Topological and Variational Methods for ODEs,

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Dedicated to Massimo Furi

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Multiple positive solutions for superlinear equations with a sign-indefinite weight

(03/06/2014 : $14^{\underline{45}}$ – $15^{\underline{15}}$)

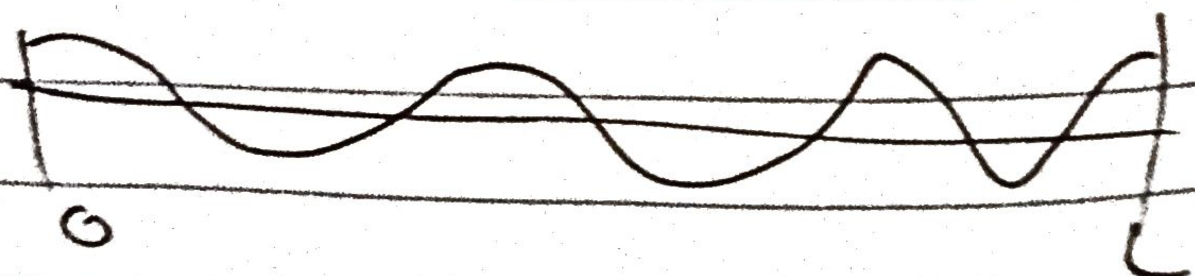
With gratitude to Massimo !!

(1)

$$\begin{cases} u'' + a(x)\, g(u) = 0 \\ u(0) = u(L) = 0 \end{cases}$$

$g : \mathbb{R}^+ \to \mathbb{R}^+ := [0, +\infty)$
g continuous,
$g(0) = 0$, $g(s) > 0$ for $s > 0$

$a : [0, L] \to \mathbb{R}$
continuous, $a(x)$ is
a weight function
which changes its sign
in $[0, L]$.



About g.

Model nonlinearity:

$g(s) = s^p$, $s \geq 0$, $p > 1$

More general:

$\frac{g(s)}{s} \to 0$ as $s \to 0^+$

$\frac{g(s)}{s} \to +\infty$ as $s \to +\infty$

"superlinear indefinite problems"

Problems of this form are considered in the literature from different points of view

1-dimensional case ODEs
N-dimensional case PDEs

$a : \overline{\Omega} \to \mathbb{R}$

$\Omega \subseteq \mathbb{R}^N$ bounded domain

$$\begin{cases} u\big|_{\partial\Omega} = 0 \\ \Delta u + a(x)\, g(u) = 0, \quad x \in \Omega \end{cases}$$

- bounded domains
- unbounded domains

$[0, L] \leftarrow \mathbb{R}^+$ or $\mathbb{R}$.

④

sign-changing weight

- $\ddot{x} + w(t)\,|x|^{p-1}x = 0$
  "nonlinear Hill equation"

Waltman, Kiguradze ...
--> '65 '67 ⟶

- Periodic solutions and complex dynamics
  Butler (1976)
  Terracini & Verzini (2000)
  Capietto, Dambrosio, Papini (2002)

- PDEs & Hamiltonian systems (Variational methods)
  Lassoued (1990) ⟶
  Alama & Tarantello, Berestycki - Capuzzo-Dolcetta & Nirenberg,
- Bandle, Pozio & Tesei (1987)
  Amann & Lopez-Gomez

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$u(x) > 0 \quad x \in \text{int}(\text{domain})$

$\boxed{\text{Positive solutions}}$

Reaction-diffusion equations, stationary solutions.

Gomez-Reñasco & Lopez-Gomez (JDE 2000)
Ackermann (2009)

$$\begin{cases} \Delta u + \lambda u + a(x) g(u) = 0 \\ u\big|_{\partial\Omega} = 0 \end{cases}$$

$g(s) = s^p, \quad p > 1$

$a(x) = a_\mu(x) = a^+(x) - \mu a^-(x)$

$N = 1, \quad \lambda < 0$ "small"
$\mu > 0$ large: $2^n - 1$
positive solutions

Existence of positive solutions.

Gaudenzi, Habets e Z (2003)

$$\limsup_{s\to 0^+} \frac{g(s)}{s} < \lambda_0 \qquad (1)$$

$$\liminf_{s\to +\infty} \frac{g(s)}{s} > \max_{i=1,\dots,m} \lambda_1^{(i)} \qquad (2)$$

$$\begin{cases} \varphi'' + \lambda a^+(x)\varphi = 0 \\ \varphi(0) = \varphi(L) = 0 \end{cases} \qquad (\lambda_0)$$

$$\begin{cases} \varphi'' + \lambda a^+(x)\varphi = 0 \\ \varphi\big|_{\partial I_i} = 0 \end{cases} \qquad (\lambda_1^{(i)})$$

I_i

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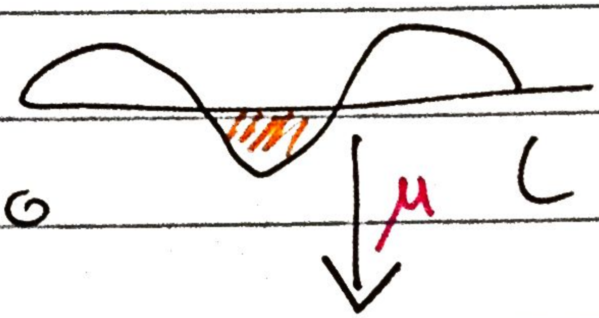
Existence of at least one positive solution (method upper/lower solutions in the reverse order)

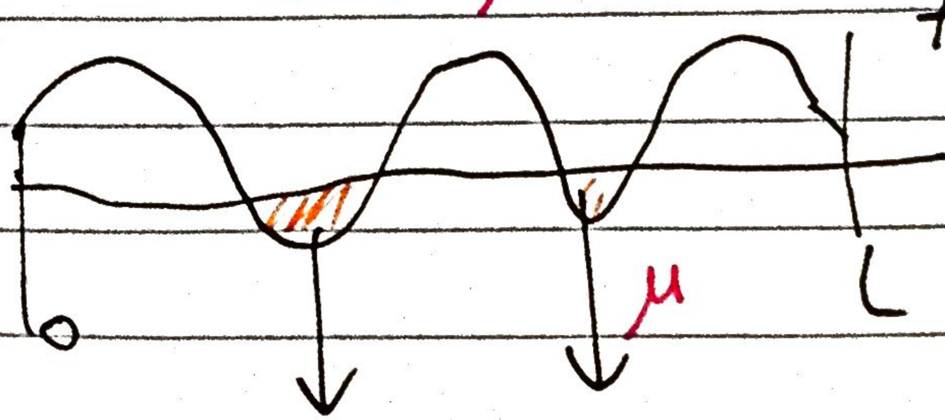
Gaudenzi - Habets & Z. (2003-4)

$g(s) = s^p$



3 positive sols

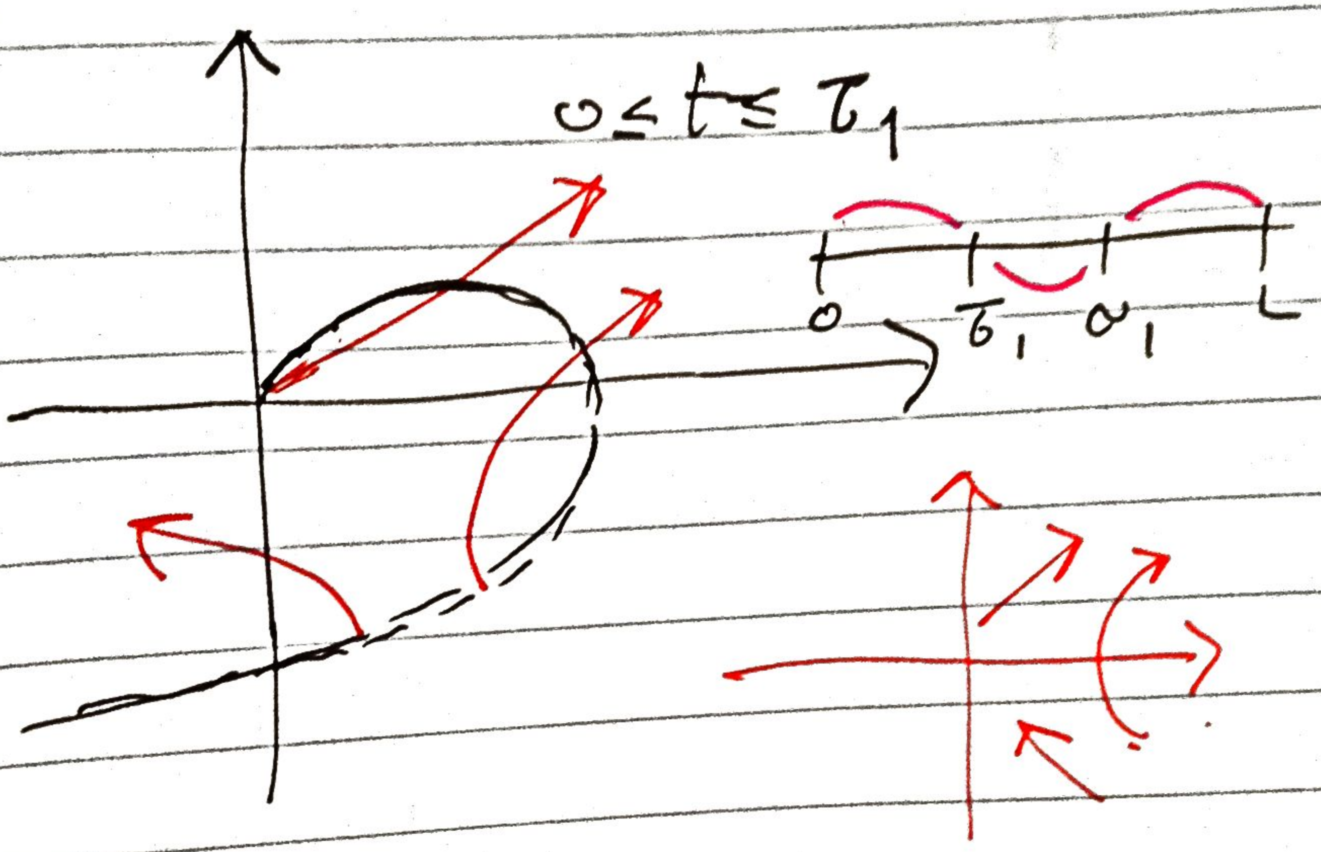
$a^+(x) - \mu a^-(x)$



7 positive sols.

Warning: for $u < 0$ the $g(u)$ is extended in usual manner !!

(method: shooting type)



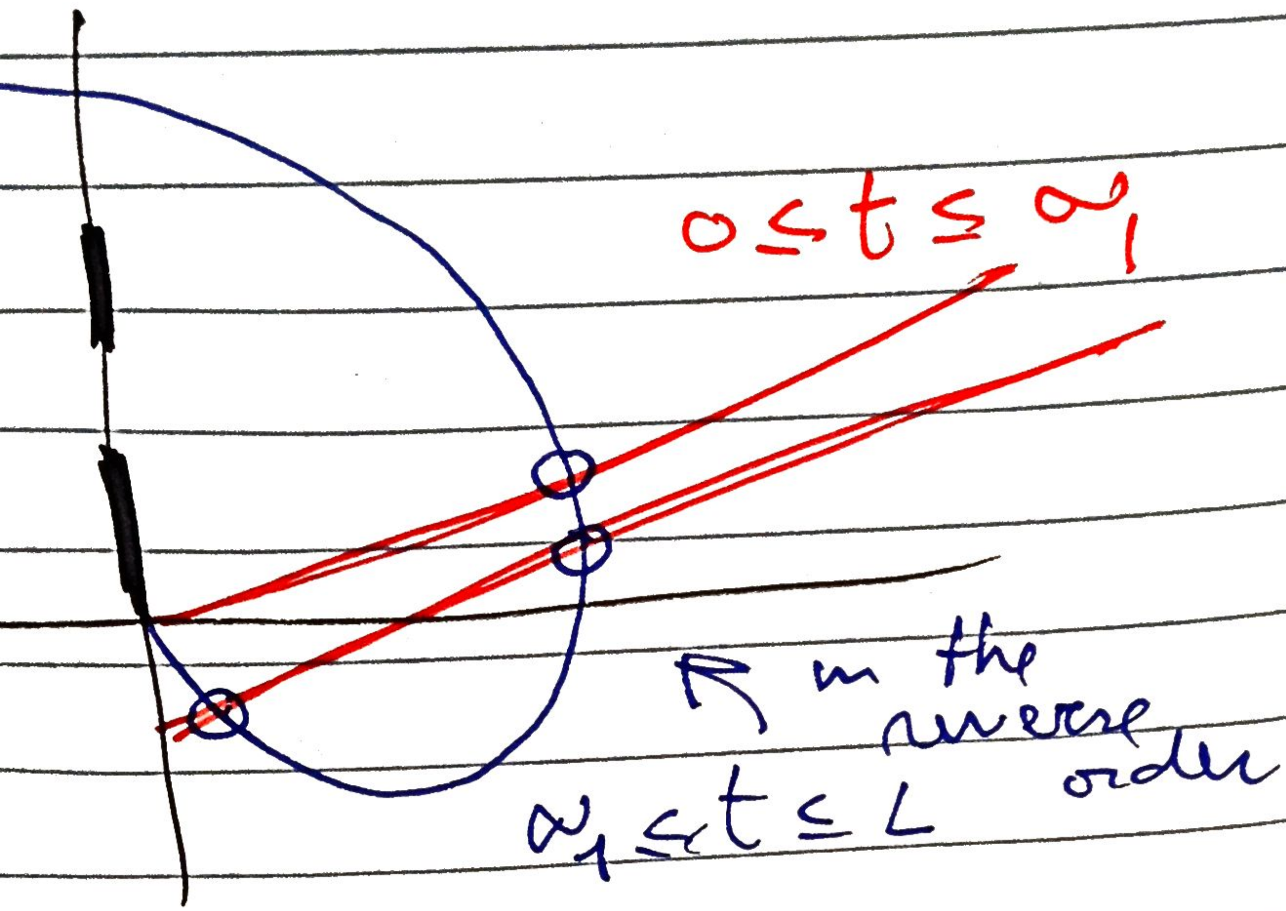
Matini & Matucci (2012)

$$\left(r(t)\, \Phi_p(u')\right)' \pm q(t)\, f(u)$$

$u(0) = 0$, $u(+\infty) = 0$

$u(t) > 0$ on $]0, +\infty)$

(Existence results)

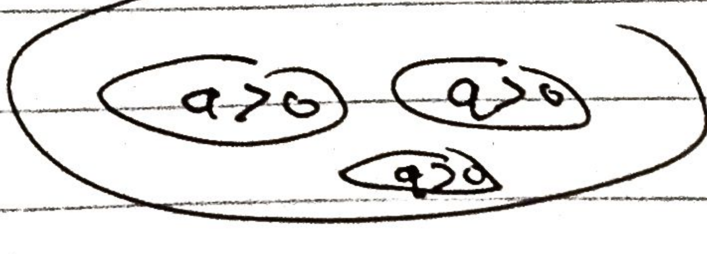


one can iterate this procedure to the case of a weight with n positive humps but, clearly the technicalities grow with the number of humps .... the argument perhaps becomes unfocused "clumsy"

3 positive solutions for the Neumann problem, by Boscaggin (JMAA 2011)

Bonheure, Gomes & Habets (2005)

$$\begin{cases} \Delta u + a_{\mu}(x)\, u^{p} = 0 & (p>1) \\ u\big|_{\partial\Omega} = 0 \end{cases}$$



also (Girão & Gomes)

$\mu >> 1$, $\quad 2^{n}-1$

positive solutions.

Via a variational approach

(11)

A degree-theoretic approach (Guglielmo Feltrin)

Main steps.

1) $$\tilde{g}(s) = \begin{cases} g(s) & \text{for } s \geq 0 \\ 0 & \text{for } s < 0 \end{cases}$$

$$\begin{cases} u'' + a(x)\,\tilde{g}(u) = 0 \\ u(0) = u(L) = 0 \end{cases}$$

Maximum principle.
A nontrivial solution must be positive and hence a solution of the original equation

2) $\begin{cases} -u'' = w(x) \\ u(0) = u(L) = 0 \end{cases}$

$$u(x) = \int_0^L g(x,\xi)\, w(\xi)\, d\xi$$

$\downarrow$

$$u(x) = \int_0^L g(x,\xi)\, \underbrace{f(\xi, u(\xi))}_{a(\cdot)\tilde{g}(u(\cdot))}\, d\xi$$

$$\boxed{u = \Phi(u)}$$

fixed point problem

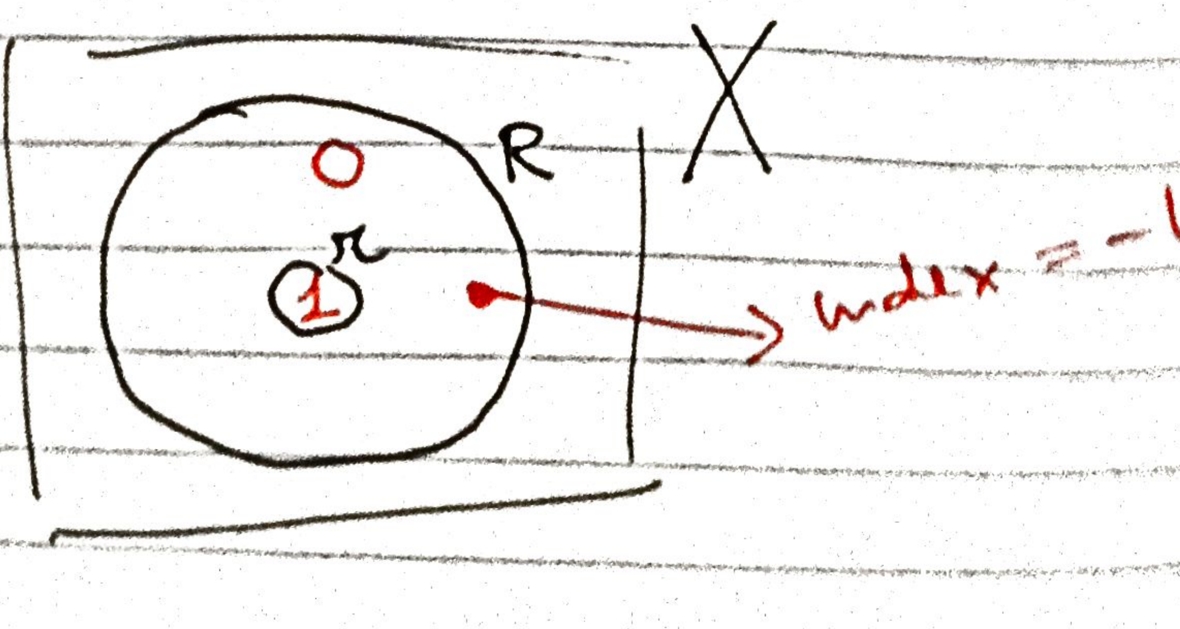
$\Phi(0) = 0$

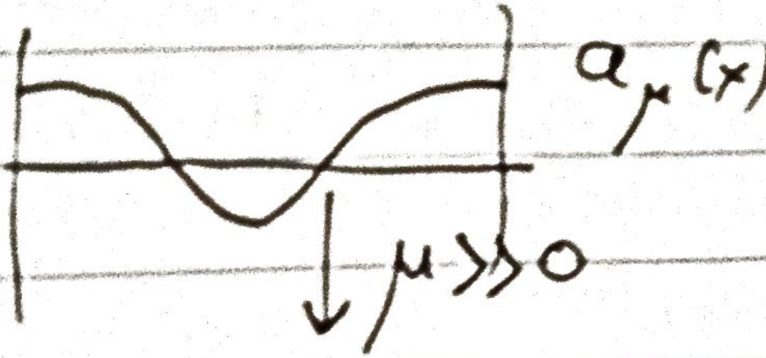
We look for non-trivial solutions.

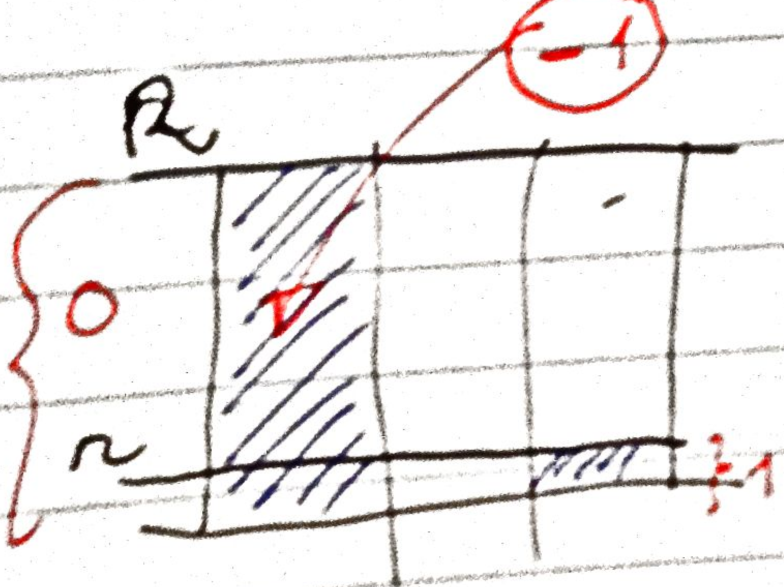
Classical case $a(x) \geq 0$, $\not\equiv 0$

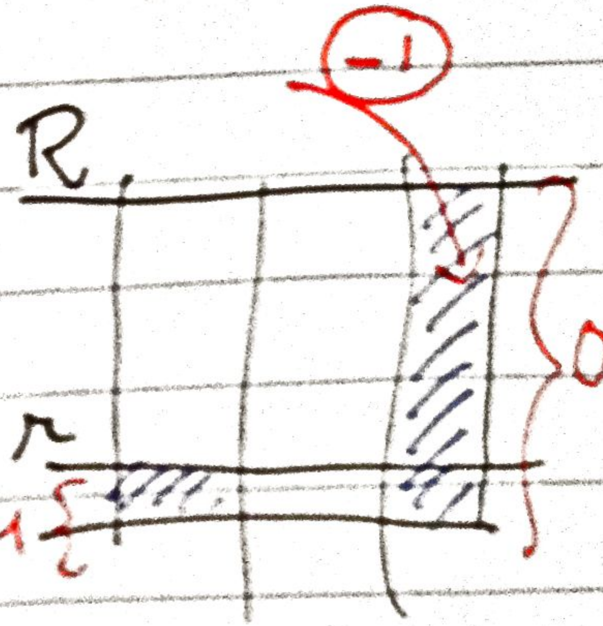
In this case one would work directly with the fixed point index in cones

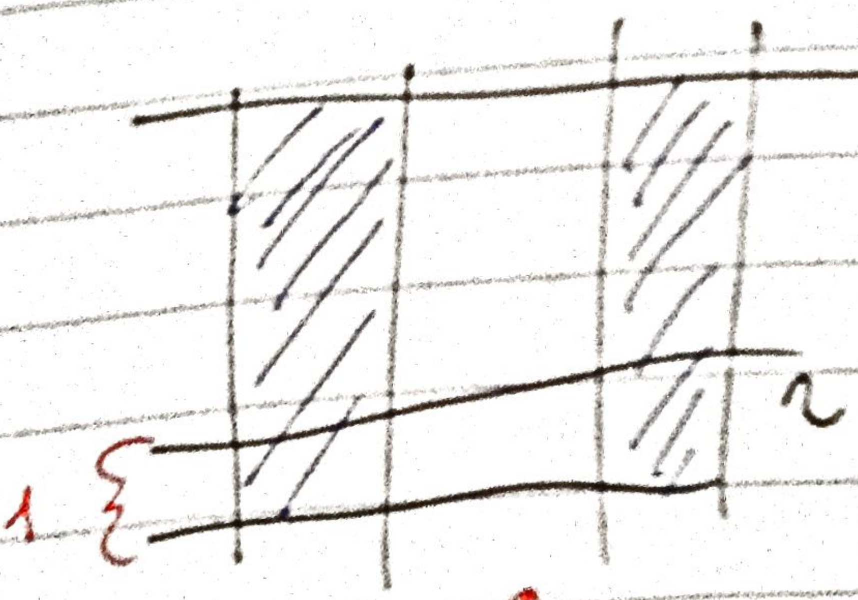
X = Banach space of continuous functions with $\|\cdot\|_\infty$ - norm





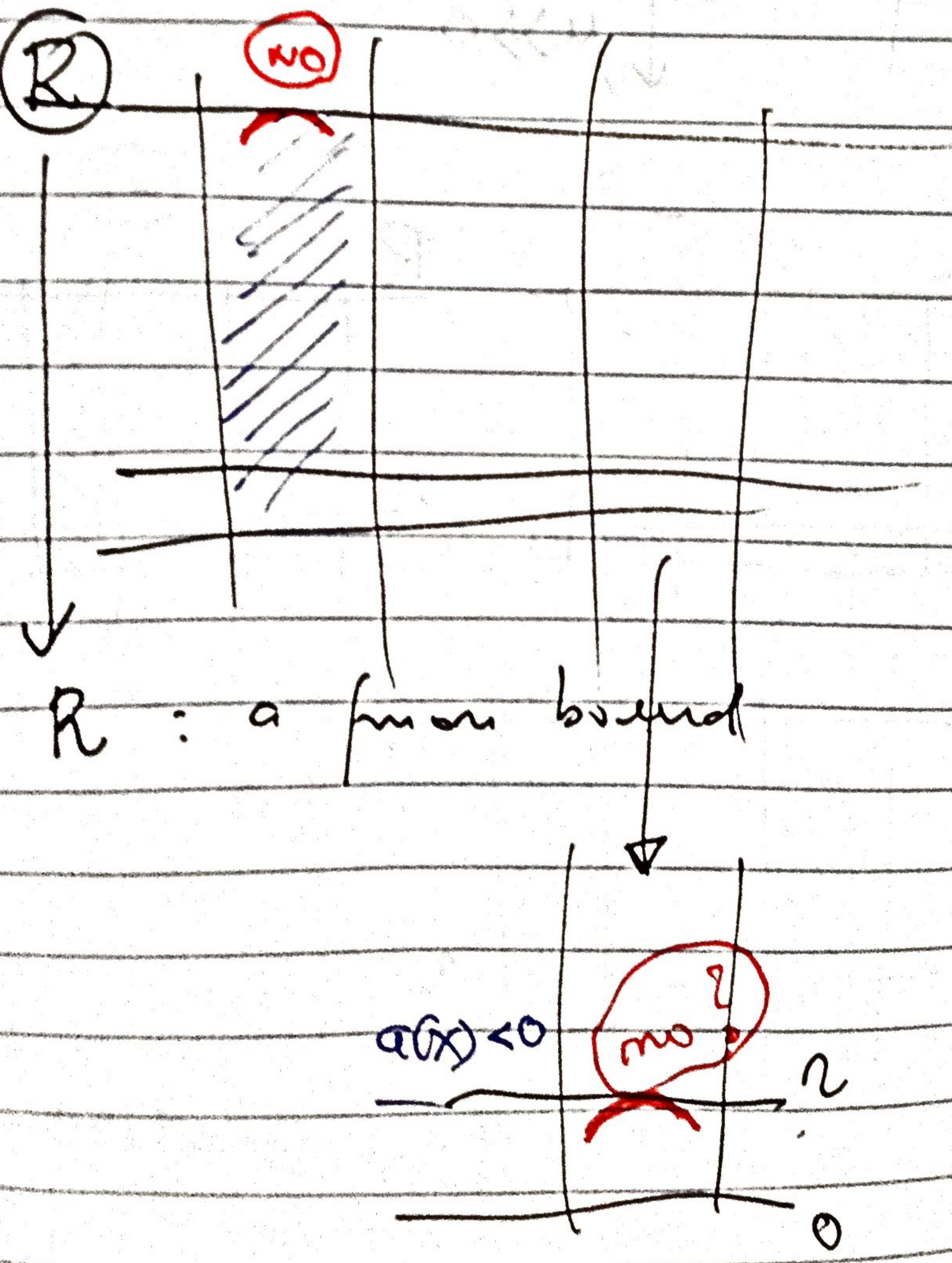


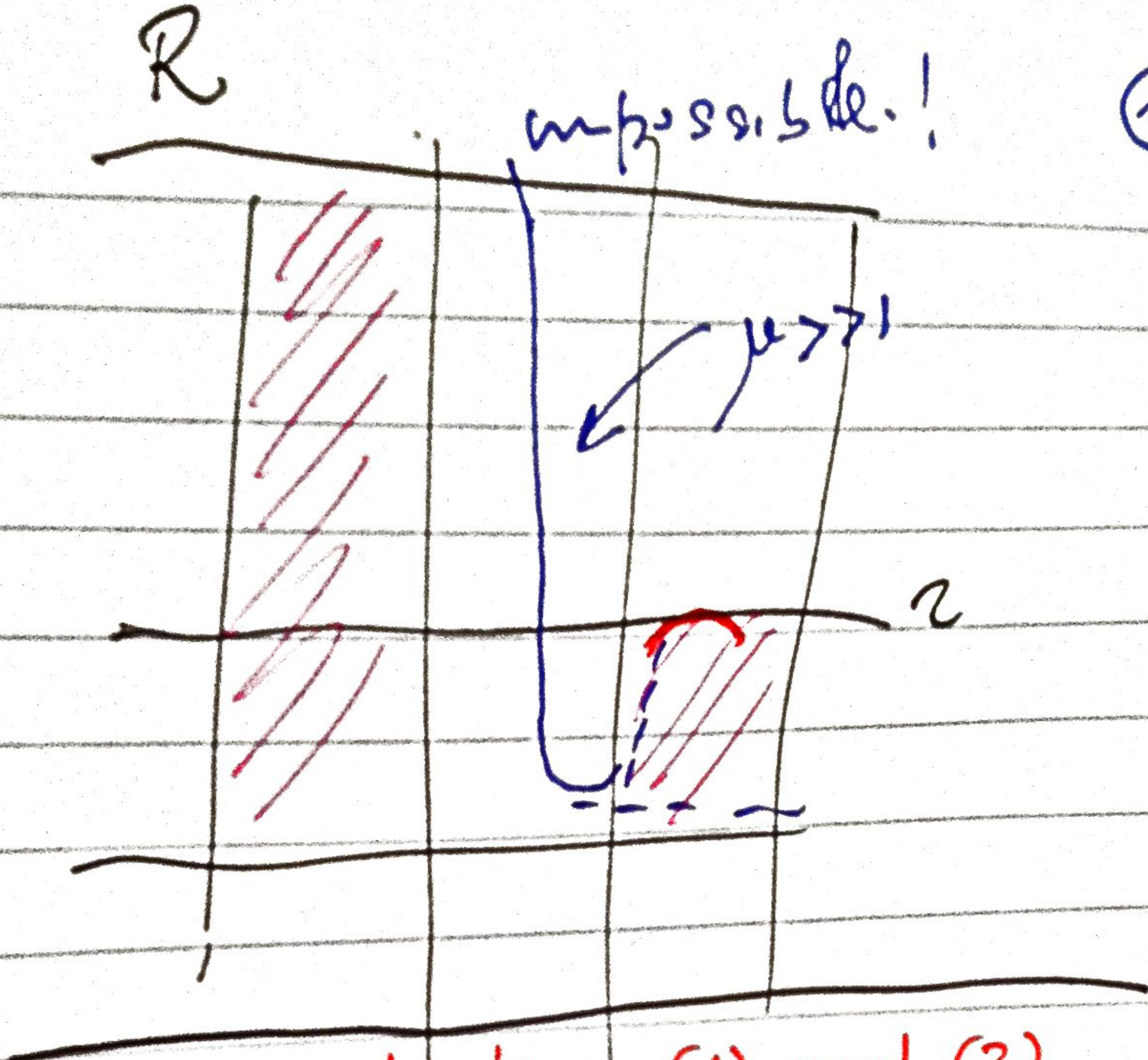




third solution!

The rôle of $\mu >> 1$





g(s) satisfies (1) and (2)

Example:

$$\begin{cases} u'' + a_\mu(x)\, g(u) = 0 \\ u(0) = u(L) = 0 \end{cases}$$

$g(s) = K s \arctan(s)$, $s \geq 0$

$K > 0$ large enough depending on $a^+(x)$.