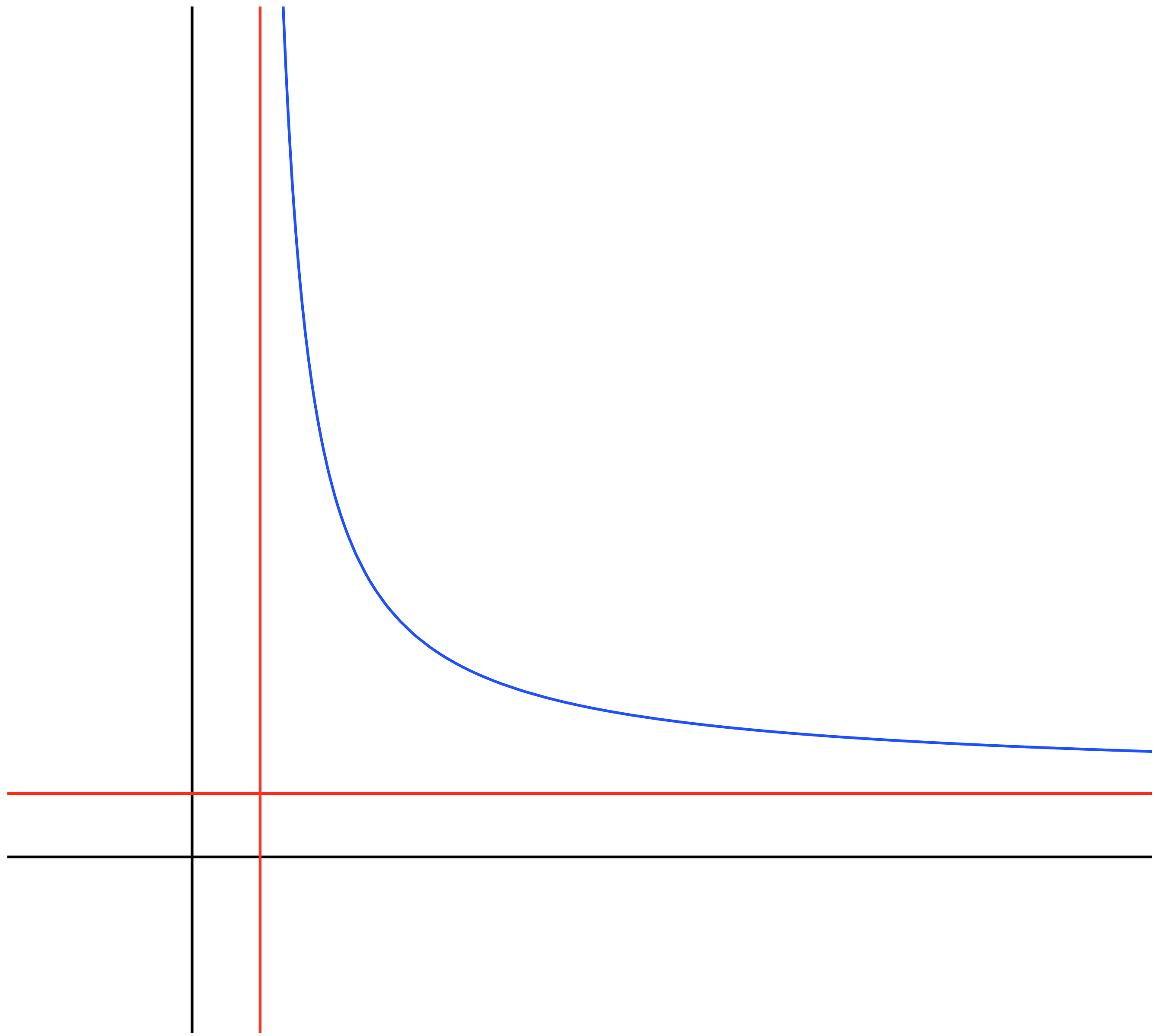


The curve $\mathcal{C}$ in dimension $N \geq 2$

Asymptotic behaviour as $r \to +\infty$ in dimension $N = 1$

Assume $N = 1$ and let $\Omega =]0, T[$. Then, we have

$$\lim_{r \to +\infty} \mu(r) > 0 \quad \text{(and hence} \quad \lim_{r \to 0^+} \nu(r) > 0).$$



The curve $\mathcal{C}$ in dimension $N = 1$

Asymptotic behaviour as $r \to +\infty$ in dimension $N = 1$

Assume $N = 1$ and let $\Omega =]0, T[$. Then, we have

$$\lim_{r\to+\infty} \mu(r) > 0 \quad \text{(and hence} \quad \lim_{r\to 0^+} \nu(r) > 0).$$

This follows from

$$\mu(r) = \min\left\{ \int_{]0,T[} |Dv| : v \in BV(]0,T[), \int_{]0,T[} v^+ \, dx = \tfrac{1}{2}, \int_{]0,T[} v^- \, dx = \tfrac{1}{2r} \right\}.$$

and

$$\operatorname*{ess\,sup}_{]0,T[} v - \operatorname*{ess\,inf}_{]0,T[} v \le \int_{]0,T[} |Dv|, \qquad \forall v \in BV(0,T).$$

However we can deduce this fact from a more precise description of $\mathcal{C}$ in case $N = 1$, which also provides the explicit value of the limit.

Explicit description of $\mathcal{C}$ in dimension $N = 1$

Assume $N = 1$ and let $\Omega =]0, T[$. Then, we have

$$\mathcal{C} = \left\{ (\mu, \nu) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+ : \frac{1}{\sqrt{\mu}} + \frac{1}{\sqrt{\nu}} = \sqrt{2T} \right\}.$$

In particular,

$$\left(\frac{2}{T}, \frac{2}{T} \right) \in \mathcal{C},$$

with $\frac{2}{T}$ the second eigenvalue c_2 of the Neumann 1-Laplacian in $]0, T[$ as defined in [Chang, 2009],

and

$$\mathcal{C} \text{ is asymptotic to the lines } \mu = \frac{1}{2T} \text{ and } \nu = \frac{1}{2T}.$$

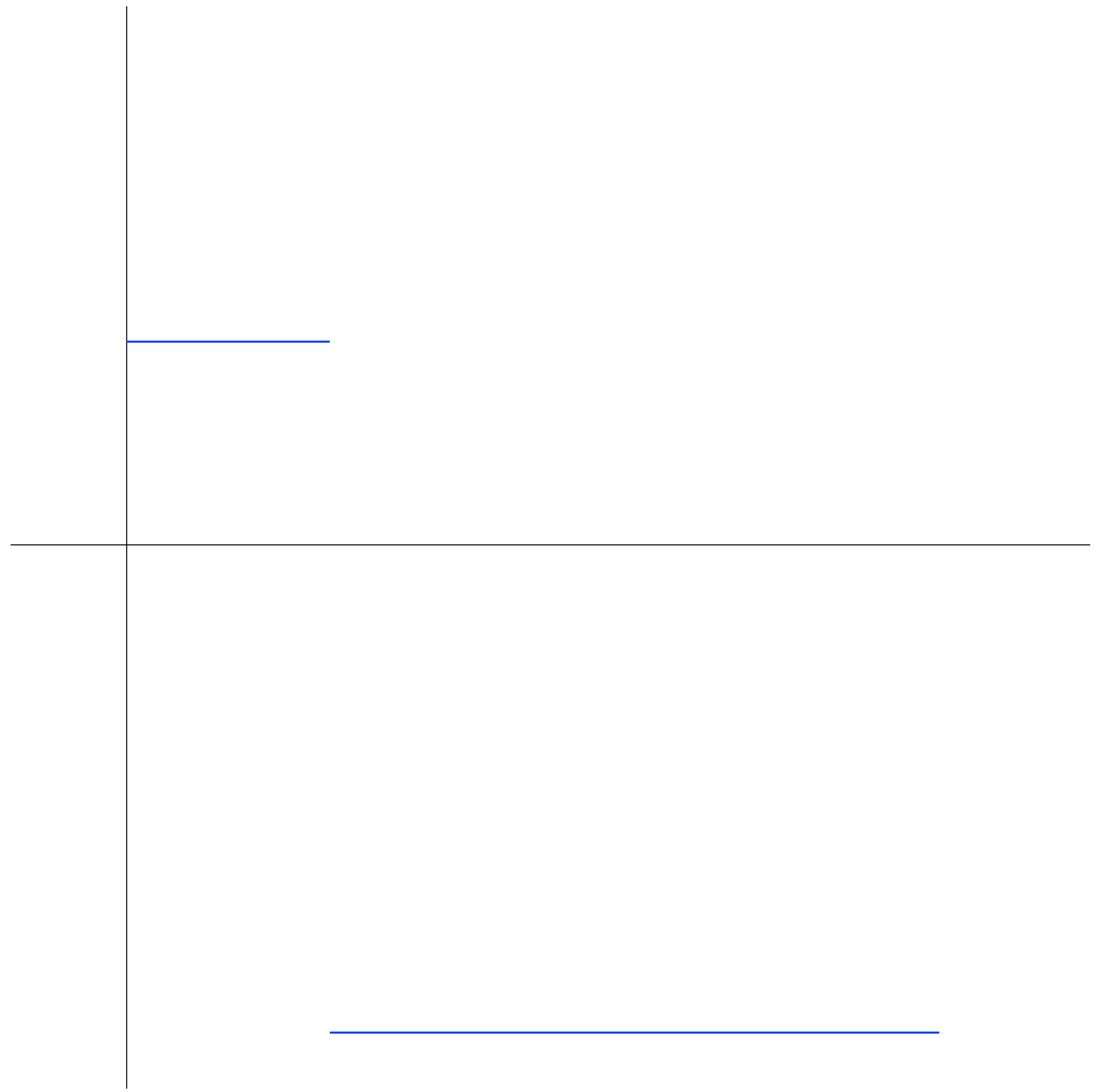
Moreover, for any given $(\mu, \nu) \in \mathcal{C}$, a function $u \in BV(0,T)$ satisfies

$$\mu \int_0^T u^+ \, dx - \nu \int_0^T u^- \, dx = 0 \quad \text{and} \quad \mu \int_0^T u^+ \, dx + \nu \int_0^T u^- \, dx = \int_{]0,T[} |Du|$$

if and only if u is a positive multiple

either of

$$\varphi(x) = \begin{cases} \dfrac{1}{T}\dfrac{1}{2\mu}\dfrac{\sqrt{\mu}+\sqrt{\nu}}{\sqrt{\nu}} & \text{if } 0 < x < \dfrac{\sqrt{\nu}}{\sqrt{\mu}+\sqrt{\nu}}T, \\ \\ -\dfrac{1}{T}\dfrac{1}{2\nu}\dfrac{\sqrt{\mu}+\sqrt{\nu}}{\sqrt{\mu}} & \text{if } \dfrac{\sqrt{\nu}}{\sqrt{\mu}+\sqrt{\nu}}T \le x < T. \end{cases}$$



The function φ

Moreover, for any given $(\mu, \nu) \in \mathcal{C}$, a function $u \in BV(0,T)$ satisfies

$$\mu \int_0^T u^+ \, dx - \nu \int_0^T u^- \, dx = 0 \quad \text{and} \quad \mu \int_0^T u^+ \, dx + \nu \int_0^T u^- \, dx = \int_{]0,T[} |Du|$$

if and only if u is a positive multiple

either of

$$\varphi(x) = \begin{cases} \dfrac{1}{T}\dfrac{1}{2\mu}\dfrac{\sqrt{\mu}+\sqrt{\nu}}{\sqrt{\nu}} & \text{if } 0 < x < \dfrac{\sqrt{\nu}}{\sqrt{\mu}+\sqrt{\nu}}T, \\ \\ -\dfrac{1}{T}\dfrac{1}{2\nu}\dfrac{\sqrt{\mu}+\sqrt{\nu}}{\sqrt{\mu}} & \text{if } \dfrac{\sqrt{\nu}}{\sqrt{\mu}+\sqrt{\nu}}T \le x < T. \end{cases}$$

or of

$$\varphi(T-x).$$

Sketch of proof.

The proof is based on a rearrangement technique:

1. we prove the validity of the asymmetric Poincaré inequality for decreasing functions whenever $\mu, \nu \in \mathbb{R}_0^+$ satisfy $\frac{1}{\sqrt{\mu}} + \frac{1}{\sqrt{\nu}} \geq \sqrt{2T}$

2. by exploiting some properties of decreasing rearrangements (area invariance, Polya-Szegö inequality), we extend the validity of the asymmetric Poincaré inequality to bounded variation functions

3. by using again the properties of decreasing rearrangements and the coarea formula, we characterize the functions yielding equality in the asymmetric Poincaré inequality

4. we show that if $\rho, \sigma \in \mathbb{R}_0^+$ satisfy $\frac{1}{\sqrt{\rho}} + \frac{1}{\sqrt{\sigma}} = \sqrt{2T}$, then $\rho = \mu(r)$, $\sigma = \nu(r)$ with $r = \frac{\sigma}{\rho}$, i.e. $(\rho, \sigma) \in \mathcal{C}$, and viceversa.

SOLVABILITY OF CAPILLARITY PROBLEMS

We turn to the study of the capillarity-type problem

$$\begin{cases} -\operatorname{div}\left(\nabla u / \sqrt{1+|\nabla u|^2}\right) = f(x,u) & \text{in } \Omega, \\ -\nabla u \cdot n / \sqrt{1+|\nabla u|^2} = \kappa(x) & \text{on } \partial\Omega. \end{cases}$$

We are going to present some statements concerning non-existence, existence and multiplicity of solutions in the space of bounded variation functions.

Our main aim is to study the case where the no convexity assumption is imposed on the associated action functional and solutions are not necessarily minimizers: this will be achieved by using the asymmetric variant of the Poincaré inequality we previously established and some tools of non-smooth critical point theory.

Here for simplicity we will restrict ourselves to the discussion of the case of homogeneous conormal boundary conditions, i.e. $\kappa = 0$.

Hereafter we assume that

(CAR) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ *satisfies the Carathéodory conditions* and

(SGC) *there exist constants* $a > 0$ *and* $q \in \,]1, 1^*[$ *and a function* $b \in L^p(\Omega)$, *with* $p > N$, *such that*

$$|f(x,s)| \leq a|s|^{q-1} + b(x)$$

for a.e. $x \in \Omega$ *and every* $s \in \mathbb{R}$.

$$\left(N \geq 2: \quad 1^* = \tfrac{N}{N-1}; \qquad N = 1: \quad 1^* = \infty \right)$$

We set

$$F(x,s) = \int_0^s f(x,\xi)\, d\xi.$$

The area functional

We define the area functional $\mathcal{J} : BV(\Omega) \to \mathbb{R}$ by

$$\mathcal{J}(v) = \int_\Omega \sqrt{1 + |Dv|^2} = \int_\Omega \sqrt{1 + |(Dv)^a|^2}\, dx + \int_\Omega |Dv|^s.$$

Here and in the sequel $m = m^a dx + m^s$ is the decomposition of any Borel measure m in its absolutely continuous and singular parts with respect to the N-dimensional Lebesgue measure.

The potential functional

We also introduce the potential functional

$$\mathcal{F} : BV(\Omega) \to \mathbb{R} \quad \text{defined by} \quad \mathcal{F}(v) = \int_\Omega F(x, v)\, dx.$$

The action functional

We define the functional $\mathcal{I} : BV(\Omega) \to \mathbb{R}$ by

$$\mathcal{I}(v) \ = \ \mathcal{J}(v) - \int_\Omega F(x,v)\,dx \ = \ \mathcal{J}(v) - \mathcal{F}(v).$$

The area functional $\mathcal{J} : BV(\Omega) \to \mathbb{R}$ is convex and (Lipschitz) continuous and the potential functional $\mathcal{F} : BV(\Omega) \to \mathbb{R}$ is C^1.

Definition of solution

A function $u \in BV(\Omega)$ is a solution of problem (P) if $\mathcal{F}'(u)$ is a subgradient at u of the functional $\mathcal{J}$,

i.e. u satisfies the variational inequality

$$\mathcal{J}(v) - \mathcal{J}(u) \geq \int_\Omega f(x,u)(v-u)\,dx,$$

for every $v \in BV(\Omega)$.

Remark 1. u is a solution of (P) if and only if u is a global minimizer in $BV(\Omega)$ of the functional $\mathcal{K}_u : BV(\Omega) \to \mathbb{R}$ defined by

$$\mathcal{K}_u(v) = \mathcal{J}(v) - \mathcal{F}'(u)v = \mathcal{J}(v) - \int_\Omega f(x,u)v\,dx.$$

Remark 2. $u \in BV(\Omega)$ satisfies the previous variational inequality, for every $v \in BV(\Omega)$, if and only if u satisfies the Euler equation

$$\int_\Omega \frac{(Du)^a\,(D\phi)^a}{\sqrt{1+|(Du)^a|^2}}\,dx + \int_\Omega S\left(\frac{Du}{|Du|}\right)\frac{D\phi}{|D\phi|}\,|D\phi|^s = \int_\Omega f(x,u)\phi\,dx$$

for every $\phi \in BV(\Omega)$ such that $|D\phi|^s$ is absolutely continuous with respect to $|Du|^s$.

Here S is the projection over S^{N-1}:

$$S(\xi) = |\xi|^{-1}\xi \quad \text{if } \xi \in \mathbb{R}^N \setminus \{0\} \quad \text{and} \quad S(\xi) = 0 \quad \text{if } \xi = 0.$$

Lower semicontinuity

The action functional $\mathcal{I} : BV(\Omega) \to \mathbb{R}$ is lower semicontinuous with respect to the L^q-convergence in $BV(\Omega)$, with $1 < q < 1^*$,

i.e. if $(v_n)_n$ is a sequence in $BV(\Omega)$ converging in $L^q(\Omega)$ to a function $v \in BV(\Omega)$, then

$$\mathcal{I}(v) \le \liminf_{n \to +\infty} \mathcal{I}(v_n).$$

A continuous projector

Fix $\mu, \nu \in \mathbb{R}_0^+$. For each $v \in L^1(\Omega)$ there exists a unique $\mathcal{P}(v) \in \mathbb{R}$ such that

$$\mu \int_\Omega (v - \mathcal{P}(v))^+ \, dx - \nu \int_\Omega (v - \mathcal{P}(v))^- \, dx = 0.$$

The map $\mathcal{P} : L^1(\Omega) \to \mathbb{R}$ such that $v \mapsto \mathcal{P}(v)$ is idempotent and continuous.

A coercivity property over cones

Assume that

(NIC) *there exists* $(\mu, \nu) \in \mathcal{C}$ *such that*

$$\operatorname*{ess\,sup}_{\Omega\times\mathbb{R}} f(x,s) < \mu \quad \textit{and} \quad \operatorname*{ess\,inf}_{\Omega\times\mathbb{R}} f(x,s) > -\nu.$$

Define the cone

$$\mathcal{W} = \mathcal{N}(\mathcal{P}) = \left\{ w \in BV(\Omega) : \mu \int_\Omega w^+ \, dx - \nu \int_\Omega w^- \, dx = 0 \right\}.$$

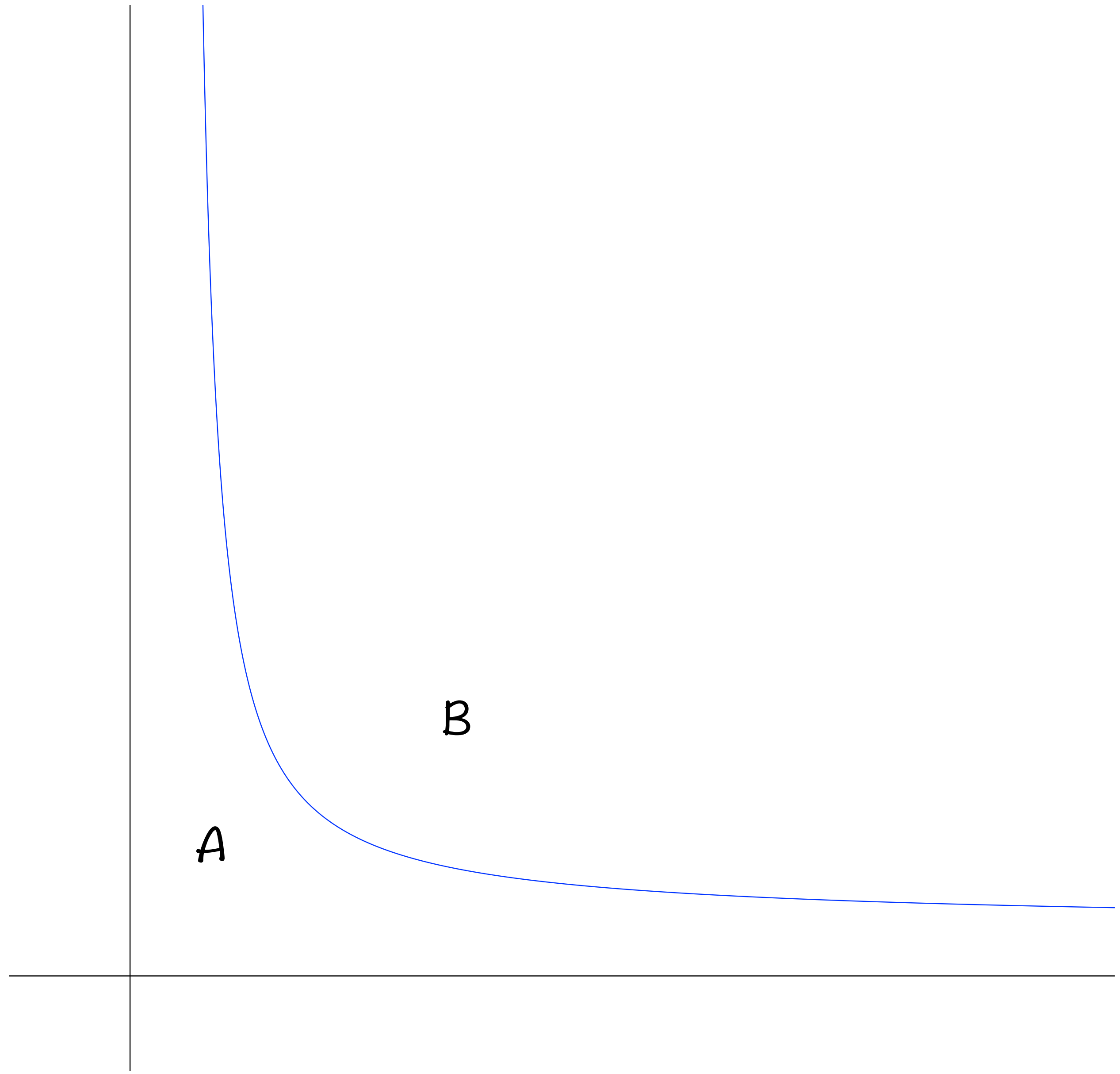
Then there exists $\eta > 0$ such that

$$\mathcal{I}(w + r) \geq \eta \int_\Omega |Dw| - \int_\Omega F(x, r) \, dx$$

for every $r \in \mathbb{R}$ and $w \in \mathcal{W}$.

This follows from the asymmetric Poincaré inequality.

Two open subsets of $\mathbb{R}_0^+ \times \mathbb{R}_0^+$



Existence vs non-existence

Here we write f in the form

$$f(x,s) = g(x,s) + e(x).$$

The following simple result shows that the existence of solutions is guaranteed when $g = 0$ and assuming that e lies, in a suitable sense, "below" the curve $\mathcal{C}$.

Existence in case $g = 0$. Let $e \in L^\infty(\Omega)$ satisfy

$$\int_\Omega e\,dx = 0 \qquad \text{and} \qquad \left(\operatorname*{ess\,sup}_\Omega e, -\operatorname*{ess\,inf}_\Omega e\right) \in \mathcal{A}.$$

Then the problem

$$\begin{cases} -\mathrm{div}\left(\nabla u / \sqrt{1+|\nabla u|^2}\right) = e(x) & \text{in } \Omega, \\ -\nabla u \cdot n / \sqrt{1+|\nabla u|^2} = 0 & \text{on } \partial\Omega \end{cases}$$

has a solution $w \in \mathcal{W}$ (i.e. such that $\mathcal{P}(w) = 0$).

Sketch of proof.
Split $v \in BV(\Omega)$ as

$$v = w + \mathcal{P}(v),$$

with $\mathcal{P}$ the projector defined above and with

$$w \in \mathcal{W} = \Big\{ w \in BV(\Omega) : \mu \int_\Omega w^+ \, dx - \nu \int_\Omega w^- \, dx = 0 \Big\}.$$

We have

$$\mathcal{I}(v) = \mathcal{I}(w) = \int_\Omega \sqrt{1 + |Dw|^2} - \int_\Omega ew \, dx.$$

The coercivity result over $\mathcal{W}$ implies that $\mathcal{I}$ is coercive on $\mathcal{W}$ and bounded from below on $BV(\Omega)$.
If $(v_n)_n$ is a minimizing sequence, then $(w_n)_n$ is a minimizing sequence too.
The coercivity result over $\mathcal{W}$ implies that $(w_n)_n$ is bounded in $BV(\Omega)$ and hence it has a subsequence converging in $L^1(\Omega)$ to some $w \in \mathcal{W}$.
The lower semicontinuity of $\mathcal{I}$ implies that w is a minimizer and therefore it is a solution.

REMARK. The condition $\int_\Omega e\,dx = 0$ is necessary for the solvability, but also (NIC), i.e. $(\operatorname{ess\,sup}_\Omega e, -\operatorname{ess\,inf}_\Omega e) \in \mathcal{A}$, cannot be dropped in general.

We show that the existence of solutions is not guaranteed if e lies, in some sense, "above" the curve $\mathcal{C}$.

NON-EXISTENCE IN CASE $g = 0$. *There exist functions $e \in L^\infty(\Omega)$, with*

$$\int_\Omega e\,dx = 0 \qquad \textit{and} \qquad (\operatorname*{ess\,sup}_\Omega e, -\operatorname*{ess\,inf}_\Omega e) \in \mathcal{B},$$

such that the problem

$$\begin{cases} -\operatorname{div}\left(\nabla u/\sqrt{1+|\nabla u|^2}\right) = e(x) & \textit{in } \Omega, \\ -\nabla u \cdot n/\sqrt{1+|\nabla u|^2} = 0 & \textit{on } \partial\Omega \end{cases}$$

has no solution.

REMARK. The conclusion is stable under perturbation.

NON-EXISTENCE IN CASE $g \neq 0$. *There exist functions* $e \in L^\infty(\Omega)$,

$$\textit{with} \qquad \int_\Omega e\,dx = 0 \qquad \textit{and} \qquad (\operatorname*{ess\,sup}_\Omega e, -\operatorname*{ess\,inf}_\Omega e) \in \mathcal{B},$$

and constants $\gamma > 0$ *such that, for any function* $g : \Omega \times \mathbb{R} \to \mathbb{R}$ *satisfying (CAR) and* $\operatorname*{ess\,sup}_{\Omega\times\mathbb{R}} |g(x,s)| \le \gamma$,

such that the problem

$$\begin{cases} -\operatorname{div}\left(\nabla u/\sqrt{1+|\nabla u|^2}\right) = g(x,u) + e(x) & \textit{in } \Omega, \\ -\nabla u \cdot n/\sqrt{1+|\nabla u|^2} = 0 & \textit{on } \partial\Omega \end{cases}$$

has no solution.

Sketch of proof.

1. Construction of e.

Fix $(\mu, \nu) \in \mathcal{C}$ and let $\varphi \in BV(\Omega) \setminus \{0\}$ be a function attaining equality in the Poincaré inequality, i.e. such that

$$\mu \int_\Omega \varphi^+ dx - \nu \int_\Omega \varphi^- dx = 0 \quad \text{and} \quad \int_\Omega |D\varphi| = \mu \int_\Omega \varphi^+ dx + \nu \int_\Omega \varphi^- dx.$$

Pick $\rho, \sigma \in \mathbb{R}_0^+$ such that

$$\sigma\,|\mathsf{supp}(\varphi^+)| = \rho\,|\mathsf{supp}(\varphi^-)|$$

and e.g.

$$\rho > \mu \quad \text{and} \quad \sigma \geq \nu.$$

Define $e \in L^\infty(\Omega)$ by

$$e = \rho\,\chi_{\mathsf{supp}(\varphi^+)} - \sigma\,\chi_{\mathsf{supp}(\varphi^-)}$$

We have

$$\int_\Omega e\,dx = 0 \qquad \text{and} \qquad (\operatorname*{ess\,sup}_\Omega e, -\operatorname*{ess\,inf}_\Omega e) \in \mathcal{B}.$$

2. Non-existence of solutions.

Fix any $u \in BV(\Omega)$.

Compute, for $t \in \mathbb{R}_0^+$,

$$\begin{aligned}\mathcal{K}_u(t\varphi) = \mathcal{J}(t\varphi) - &\int_\Omega (g(x,u)+e)t\varphi\,dx \\ &\le |\Omega| - k\Big((\rho-\mu-\gamma)\int_\Omega \varphi^+\,dx + (\sigma-\nu-\gamma)\int_\Omega \varphi^-\,dx\Big).\end{aligned}$$

Take $\gamma > 0$ so small that

$$(\rho-\mu-\gamma)\int_\Omega \varphi^+\,dx + (\sigma-\nu-\gamma)\int_\Omega \varphi^-\,dx > 0.$$

We infer that

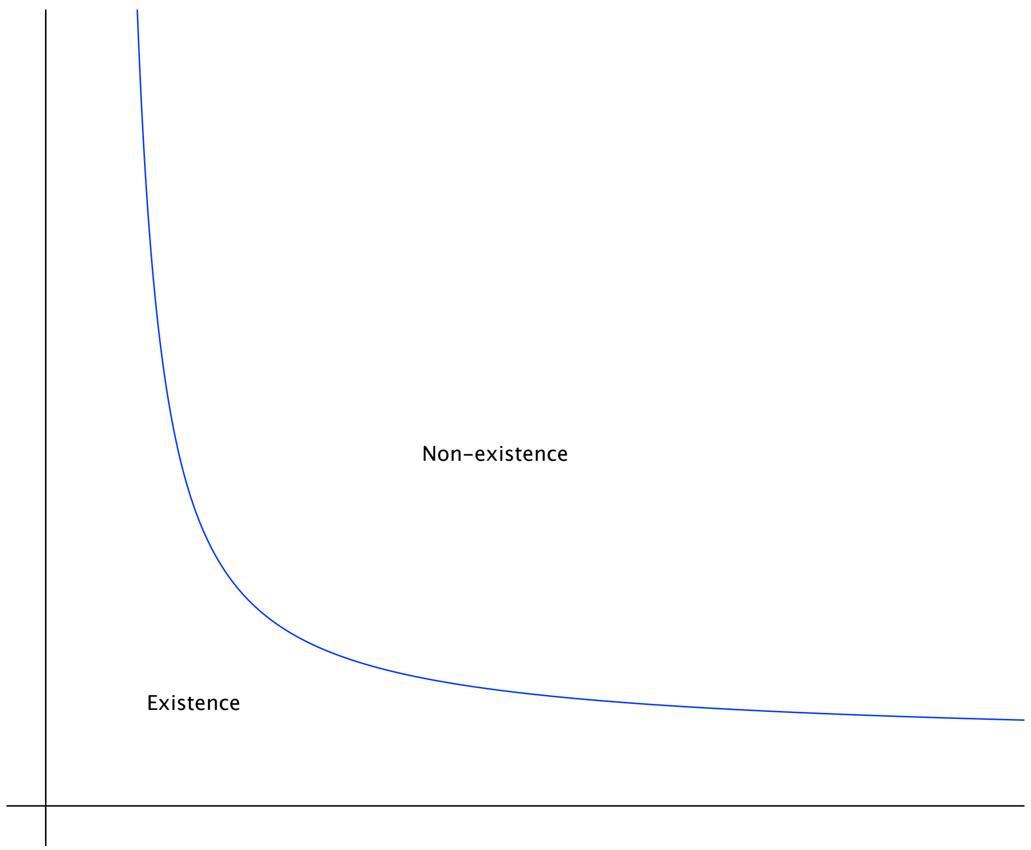
$$\inf_{v \in BV(\Omega)} \mathcal{K}_u(v) = -\infty.$$

Therefore u is not a solution of the problem

$$\begin{cases} -\mathrm{div}\Big(\nabla u/\sqrt{1+|\nabla u|^2}\Big) = g(x,u)+e(x) & \text{in } \Omega, \\ -\nabla u \cdot n/\sqrt{1+|\nabla u|^2} = 0 & \text{on } \partial\Omega. \end{cases}$$

Conclusions

$$
\begin{cases}
-\operatorname{div}\left(\nabla u/\sqrt{1+|\nabla u|^2}\right) = e(x) & \text{in } \Omega, \\
-\nabla u \cdot n/\sqrt{1+|\nabla u|^2} = 0 & \text{on } \partial\Omega
\end{cases}
$$



- Non-existence is stable under small perturbations.
- Existence is not stable under perturbations.
- We keep existence if we assume some structure on the perturbation.

EXISTENCE RESULTS

Let us consider the problem

$$(P) \qquad \begin{cases} -\mathrm{div}\left(\nabla u / \sqrt{1+|\nabla u|^2}\right) = f(x,u) & \text{in } \Omega, \\ -\nabla u \cdot n / \sqrt{1+|\nabla u|^2} = 0 & \text{on } \partial\Omega. \end{cases}$$

A necessary condition in order a solution u exists is that

$$\int_\Omega f(x,u)\,dx = 0.$$

This implies that, if non-zero, f must change sign in $\Omega \times \mathbb{R}$:

we are going to assume some hypotheses which imply this condition and also yield some nice geometry of the action functional.

Coercivity of the averaged primitive

THEOREM. *Assume*

(NIC) *there exists* $(\mu, \nu) \in \mathcal{C}$ *such that*

$$\operatorname*{ess\,sup}_{\Omega\times\mathbb{R}} f(x,s) < \mu \quad \textit{and} \quad \operatorname*{ess\,inf}_{\Omega\times\mathbb{R}} f(x,s) \geq -\nu$$

and

(ALP+) $$\lim_{s\to\pm\infty} \int_\Omega F(x,s)\,dx = +\infty.$$

Then problem (P) has at least one solution.

The solution is obtained by a minimax procedure based on a version of the MPL in $BV(\Omega)$ for non-differentiable functional.

Technically, the failure of the Palais-Smale condition in BV require some delicate estimates in order to prove the convergence of a sequence of almost sub-critical points to a subcritical point.

Anticoercivity of the averaged primitive

THEOREM *Assume*

(NIC) *there exists* $(\mu, \nu) \in \mathcal{C}$ *such that*

$$\operatorname*{ess\,sup}_{\Omega\times\mathbb{R}} f(x,s) < \mu \quad \textit{and} \quad \operatorname*{ess\,inf}_{\Omega\times\mathbb{R}} f(x,s) > -\nu$$

and

(ALP-)
$$\lim_{s\to\pm\infty} \int_\Omega F(x,s)\,dx = -\infty.$$

Then problem (P) has at least one solution.

The solution is found by minimization: conditions (ALP-) and (NIC) imply that $\mathcal{I}$ is coercive and bounded from below.

In the light of the previous non-existence results, (NIC) cannot be omitted.

One-sided conditions

In dimension $N = 1$ the two-sided non-interference condition (NIC) can be replaced by one-sided conditions, which allow f to be unbounded from above or from below. This peculiarity is related to the asymptotic behaviour of the curve $\mathcal{C}$ which differs in the case $N = 1$ from the case $N \geq 2$.

THEOREM *Suppose* $N = 1$ *and let* $\Omega =]0, T[$.

Assume

(OSC) $\quad \operatorname*{ess\,inf}_{]0,T[\times\mathbb{R}} f(x,s) > -\frac{1}{2T} \quad$ *or* $\quad \operatorname*{ess\,sup}_{]0,T[\times\mathbb{R}} f(x,s) < \frac{1}{2T}$.

Suppose that (ALP+) or (ALP-) holds.

Then problem (P) has at least one solution.

MULTIPLICITY RESULTS

We finally discuss the existence of multiple solutions.

Under (NIC) the multiplicity of solutions can be proved, whenever the averaged primitive

$$s \mapsto \int_\Omega F(x,s)\,dx$$

exhibits an oscillatory behaviour at infinity, like, e.g.,

$$\limsup_{s\to\pm\infty} \int_\Omega F(x,s)\,dx = +\infty$$

and

$$\liminf_{s\to\pm\infty} \int_\Omega F(x,s)\,dx = -\infty.$$

An Infinite Multiplicity Result

Assume (NIC) and

(AOSC) $$\limsup_{s\to\pm\infty}\int_\Omega F(x,s)\,dx > -\infty \quad \textit{and} \quad \liminf_{s\to\pm\infty}\int_\Omega F(x,s)\,dx = -\infty.$$

Then problem (P) has three sequences $(u_n^{(1)})_n$, $(u_n^{(2)})_n$ *and* $(u_n^{(3)})_n$ *of solutions such that*

$$\lim_{n\to+\infty}\mathcal{I}(u_n^{(1)}) = +\infty, \quad \limsup_{n\to+\infty}\mathcal{I}(u_n^{(2)}) < +\infty, \quad \limsup_{n\to+\infty}\mathcal{I}(u_n^{(3)}) < +\infty$$

and

$$\lim_{n\to+\infty}\mathcal{P}(u_n^{(2)}) = +\infty \quad \textit{and} \quad \lim_{n\to+\infty}\mathcal{P}(u_n^{(3)}) = -\infty.$$

Here the previously cited MPL is used in its full power both to prove the existence of solutions and to localize them: to prove localization a careful study of the behaviour of a sequence of “minimizing” paths is needed.

Some questions

- Even though the functional and the constraints are not smooth, do the minimizers in the Poincaré inequality satisfy any Euler equation (or inclusion)?
- Which are the relations, in dimension $N \geq 2$, of $\mathcal{C}$ with the second eigenvalue of the 1-Laplace operator, defined using Lusternik-Schnirelmann theory?
- Does any antimaximum principle hold for the 1-Laplacian, possibly with reference to the asymptotic behaviour of $\mathcal{C}$?
- What about regularity of solutions, which are not minimizers? If u is a non-regular solution, what about the singular part of Du? Is u SBV?
- How to extend these results to the Dirichlet problem?

Thanks for your attention!

CONGRATULAZIONI,

MASSIMO!