RECONSTRUCTION OF TWISTED POLYTOPES AND APPLICATIONS

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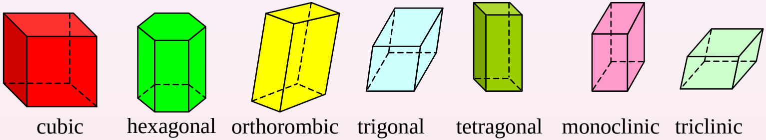
Convexity is a natural geometric assumption, and, also, it is frequently involved in natural shapes.
In particular we are mainly concerned with convex polytopes.

Convexity in applications.

The main reason is that crystals can be grouped in polyhedral classes, depending on the symmetries of their primitive cell.



Applications of discrete tomography to reconstruction of crystals has received considerable attention.
(For instance Salzberg-Figueroa; Batenburg-Palenstijn; Schwander; Tijdeman-te Riele; Baake, Gritzmann, Huck, Langfeld, and Lord)


mint


maple


tomato


grape-wine


edelweiss


sea star

Beyond convexity.

quartz

amethyst

diamond

quartz

amethyst

diamond

Problem. Find uniqueness results for special (non-convex) clusters of convex polytopes.

Getting uniqueness.

- Uniqueness results are known for convex bodies in $\mathbb{R}^2, \mathbb{Z}^2$.

Gardner-McMullen, 1980; Gardner-Gritzmann, 1997

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• Very few results are known in higher dimensions (mainly in the non-uniqueness direction).
Positive results in $\mathbb{Z}^n$ for X-rays in coordinate directions by Fishburn et al., 1991; Vallejo, 1997-1998-2002.
Counterexamples by Volčič, J.Wills and R.Gardner (see Gardner's book). Also by [Fishburn, Lagarias, Reeds, and Shepp, 1990

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Let $\mathcal{H} = \{H_i : 1 \leq i \leq m\}$ be a set of subspaces of $\mathbb{R}^n$. A bounded set $E \subset \mathbb{R}^n$ is called *$\mathcal{H}$-additive* if

$$E = \left\{ x \in \mathbb{R}^n : \sum_i f_i(x) > 0 \right\},$$

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Remark. The results also holds in the lattice $\mathbb{Z}^n$, provided $\mathcal{H}$ is a so-called *Radon Base*. Fishburn-Shepp, 1999

Additivity of polytopes.

Theorem. Let P be a non-degenerate n-dimensional convex polytope, $n \geq 2$. Then P is $\mathcal{H}$-additive with respect to the set $\mathcal{H}$ of the $n-1$ dimensional spaces parallel to its facets.

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- For each $j \in \{1, ..., m\}$, define the following function on $\mathbb{R}^n$

$$f_j(x) = \begin{cases} -(m-1)/m & \text{if } \; x \in B_j^- \\ 1/m & \text{if } \; x \in \text{cl}(B_j^+). \end{cases}$$

$B_j^{\pm}$=open half-spaces bounded by the hyperplane of the j-th facet.

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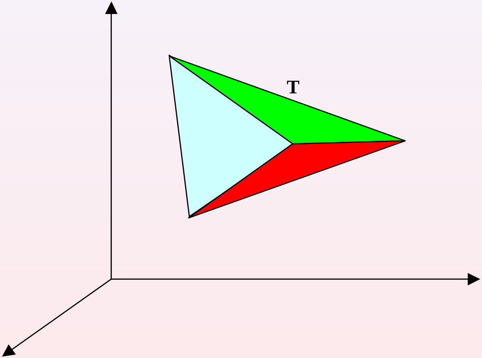

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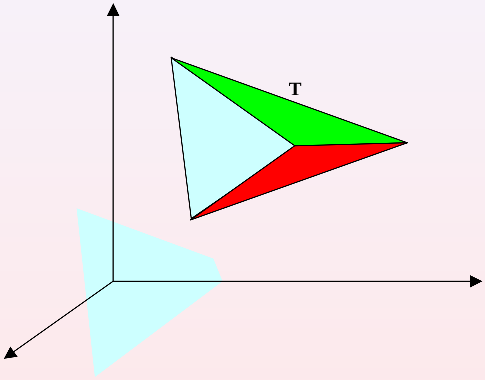

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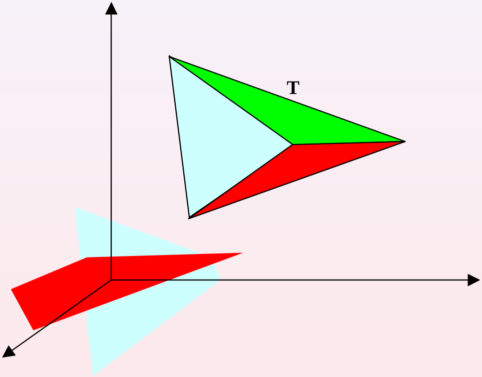

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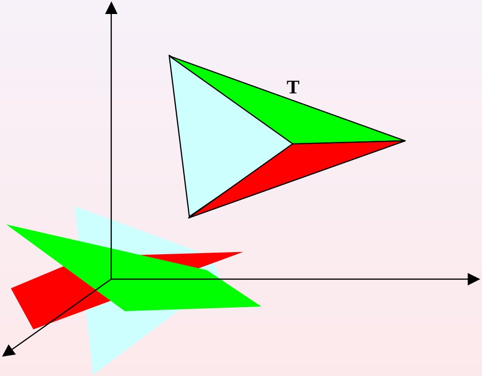

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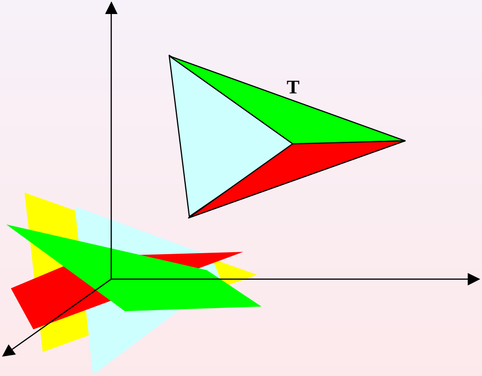

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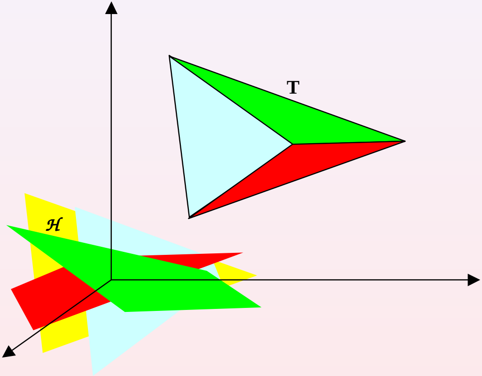

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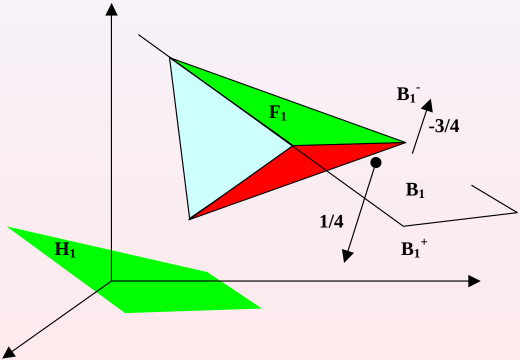

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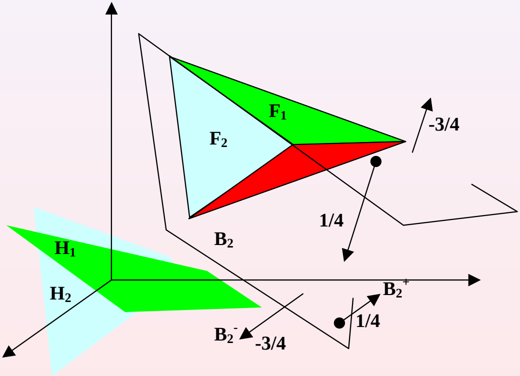

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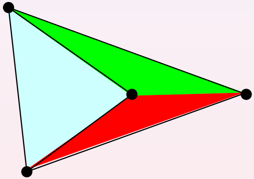

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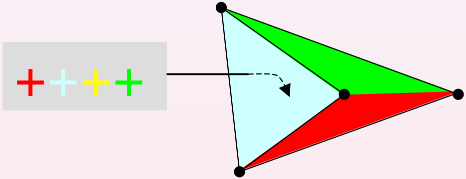

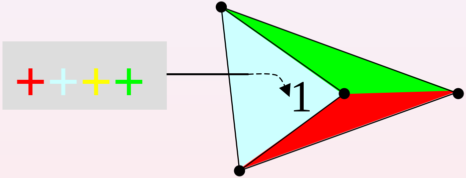
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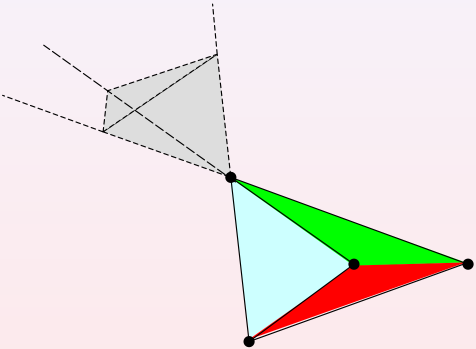

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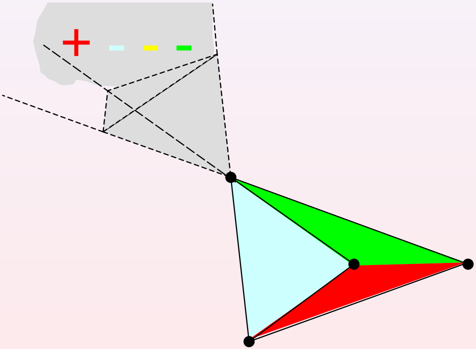

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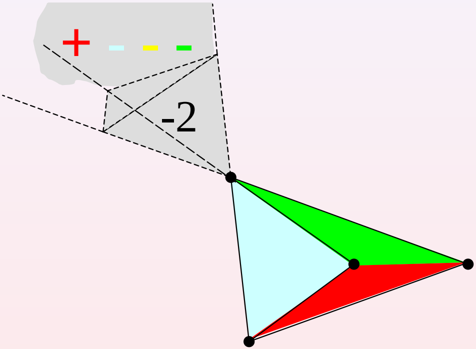

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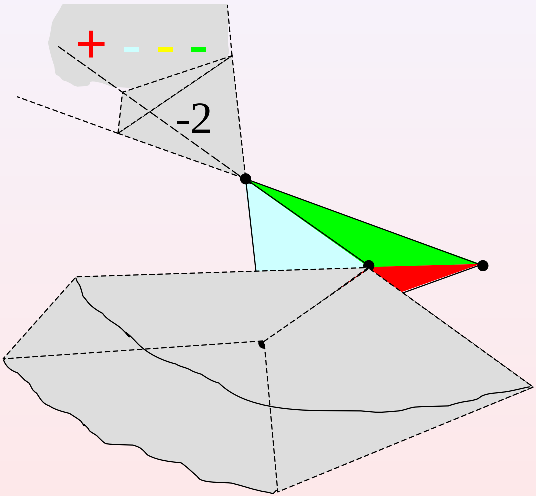

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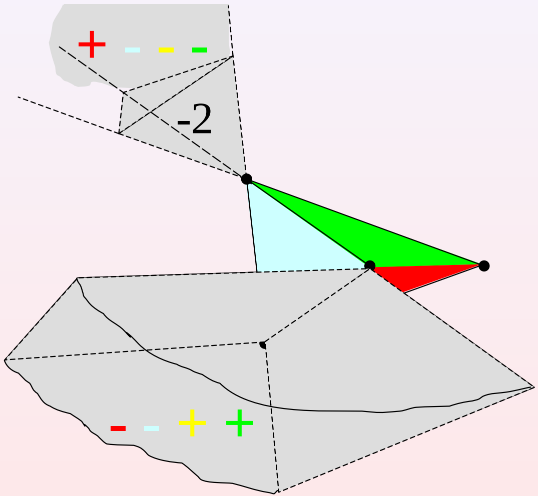

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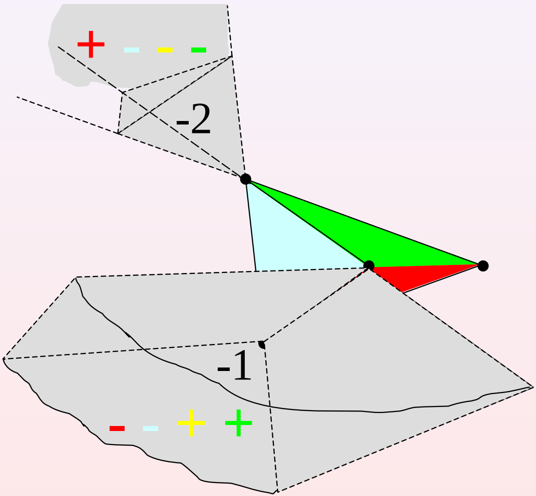

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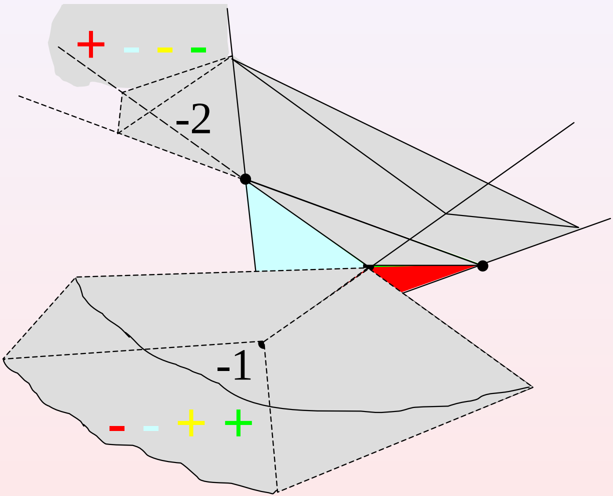

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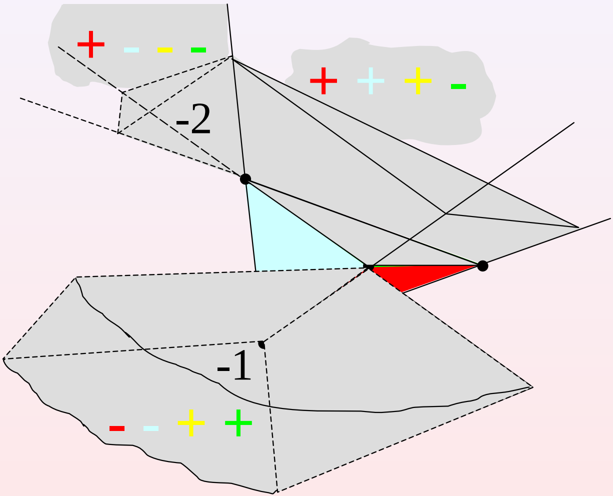

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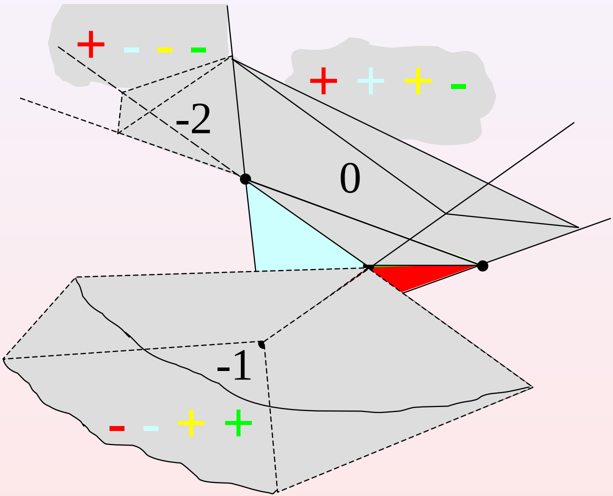

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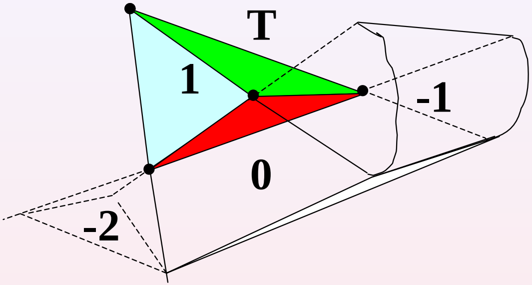

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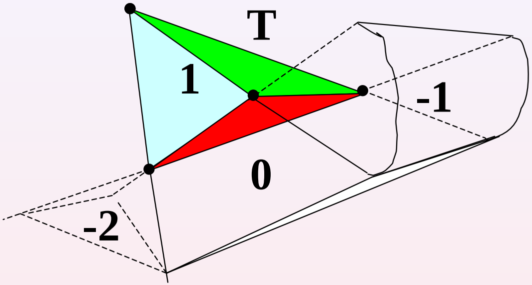

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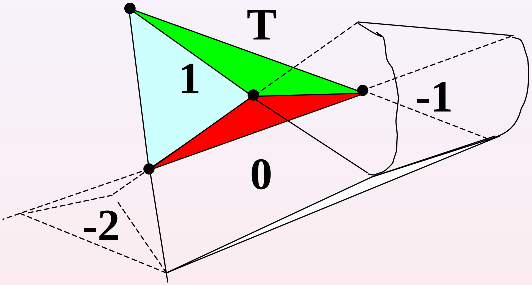

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For any $P \in \mathbb{R}^n$, if p belongs to the skew back-projection of a k-dimensional face of P, then it proves to be $f(p) = k + 1 - n$, and the additivity of P still follows.

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For a set S of directions in $\mathbb{R}^n$, let $\mathcal{P}_S$ be the set of convex polytopes whose facets are parallel to some direction in S.

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Theorem. Let S be a set of non-parallel directions in $\mathbb{R}^n$, $n \geq 2$, and let $P \in \mathcal{P}_S$. Then P is uniquely determined among all measurable sets by its (1-dimensional) X-rays in the directions in S.

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Proof. Let $E \subset \mathbb{R}^n$ a measurable set with the same X-rays as P in the directions in S.

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Proof. Let $E \subset \mathbb{R}^n$ a measurable set with the same X-rays as P in the directions in S.

Let $\mathcal{H}_P$ be the set of $n-1$-dimensional bounding subspaces of P.

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Since $P \in \mathcal{P}_S$ then, for each $H \in \mathcal{H}_P$ there exists $\mathbf{u}_H \in S \cap H$ such that

$$\lambda_1(L(x, \mathbf{u}_H) \cap E) = \lambda_1(L(x, \mathbf{u}_H) \cap P),$$

for all $x \in \mathbb{R}^n$ (λ_1=1-dimensional Lebesgue measure).

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By the Cavalieri principle we get $\lambda_{n-1}(E \cap F) = \lambda_{n-1}(P \cap F)$ for all hyperplane F parallel to H, for all $H \in \mathcal{H}_P$ (λ_{n-1}=$n-1$ dimensional Lebesgue measure).

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Remark. The result also holds in the n-dimensional integer lattice $\mathbb{Z}^n$ ($\mathcal{H}$ is a Radon base).

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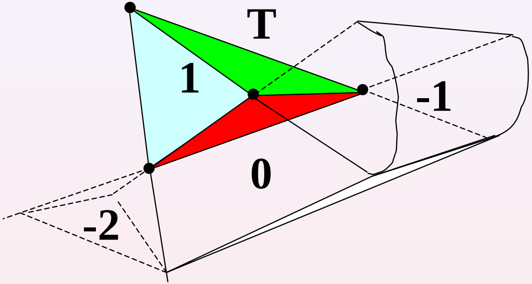

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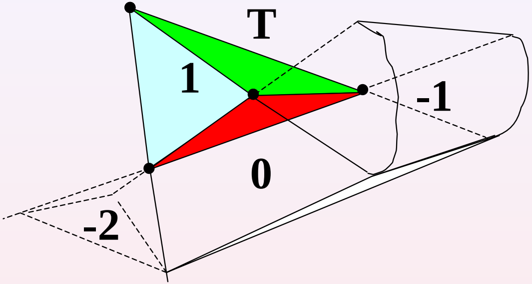

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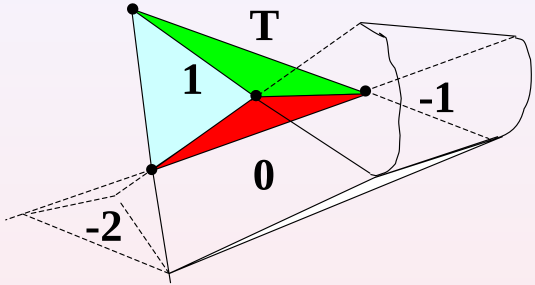

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If the polytopes are carefully selected, the resulting cluster of twisted polytopes is still an additive set.

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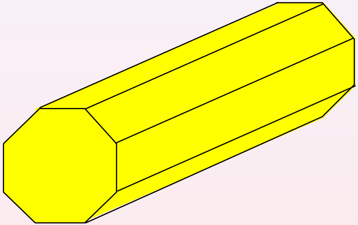

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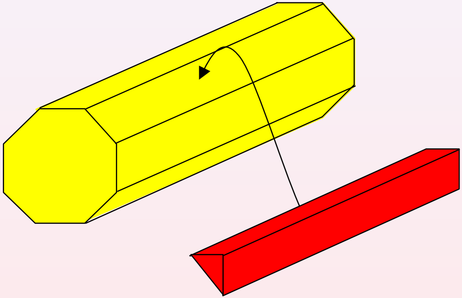

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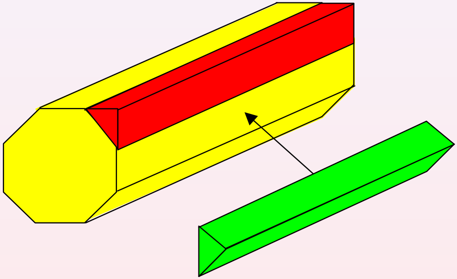

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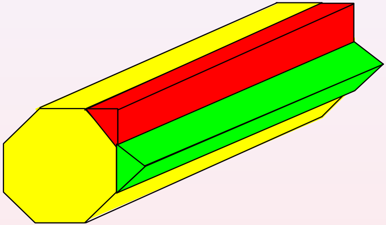

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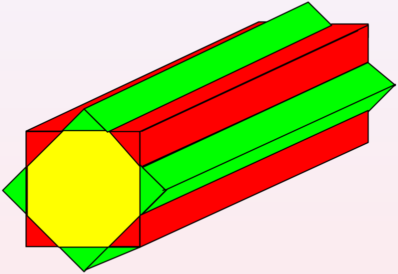

Theorem. Let $C \subset \mathbb{R}^n$ be a cluster of twisted polytopes $P_1, ..., P_r$. Denote by $\mathcal{H}$ the set of all the bounding subspaces of $P_1, ..., P_r$. Then C is $\mathcal{H}$-additive.

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The result also holds in the n-dimensional integer lattice $\mathbb{Z}^n$.

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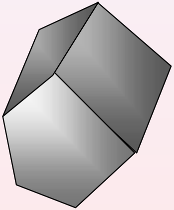

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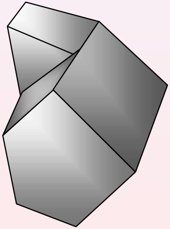

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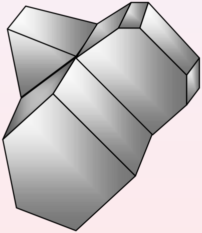

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This is related to the notion of inscribability.

For a finite set $\mathcal{D}$ of directions in $\mathbb{R}^2$, a convex body $K \subset \mathbb{R}^2$ is $\mathcal{D}$-*inscribable* if its interior is the union of interiors of convex polygons inscribed in K, each of whose edges is parallel to some direction in $\mathcal{D}$.

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Theorem Let $\mathcal{D}$ be a finite set of at least two nonparallel lattice directions. Then the class of non-degenerate $\mathcal{D}$-inscribable sets is $\mathcal{D}$-unique. [D.-Peri, 2011]
This seems to be interesting in view of applications to sections of non-convex bodies