Bifurcation of critical points for continuous families of functionals of Fredholm type

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Firenze 3-4 giugno 2014

Overview

I will review a joint paper with Nils Waterstraat which improves in several ways an old result about the relation of the spectral flow with the variational bifurcation and will discuss some of its applications in problems where strongly indefinite functionals arise.

Our main goal here is to use topological invariants in order to obtain bifurcation criteria directly in terms of the coefficients of the linearised equation at the trivial branch.

CONTENTS:

- Morse index and variational bifurcation
- Spectral flow and its properties
- The main bifurcation Thm.
- The several parameter case
- A comparison principle
- Bifurcation of periodic orbits of Hamiltonian systems.

Morse index and bifurcation

A general principle of variational bifurcation states:

The change in Morse index along a trivial branch of critical points entails bifurcation of critical points from the branch.

More precisely:

- If $\psi\colon [a,b] \times U \to \mathbb{R}$ is C^2
- 0 is a critical point of $\psi_\lambda = \psi(\lambda, -)$ for all $\lambda \in [a,b]$
- The Hessian $H\psi_\lambda(0)$ is essentially positive.
- 0 is a non degenerate critical point of $\psi_\lambda,\ \lambda = a, b$.

Then bifurcation of critical points of ψ_λ arise in (a,b) whenever

$$\Delta m = m(H\psi_a, 0) - m(H\psi_b, 0) \neq 0.$$

Recall that a point $\lambda \in [a,b]$ is a point of bifurcation from the trivial branch of critical points of the family ψ, if every neighborhood of $(\lambda, 0) \in [a,b] \times U$ contains a solution (λ, x) of the equation $\nabla\psi_\lambda(x) = 0$ with $x \neq 0$.

Spectral flow

Given a path $L_\lambda;\ \lambda \in [a,b]$ of self–adjoint Fredholm operators
A natural substitute to Δm which allows to extend the above result to the case of strongly indefinite functionals is the *spectral flow*.

> *" Given a path $L_\lambda;\ \lambda \in [a,b]$ of self–adjoint Fredholm operators, invertible at the end points, the spectral flow $sf(L)$ is the number of negative eigenvalues of L_a that become positive as λ goes from a to b minus the number of positive eigenvalues that become negative."*

- $sf(L) = \Delta m$ whenever the Morse index is defined.
- $sf(L)$ is homotopy invariant.
- $sf(L)$ is additive under paths concatenation.
- $sf(L) = m_{rel}(L_a, L_b)$ if $L_\lambda - L_a$ is a compact operator.

A geometric definition

Let X be separable Hilbert space. Let $\Phi_{SA} = SA(X) \cap \Phi(X)$.
The set $\Sigma_k = \{T \in \Phi_{SA} / \dim \ker T = k\}$ is a submanifold of Φ_{SA} of codimension k^2.
Given a path $L \colon [a, b] \to \Phi_{SA}$ with invertible end points, its spectral flow is the *intersection number* of the path L with the stratified set

$$\Sigma = \cup_{k \geq 1} \Sigma_k.$$

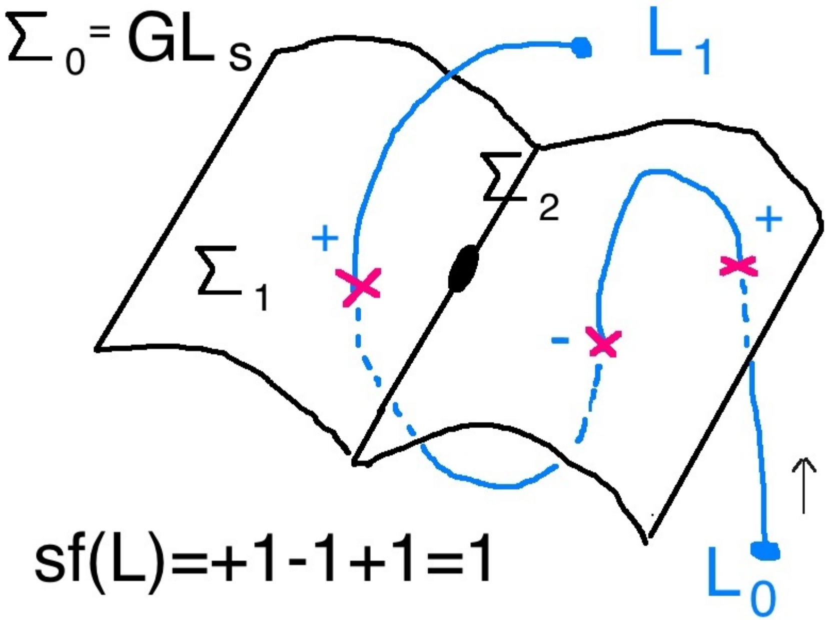
of singular self–adjoint Fredholm operators.

If the path L is transversal to all of Σ_k, then

$$sf(L) = \sum_{\lambda \in L^{-1}(\Sigma_1)} \operatorname{sgn} \dot{\mu}_1(\lambda)$$

This extends to continuous paths by approximation with smooth transversal paths.

Illustration 1



The Bifurcation Theorem.

Theorem (1)

Let U be a neighborhood of 0 in a separable Hilbert space X and let $\psi\colon [a,b]\times U \to \mathbb{R}$ be a continuous family of C^2 functionals. Assume that, for all $\lambda \in [a,b]$, $\nabla\psi_\lambda(0) = 0$ and the Hessian L_λ of ψ_λ at 0 is Fredholm with L_a and L_b being invertible. Then:

i) *If $sf(L) \neq 0$, the interval (a,b) contains at least one point of bifurcation of critical points of ψ_λ from the trivial branch.*

ii) *If $\Sigma(L) = L^{-1}(\Sigma) = \{\lambda \in [a,b] \mid \dim\ker L_\lambda \geq 1\}$ is a finite subset of (a,b) then the family ψ possesses at least $|sf(L)|/d(L)$ bifurcation points in (a,b), where*

$$d(L) = max\{\dim\ker L_\lambda : \lambda \in [a,b]\}$$

is the order of degeneracy of the path L.

Remarks

The hypothesis in *ii*) is verified if either the path L of Hessians is real analytic or it is differentiable and has only regular crossing.

A regular crossing is a point $\lambda \in \Sigma(L)$ *at which the crossing form* $\mathcal{Q}(\lambda)h = \langle \dot{L}_\lambda h, h \rangle; \; h \in \ker L_\lambda$, *is non-degenerate. In this case*

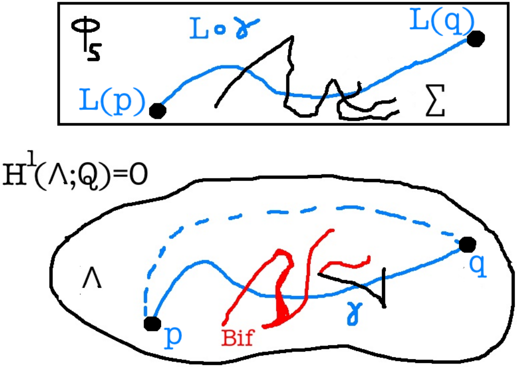
$$sf(L, I) = \sum_{\lambda \in \Sigma(L)} \operatorname{sig} \mathcal{Q}(\lambda).$$

L has regular crossings if the quadratic form $\mathcal{Q}(\lambda)$ is either always positive definite or negative definite (e.g., if $\dot{L}$ is either positive definite or negative definite). In this case the spectral flow is $\pm$ the sum of the dimensions of the kernels.

If K is a compact, self–adjoint operator, then $L_\lambda = \mathrm{Id} - \lambda K$ is a path as above. Thus, if $\nabla\phi$ is compact, $\nabla\phi(0) = 0$, then any characteristic value of $K = D\nabla(\phi)(0)$ is a bifurcation point for solutions for the equation $x - \lambda\nabla\phi(x) = 0$.

Multiparameter bifurcation

$\psi : \Lambda \times U \to \mathbb{R}$ = a continuous family of Fredholm C^2 functionals $\nabla\psi_\lambda(0) = 0$, for all $\lambda \in \Lambda$. $L = D\nabla\psi_\lambda(0)$ = the family of Hessians at 0. $\Sigma(L) = L^{-1}(\Sigma)$. $Bif = B(\psi)$ = the set of bifurcation points.



$$sf(L \circ \gamma) \neq 0 \implies Bif \text{ disconnects } \Lambda$$

Multiparameter bifurcation Thm

Theorem (1m)

Let ψ be as above. If (H2) holds, then:

i) *If there exists an admissible path γ in Λ such that $sf(L,\gamma) \neq 0$, then $\Lambda \setminus B(\psi)$ is disconnected.*

ii) *If there exists a sequence of admissible paths γ_n, $n \in \mathbb{N}$, such that $\lim_{n\to\infty} |sf(L,\gamma_n)| = \infty$, then $\Lambda \setminus B(\psi)$ has infinitely many path components.*

iii) *If $\Sigma(L) = B(\psi)$, any admissible path γ such that $L \circ \gamma$ has only isolated singular points will cross at least $\frac{|sf(L,\gamma)|}{d(L\circ\gamma)} + 1$ components of $\Lambda \setminus B(\psi)$.*

Remark since no subset of covering dimension strictly smaller than $n-1$ can disconnect a topological n-manifold, it follows that, if the parameter space Λ is a topological n-manifold, then the covering dimension of $B(\psi)$ is at least $n-1$.

Nils Waterstraat Thm.

Theorem

If M is a closed spin manifold of dimension $n \equiv 3 \mod 4$. Then the space of metrics g on M such that the Dirac operator D_g is invertible, if nonempty, has infinitely many path components.

Proved previously by Dahl for $n > 7$ in a different way.

Comparison

We say that $T \geq S$ if $T - S$ is a positive operator. If T and S have a Morse index then $T \geq S$ implies $m(T) \leq m(S)$. The following property of the spectral flow is an extension of this.

Theorem (2)

Let $H\colon [0,1] \times [a,b] \to \Phi_{SA}$ be a homotopy such that $H(\cdot, a)$ is non-increasing and $H(\cdot, b)$ is non-decreasing, then

$$\mathrm{sf}(H_0) \leq \mathrm{sf}(H_1).$$

Corollary (3)

Let $L, M\colon [a,b] \to \Phi_{SA}$ be such that $L_\lambda - M_\lambda$ is compact for each $\lambda \in I$. If $L_a \leq M_a$ and $L_b \geq M_b$, then

$$sf(M) \leq sf(L).$$

Proof: Take $H(t,\lambda) = M_\lambda + t(L_\lambda - M_\lambda)$

Remark: the generalized spectral flow

Given any path $L\colon [a,b] \to \Phi_{SA}$ (possibly with noninvertible end points), we define $\mathrm{sf}(L) = \mathrm{sf}(L + \delta\,\mathrm{Id})$, *where* $\delta > 0$ *such that* $L_i + \lambda\,\mathrm{Id};\ i = a, b$ *is invertible for* $0 < \lambda \leq \delta$.

Clearly, the right hand side does not depend on the choice of δ. The resulting function is additive under concatenation and is homotopy invariant under homotopies keeping the end-points fixed.

The comparison principle extends without any restriction to this more general case.

Bifurcation of periodic orbits of Hamiltonian systems

Let us consider a family of Hamiltonian systems parametrized by $\lambda \in \Lambda$.

$$\begin{cases} J\,u'(t) + \nabla_u \mathcal{H}(\lambda, t, u(t)) = 0, & t \in [0, 2\pi] \\ u(0) = u(2\pi). \end{cases} \tag{1}$$

Here

- J denotes the standard symplectic matrix.
- Λ is a connected topological space
- $\mathcal{H} : \Lambda \times \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R}$ is a continuous function 2π-periodic in t, such that each $\mathcal{H}_\lambda$ is C^2 with its first and second partial derivatives depend continuously on (λ, t, u).
- $\mathcal{H}(\lambda, t, 0) = 0$ for all $(\lambda, t) \in \Lambda \times [0, 2\pi]$.

In order to work with bounded operators we will study weak solutions of the equation belonging to $H^{\frac{1}{2}} = H^{\frac{1}{2}}(S^1, \mathbb{R}^{2n})$.

Bifurcation of periodic orbits I

We extend the bilinear form

$$(u, v) \rightsquigarrow \int_0^{2\pi} \langle \mathrm{J}\, u'(t), v(t) \rangle \, dt$$

to a form $\Gamma : H^{\frac{1}{2}} \times H^{\frac{1}{2}} \to \mathbb{R}$ and consider

$$\psi : \Lambda \times H^{\frac{1}{2}}(S^1, \mathbb{R}^{2n}) \to \mathbb{R}, \quad \psi_\lambda(u) = \frac{1}{2}\Gamma(u, u) + \int_0^{2\pi} \mathcal{H}(\lambda, t, u(t)) \, dt.$$

Under the natural growth assumptions each ψ_λ is C^2 and the critical points of ψ_λ are the weak solutions of the Hamiltonian system (1).

The Hessian L_λ of ψ_λ the at 0 is defined by

$$\langle L_\lambda u, v \rangle_{H^{\frac{1}{2}}} = \Gamma(u, v) + \int_0^{2\pi} \langle A_\lambda(t) u(t), v(t) \rangle \, dt,$$

where $A_\lambda(t) = D_u \nabla_u \mathcal{H}_\lambda(t, 0)$.

L_λ is Fredholm by compactness of the embedding of $H^{\frac{1}{2}}$ into L^2.

Bifurcation of periodic orbits II

Assumption (H_1) : $A_\lambda(t) \equiv A_\lambda$ *for all* $\lambda \in \Lambda$.

In this case, L_λ is invertible if and only if the matrix A_λ is non-resonant, i.e., the spectrum of $\mathrm{J}\, A_\lambda$ does not contain integral multiples of $i = \sqrt{-1}$.
Taking $\Lambda = [a, b]$ and assuming that A_a, A_b are non resonant, the spectral flow $sf(L)$ can be computed as follows:

Consider the sequence of $4n \times 4n$ matrices:

$$L^0(A) = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, \quad L^k(A) = \begin{pmatrix} \frac{1}{k}A & \mathrm{J} \\ -\mathrm{J} & \frac{1}{k}A \end{pmatrix}, \ k \in \mathbb{N}.$$

Define the index of the matrix A by

$$i(A) = \frac{1}{2} \sum_{k=0}^{\infty} \operatorname{sgn} L^k(A).$$

Then, $\quad sf(L) = i(A_b) - i(A_a)$(see [S], [FPR]).

Bifurcation of periodic orbits III

Theorem (4)

Assume that the Hamiltonian system (1) verifies (H1) and that A_λ is invertible for all λ.

i) *If $i(A_a) \neq i(A_b)$, then any neighbourhood of the stationary branch $[a, b] \times \{0\}$ in $[a, b] \times H^{1/2}$ contains solutions of the form (λ, u) with u non-constant and 2π-periodic.*

ii) *If A_λ is non resonant for all but a finite number of λ then there are at least $|i(A_a) - i(A_b)|/2n$ bifurcation points of periodic orbits in the interval $[a, b]$.*

Multiparameter bifurcation of periodic orbits

As an immediate consequence of Theorem $(1m)$ we obtain:

Theorem

Assume that $(\mathrm{H1})$ *and* $(\mathrm{H2})$ *hold.*

i) *If there exist* $\lambda, \mu \in \Lambda$ *such that the matrices* A_λ, A_μ *are non-resonant and* $i(A_\lambda) \neq i(A_\mu)$, *then* $\Lambda \setminus B(\psi)$ *is disconnected.*

ii) *If there exists a sequence* $\{\lambda_n\}_{n\in\mathbb{N}} \subset \Lambda$ *such that* A_{λ_n} *is non-resonant for all* $n \in \mathbb{N}$ *and* $\lim_{n\to\infty} |i(A_{\lambda_n})| = \infty$, *then* $\Lambda \setminus B(\psi)$ *has infinitely many path components.*

iii) *If* $B(\psi) = \{\lambda : A_\lambda \text{ is resonant}\}$, *then any path* γ *joining two non-resonant parameters* λ *and* μ *such that* $A \circ \gamma$ *has only isolated resonant points must cross at least* $\frac{|i(A_\lambda) - i(A_\mu)|}{2n} + 1$ *components of* $\Lambda \setminus B(\psi)$.

Bifurcation of periodic orbits IV

Time depending systems $Ju'(t) - A_\lambda(t)u(t) = 0$ do not have an index defined in terms of the coefficients as before. The spectral flow $sf(L, I)$ can still be computed as the relative Conley-Zehnder index of the path $\{P_\lambda\}_{\lambda\in I}$ of Poincaré monodromy operators (cf. [FPR II]). But the monodromy operator can be only obtained by integrating the linearization and cannot be considered as given directly by our data. Floquet theory leads to the same problem. However, using the *comparison principle* for the spectral flow, we still are able to detect bifurcation and estimate from below the number of bifurcation points directly from the coefficient matrix of the system.

Here we will consider only the case $\Lambda = [a, b]$ taking $a = 0, b = 1$.

Bifurcation of periodic orbits V

Let $Ju'(t) + A_\lambda(t)u(t) = 0$ be the linearised equation at 0. In order to estimate $sf(L)$ we will use the *numerical range* of $A_i(t), i = 0, 1$.

Let $\{\mu_1^i(t) \leq \mu_2^i(t) \leq \cdots \leq \mu_{2n}^i(t)\}$ be the eigenvalues of $A_i(t)$. For $i = 0, 1$, set

$$\mu_i^- = \inf_t \{\mu_1^i(t)\}, \quad \mu_i^+ = \sup_t \{\mu_{2n}^i(t)\}.$$

Then

$$\mu_i^- \,\mathrm{Id} \leq A_i(t) \leq \mu_i^+ \,\mathrm{Id}, \quad i = 0, 1. \tag{2}$$

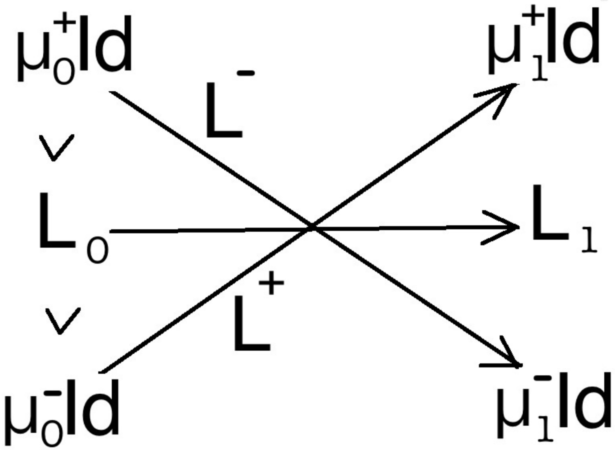
Let $A_\lambda^\pm = [\lambda\mu_1^\pm + (1-\lambda)\mu_0^\mp]\,\mathrm{Id}$ and let $L^\pm$ be the path of operators on $H^{\frac{1}{2}}$ defined by:

$$\langle L_\lambda^\pm u, v \rangle_{H^{\frac{1}{2}}} = \Gamma(u, v) + \int_0^{2\pi} \langle A_\lambda^\pm u(t), v(t) \rangle \, dt.$$

Then $L_\lambda - L_\lambda^\pm$ is compact for all λ.

By the corollary of the comparison theorem,

$$sf(L^-) \leq sf(L) \leq sf(L^+). \tag{3}$$



Bifurcation of periodic orbits VII

The spectral flows of $L^{\pm}$ are easy to compute.

Given real numbers μ and ν, define

$$\Delta(\mu,\nu) = \begin{cases} \#\{i \in \mathbb{Z} : \mu \leq i < \nu\} & \text{if } \mu \leq \nu \\ -\#\{i \in \mathbb{Z} : \nu \leq i < \mu\} & \text{if } \nu \leq \mu. \end{cases} \tag{4}$$

Then

$$sf(L^{\pm}) = 2n\,\Delta(\mu_0^{\mp}, \mu_1^{\pm})$$

By (3) and (4),

$$2n\,\Delta(\mu_0^{+}, \mu_1^{-}) \leq sf(L) \leq 2n\,\Delta(\mu_0^{-}, \mu_1^{+}).$$

From Theorem (1) we conclude that

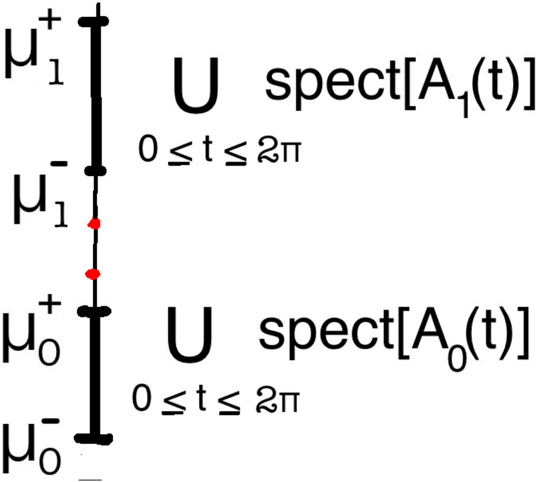
Bifurcation of periodic orbits VI

Theorem (6)

i) *The interval* $[a, b]$ *contains some bifurcation point of* 2π*-periodic orbits from the stationary branch if either* $\mu_0^+ < \mu_1^-$ *and* $\Delta(\mu_0^+, \mu_1^-) > 0$ *or* $\mu_1^+ < \mu_0^-$ *and* $\Delta(\mu_0^-, \mu_1^+) < 0$

ii) *If moreover, the linearization of the problem* (1) *along the stationary branch admits only trivial solutions for all but a finite number of values of* $\lambda \in [0, 1]$, *then the family* (1) *has at least* $\Delta(\mu_0^+, \mu_1^-)$ *points of bifurcation of periodic solutions from the stationary branch in the first case and at least* $-\Delta(\mu_0^-, \mu_1^+)$ *bifurcation points in the second.*

Illustration 2



Thank you and congratulations to Massimo!