Polygonal solutions for a shape optimisation problem

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in collaboration with

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Geometric Formulation

$$\min_{\Omega \in \mathscr{C}_{a,b}} \lambda|\Omega| - P(\Omega),$$

where:

$$\mathscr{C}_{a,b} = \{K \subseteq \mathbb{R}^2 \ : \ K \text{ convex, } D_a \subseteq K \subseteq D_b\}.$$

Let us define $\mathrm{J}_\lambda(\Omega) = \lambda|\Omega| - P(\Omega) \rightsquigarrow \min_{\Omega \in \mathscr{C}_{a,b}} \mathrm{J}_\lambda(\Omega)$.

Notice: For every $\lambda \geq 0$ there exists a solution Ω_λ.

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The limit cases $\lambda = 0, +\infty$

- $\lambda = 0$, $\mathrm{J}_0(\cdot)$ attains its minimum on D_b
  ($\rightsquigarrow$ maximise $P(\cdot)$!)
- $\lambda = +\infty$, $\mathrm{J}_{+\infty}(\cdot)$ attains its minimum on D_a
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Analytic Formulation

We can rewrite the functional J_λ in terms of h_Ω, the support function of Ω:

$$\mathrm{J}_\lambda(h_\Omega) = \frac{\lambda}{2} \int_0^{2\pi} (h_\Omega^2 - h_\Omega'^2)\, d\theta - \int_0^{2\pi} h_\Omega \, d\theta.$$

Moreover the class $\mathscr{C}_{a,b}$ is:

$$\mathscr{C}_{a,b} = \{h_K \in W^{1,2}(0, 2\pi) \;:\; a \leq h_K \leq b,\; h_K'' + h_K \geq 0 \,\forall\theta \in [0, 2\pi]\}.$$

Minima of J_λ are locally polygons in $D_b \setminus \overline{D_a}$

Thm. Let $\Omega \in \mathscr{C}_{a,b}$; if for some $\omega \subseteq [0, 2\pi]$, $supp(h''_\Omega + h_\Omega) \cap \omega$ contains at least 3 directions, corresponding to support sets in the interior of the ring, then $\exists v$ compactly supported in ω s.t. $h_\Omega + tv$ is the support function of a convex set in $\mathscr{C}_{a,b}$, for $t \in (-\delta, \delta)$.

[J. Lamboley, A. Novruzi 2009]

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Corollary If $supp(h'' + h) \cap (0, \pi)$ has at least 3 directions, corresponding to support sets in the ring, then Ω cannot be optimal.

Hence: every minimum is a polygon inside the ring.

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[Proof of the Corollary.]

Let Ω be a minimum for J_λ; assume it is not a polygon in the ring
$\rightsquigarrow$ $supp(h''_\Omega + h_\Omega) \cap (0, \varepsilon)$ contains at least 3 points for some $\varepsilon < \pi$.
Hence: we can make perturbations Ω^t s.t. $h_{\Omega^t} = h_\Omega + tv$, $|t| < \delta$.
By second order optimality conditions

$$0 \leq \langle \mathrm{J}''_\lambda(\Omega)v, v\rangle = \lambda \int_0^{2\pi} v^2 - v'^2.$$

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But using Poincaré inequality we get:

$$0 \le \lambda \int_0^{\varepsilon} v^2 - v'^2 \le \lambda \left(\frac{\varepsilon^2}{\pi^2} - 1\right) \int_0^{\varepsilon} v'^2.$$

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Minima of J_λ are **locally polygons** in $D_b \setminus \overline{D_a}$

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When is the solution a polygon?

- **Thm.** Ω_λ is a polygon if and only if $\frac{1}{2b} < \lambda < \frac{2}{a}$.

  Moreover:
  $\Omega_\lambda = D_b$ for $0 \le \lambda \le \frac{1}{2b}$ and $\Omega_\lambda = D_a$ for $\frac{2}{a} < \lambda \le +\infty$;
  if $\lambda = \frac{2}{a}$ there exist infinite solutions, circumscribed to D_a.

[CB, A. Henrot 2011]

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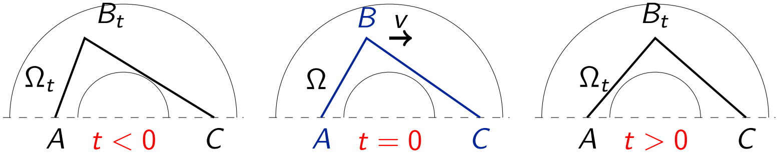
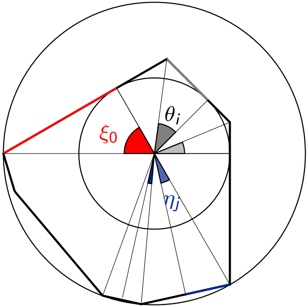
"Free sides" are not allowed



There cannot exist two "free" sides $\overline{AB}, \overline{BC}$ otherwise $\rightsquigarrow$ consider the set Ω_t obtained as a perturbation of Ω_λ moving the vertex B along the direction $v = \overrightarrow{AC}$ for a time $t \in (-\delta, \delta)$ (N.B. all the other vertices are fixed!)

$\rightsquigarrow$ contradiction!

Reduction to a finite dimensional problem



Classes of (central) angles:

$\xi_0 \in \mathscr{A}_b^a$, $\cos \xi_0 = \frac{a}{b}$
$\theta_i \in \mathscr{A}^a$, $\cos \theta_i > \frac{a}{b}$
$\eta_j \in \mathscr{A}_b$, $\cos \eta_j > \frac{a}{b}$.

Corresponding classes of sides:

$\mathscr{A}_b^a \leftrightsquigarrow \mathscr{L}_b^a$,
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Thm. For $\frac{1}{2b} < \lambda < \frac{2}{a}$, $\partial\Omega_\lambda = \cup l_i$ with $l_i \in \mathscr{L}_b^a \cup \mathscr{L}^a \cup \mathscr{L}_b$.
Moreover Ω_λ is either a regular or a quasi-regular polygon.

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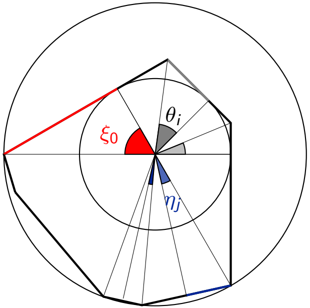

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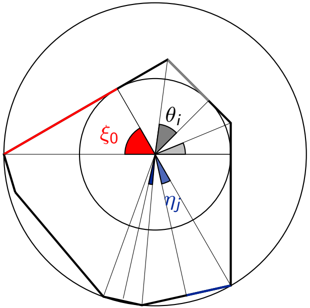

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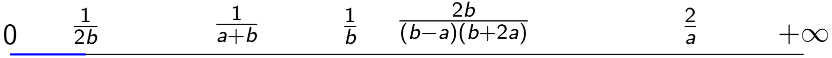
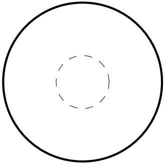
[CB, A. Henrot 2011]

An example: $a = 1, b = 3$





$$0 \leq \lambda < \frac{1}{6}$$

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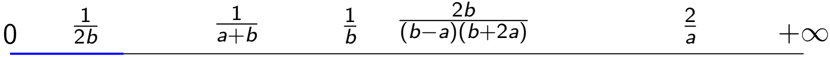

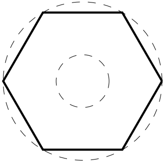

$$0.1792 < \lambda < 0.1847$$

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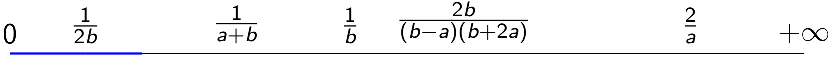

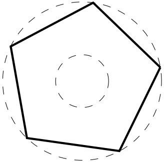

$$0.1847 < \lambda < 0.19506$$

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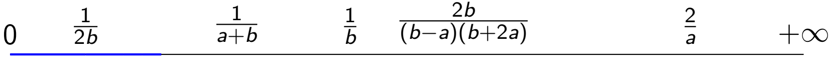

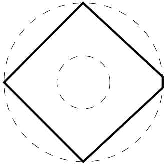

$$0.19506 < \lambda < 0.19525$$

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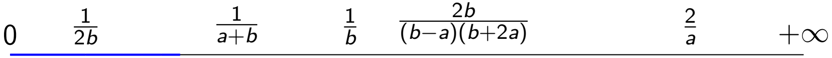
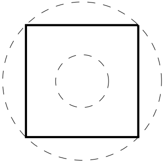
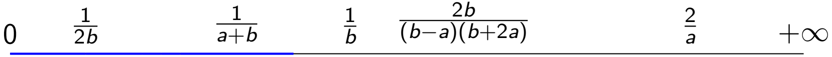
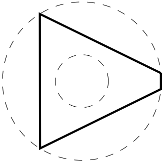




$$0.19525 < \lambda < 0.2187$$

An example: $a = 1, b = 3$





$$0.2187 < \lambda < 0.2222$$

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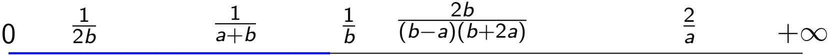
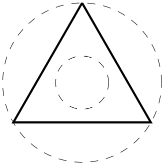
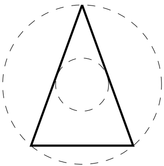




$$0.2222 < \lambda < 0.3080$$

An example: $a = 1, b = 3$

$0 \quad \frac{1}{2b} \quad \frac{1}{a+b} \quad \frac{1}{b} \quad \frac{2b}{(b-a)(b+2a)} \quad \frac{2}{a} \quad +\infty$



$$0.3080 < \lambda < 0.6$$

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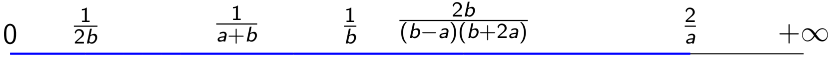

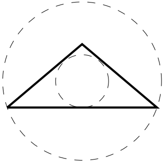

$$0.6 < \lambda < 2$$

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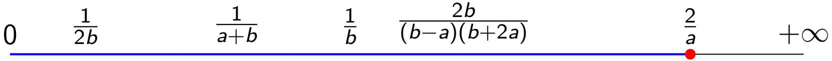

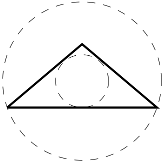

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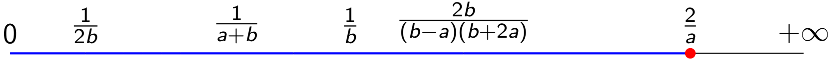

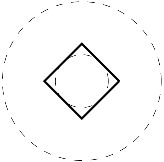

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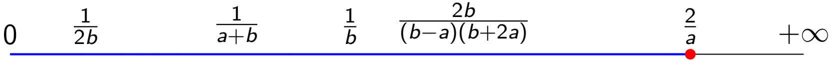

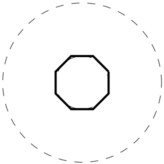

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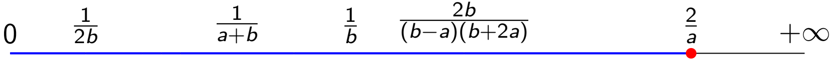

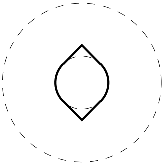

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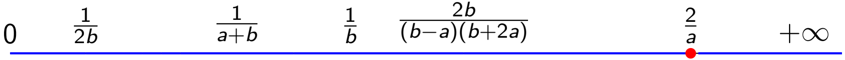
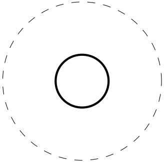
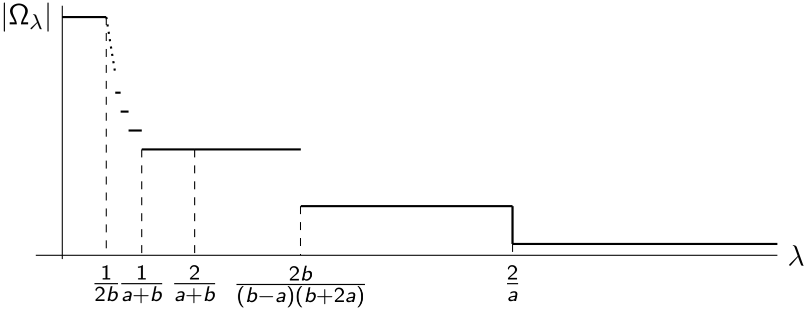




$$2 < \lambda < +\infty$$

An example: analysis of $|\Omega_\lambda|$



A Bonnesen-Fenchel type inequalitiy

For every planar convex set Ω of inradius $r(\Omega)$ it holds

$$P(\Omega) \leq 2\frac{|\Omega|}{r(\Omega)}.$$

[T. Bonnesen- W. Fenchel 1929]

Proof.

We can assume $D_r \subset \Omega$; moreover $\exists R > r$ s.t. $\Omega \subset D_R \rightsquigarrow \Omega \in \mathscr{C}_{r,R}$. Take $\lambda = \frac{2}{r} \rightsquigarrow \Omega_{\frac{2}{r}} = D_r \rightsquigarrow \mathrm{J}_{\frac{2}{r}}(\Omega) \geq \mathrm{J}_{\frac{2}{r}}(D_r)$

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$$\frac{2}{r}|\Omega| - P(\Omega) \geq \frac{2}{r}|D_r| - P(D_r) = 0.$$

Moreover: equality holds for every convex set Ω whose boundary is composed by arcs of D_r and tangent segments to it.

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Favard type inequalities

For every planar convex set Ω, whose circumradius is $R(\Omega)$, it holds

$$|\Omega| \geq R(\Omega)(P(\Omega) - 4R(\Omega)),$$

and equality holds for linear segments.

[J. Favard 1929]

Moreover

$$|\Omega| \geq R(\Omega)(2P(\Omega) - 3\pi R(\Omega)),$$

and equality holds if Ω is a ball.

[CB, A. Henrot 2011]

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