On the distribution of the ψ_2-norm of linear functionals on isotropic convex bodies

Joint work with G. Paouris and P. Valettas

Cortona 2011

June 16, 2011

Isotropic convex bodies

Isotropic convex bodies

A convex body K in $\mathbb{R}^n$ is called isotropic if it has volume 1, it is centered (i.e. it has its center of mass at the origin), and there exists a constant $L_K > 0$ such that

$$\int_K \langle x, \theta \rangle^2 dx = L_K^2$$

for every $\theta \in S^{n-1}$.

Isotropic convex bodies

Isotropic convex bodies

A convex body K in $\mathbb{R}^n$ is called isotropic if it has volume 1, it is centered (i.e. it has its center of mass at the origin), and there exists a constant $L_K > 0$ such that

$$\int_K \langle x, \theta \rangle^2 dx = L_K^2$$

for every $\theta \in S^{n-1}$.

Hyperplane conjecture

There exists an absolute constant $C > 0$ such that $L_K \leq C$ for every isotropic convex body K.

Bourgain: $L_K \leq c\sqrt[4]{n} \log n$, Klartag: $L_K \leq c\sqrt[4]{n}$.

Subgaussian directions

ψ_α-norm

If $\alpha \in [1, 2]$ and $f : K \to \mathbb{R}$ is bounded, then

$$\|f\|_{\psi_\alpha} = \inf \left\{ b > 0 : \int_K \exp\left((|f(x)|/b)^\alpha \right) \, dx \leq 2 \right\}.$$

Subgaussian directions

ψ_α-norm

If $\alpha \in [1,2]$ and $f : K \to \mathbb{R}$ is bounded, then

$$\|f\|_{\psi_\alpha} = \inf \left\{ b > 0 : \int_K \exp\left((|f(x)|/b)^\alpha\right)\, dx \leq 2 \right\}.$$

Subgaussian directions

We say that $\theta \in S^{n-1}$ is a subgaussian direction for K with constant $r > 0$ if

$$\|\langle \cdot, \theta\rangle\|_{\psi_2} \leq r \|\langle \cdot, \theta\rangle\|_2 = rL_K.$$

Subgaussian directions

ψ_α-norm

If $\alpha \in [1,2]$ and $f : K \to \mathbb{R}$ is bounded, then

$$\|f\|_{\psi_\alpha} = \inf \left\{ b > 0 : \int_K \exp\left((|f(x)|/b)^\alpha \right) dx \leq 2 \right\}.$$

Subgaussian directions

We say that $\theta \in S^{n-1}$ is a subgaussian direction for K with constant $r > 0$ if

$$\|\langle \cdot, \theta \rangle\|_{\psi_2} \leq r \|\langle \cdot, \theta \rangle\|_2 = r L_K.$$

Equivalent description: for all $t \geq 1$,

$$|\{x \in K : |\langle x, \theta \rangle| \geq c t L_K\}| \leq e^{-\frac{t^2}{r^2}}.$$

Question (existence)

Milman

Does every isotropic convex body K have at least one subgaussian direction with constant $r \leq C$?

Question (existence)

Milman

Does every isotropic convex body K have at least one subgaussian direction with constant $r \leq C$?

Unconditional case: Bobkov-Nazarov

If K is an isotropic 1-unconditional convex body in $\mathbb{R}^n$, then

$$\|\langle \cdot, \theta \rangle\|_{\psi_2} \leq c\sqrt{n}\|\theta\|_\infty$$

for every $\theta \in S^{n-1}$.
Then, the diagonal direction is a subgaussian direction.

Answer in the general case

Theorem (G-Paouris-Valettas)

Let K be an isotropic convex body in $\mathbb{R}^n$. There exists $\theta \in S^{n-1}$ such that

$$|\{x \in K : |\langle x, \theta \rangle| \geq ctL_K\}| \leq e^{-\frac{t^2}{\log(t+1)}}$$

for all $t \geq 1$, where $c > 0$ is an absolute constant.

Answer in the general case

Theorem (G-Paouris-Valettas)

Let K be an isotropic convex body in $\mathbb{R}^n$. There exists $\theta \in S^{n-1}$ such that

$$\left|\{x \in K : |\langle x, \theta \rangle| \geq ctL_K\}\right| \leq e^{-\frac{t^2}{\log(t+1)}}$$

for all $t \geq 1$, where $c > 0$ is an absolute constant.

There were previous results by Klartag and G-Pajor-Paouris in 2006.

Answer in the general case

Theorem (G-Paouris-Valettas)

Let K be an isotropic convex body in $\mathbb{R}^n$. There exists $\theta \in S^{n-1}$ such that

$$|\{x \in K : |\langle x, \theta \rangle| \geq ctL_K\}| \leq e^{-\frac{t^2}{\log(t+1)}}$$

for all $t \geq 1$, where $c > 0$ is an absolute constant.

There were previous results by Klartag and G-Pajor-Paouris in 2006.

Equivalent form

There exists $\theta \in S^{n-1}$ such that

$$\|\langle \cdot, \theta \rangle\|_{\psi_2} \leq c\sqrt{\log n} L_K.$$

The body $\Psi_2(K)$

$\Psi_2(K)$

We define a convex body $\Psi_2(K)$ with support function

$$h_{\Psi_2(K)}(\theta) := \sup_{2 \leq q \leq n} \frac{\|\langle \cdot, \theta \rangle\|_q}{\sqrt{q}}.$$

One can check that

$$h_{\Psi_2(K)}(\theta) \simeq \|\langle \cdot, \theta \rangle\|_{\psi_2}.$$

The body $\Psi_2(K)$

$\Psi_2(K)$

We define a convex body $\Psi_2(K)$ with support function

$$h_{\Psi_2(K)}(\theta) := \sup_{2\leq q\leq n} \frac{\|\langle \cdot, \theta\rangle\|_q}{\sqrt{q}}.$$

One can check that

$$h_{\Psi_2(K)}(\theta) \simeq \|\langle \cdot, \theta\rangle\|_{\psi_2}.$$

Theorem (G-Paouris-Valettas)

Let K be an isotropic convex body in $\mathbb{R}^n$. Then,

$$c_1 L_K \leq \left(\frac{|\Psi_2(K)|}{|B_2^n|}\right)^{1/n} \leq c_2\sqrt{\log n}L_K,$$

where $c_1, c_2 > 0$ are absolute constants.

L_q-centroid bodies

Definition

For every $q \geq 1$ and $y \in \mathbb{R}^n$ we define

$$h_{Z_q(K)}(y) := \|\langle \cdot, y\rangle\|_{L_q(K)} = \left(\int_K |\langle x, y\rangle|^q dx\right)^{1/q}.$$

L_q-centroid bodies

Definition

For every $q \geq 1$ and $y \in \mathbb{R}^n$ we define

$$h_{Z_q(K)}(y) := \|\langle \cdot, y \rangle\|_{L_q(K)} = \left(\int_K |\langle x, y \rangle|^q dx \right)^{1/q}.$$

Basic facts

- K is isotropic if and only if $Z_2(K) = L_K B_2^n$.
- $Z_1(K) \subseteq Z_p(K) \subseteq Z_q(K) \subseteq Z_\infty(K)$ for every $1 \leq p \leq q \leq \infty$, where $Z_\infty(K) = \mathrm{conv}\{K, -K\}$.
- $Z_q(K) \subseteq c\frac{q}{p} Z_p(K)$ for all $1 \leq p < q$.
- If K is centered, then $Z_q(K) \supseteq c_1 K$ for all $q \geq n$.

Covering numbers of $Z_q(K)$

Theorem (G-Paouris-Valettas)

Let K be an isotropic convex body in $\mathbb{R}^n$, let $1 \leq q \leq n$ and $t \geq 1$. Then,

$$\log N\left(Z_q(K), c_1 t\sqrt{q} L_K B_2^n\right) \leq c_2 \frac{n}{t^2} + c_3 \frac{\sqrt{qn}}{t},$$

where $c_1, c_2, c_3 > 0$ are absolute constants.

Note that the upper bound is of the order n/t^2 if $t \leq \sqrt{n/q}$ and of the order $\sqrt{qn}/t$ if $t \geq \sqrt{n/q}$.

Covering numbers of projections

In an analogous way we can obtain upper bounds for the covering numbers of $P_F(Z_q(K))$, where $F \in G_{n,k}$.

Proposition

Let K be an isotropic convex body in $\mathbb{R}^n$. For every $1 \leq q < k \leq n$, for every $F \in G_{n,k}$ and every $t \geq 1$, we have

$$\log N\left(P_F(Z_q(K)), t\sqrt{q}L_K B_F\right) \leq \frac{c_1 k}{t^2} + \frac{c_2\sqrt{qk}}{t},$$

where $c_1, c_2 > 0$ are absolute constants. Also, for every $k \leq q \leq n$, $F \in G_{n,k}$ and $t \geq 1$,

$$\log N\left(P_F(Z_q(K)), t\sqrt{q}L_K B_F\right) \leq \frac{c_3\sqrt{qk}}{t},$$

where $c_3 > 0$ is an absolute constant.

ψ_2-directions in a given subspace

Using these bounds we can prove the existence of directions with relatively small ψ_2-norm on any subspace of $\mathbb{R}^n$. The dependence is better as the dimension increases.

Theorem (G-Paouris-Valettas)

Let K be an isotropic convex body in $\mathbb{R}^n$.

- *For every $1 \leq k \leq n/\log n$ and every $F \in G_{n,k}$, there exists $\theta \in S_F$ such that*

$$\|\langle \cdot, \theta\rangle\|_{\psi_2} \leq C\sqrt{n/k}\, L_K$$

- *For every $n/\log n \leq k \leq n$ and every $F \in G_{n,k}$, there exists $\theta \in S_F$ such that*

$$\|\langle \cdot, \theta\rangle\|_{\psi_2} \leq C\sqrt{\log n}\, L_K.$$

Sketch of the proof

We consider the convex body

$$T = \mathrm{conv}\left(\bigcup_{i=1}^{\lfloor \log_2 n \rfloor} \frac{Z_{2^i}(K)}{2^{i/2} L_K}\right) \simeq \Psi_2(K).$$

Our aim: to show that $N(P_F(T), \sqrt{n/k} B_F) \leq e^{ck}$.

Sketch of the proof

We consider the convex body

$$T = \mathrm{conv}\left(\bigcup_{i=1}^{\lfloor \log_2 n \rfloor} \frac{Z_{2^i}(K)}{2^{i/2} L_K}\right) \simeq \Psi_2(K).$$

Our aim: to show that $N(P_F(T), \sqrt{n/k} B_F) \leq e^{ck}$. Then,

$$|P_F(T)| \leq |C\sqrt{n/k}\, B_F|,$$

Sketch of the proof

We consider the convex body

$$T = \mathrm{conv}\left(\bigcup_{i=1}^{\lfloor \log_2 n \rfloor} \frac{Z_{2^i}(K)}{2^{i/2} L_K}\right) \simeq \Psi_2(K).$$

Our aim: to show that $N(P_F(T), \sqrt{n/k} B_F) \leq e^{ck}$. Then,

$$|P_F(T)| \leq |C\sqrt{n/k}\, B_F|,$$

and hence, there exists $\theta \in S_F$ such that

$$h_T(\theta) = h_{P_F(T)}(\theta) \leq C\sqrt{n/k},$$

Sketch of the proof

We consider the convex body

$$T = \mathrm{conv}\left(\bigcup_{i=1}^{\lfloor \log_2 n \rfloor} \frac{Z_{2^i}(K)}{2^{i/2} L_K}\right) \simeq \Psi_2(K).$$

Our aim: to show that $N(P_F(T), \sqrt{n/k} B_F) \leq e^{ck}$. Then,

$$|P_F(T)| \leq |C\sqrt{n/k}\, B_F|,$$

and hence, there exists $\theta \in S_F$ such that

$$h_T(\theta) = h_{P_F(T)}(\theta) \leq C\sqrt{n/k},$$

which implies

$$\|\langle \cdot, \theta\rangle\|_{L_q(K)} \leq C\sqrt{q}\,\sqrt{n/k}\, L_K$$

for every $1 \leq q = 2^i \leq n$.

Sketch of the proof (contd.)

Recall that we want to show that $N(P_F(T), \sqrt{n/k}B_F) \leq e^{ck}$, where

$$P_F(T) = \mathrm{conv}\left(\bigcup_{i=1}^{\lfloor \log_2 n \rfloor} P_F\left(\frac{Z_{2^i}(K)}{2^{i/2}L_K}\right)\right).$$

Sketch of the proof (contd.)

Recall that we want to show that $N(P_F(T), \sqrt{n/k}B_F) \leq e^{ck}$, where

$$P_F(T) = \mathrm{conv}\left(\bigcup_{i=1}^{\lfloor \log_2 n \rfloor} P_F\left(\frac{Z_{2^i}(K)}{2^{i/2}L_K}\right)\right).$$

We will use a standard fact: If $A_1, \dots, A_s$ are subsets of RB_2^k, then for every $t > 0$ we have

$$N(\mathrm{conv}(A_1 \cup \cdots \cup A_s), 2tB_2^k) \leq \left(\frac{cR}{t}\right)^s \prod_{i=1}^{s} N(A_i, tB_2^k).$$

Sketch of the proof (contd.)

- We apply this to the sets $A_i = P_F\left(\frac{Z_{2^i}(K)}{2^{i/2}L_K}\right)$.

Then, for every $t \geq 1$ we can write

$$
\begin{aligned}
N(P_F(T), 2tB_F) &\leq (c_2\sqrt{n})^{\lfloor \log_2 n \rfloor} \left[\prod_{i=1}^{\lfloor \log_2 n \rfloor} N\left(P_F\left(\frac{Z_{2^i}(K)}{2^{i/2}L_K} \right), tB_F \right) \right] \\
&\leq e^{c_3 \log^2 n} \exp\left(C \sum_{i=1}^{\lfloor \log_2 n \rfloor} \frac{2^{i/2}\sqrt{k}}{t} + C \sum_{t^2 \leq 2^i \leq k} \frac{k}{t^2} \right) \\
&\leq e^{c_3 \log^2 n} \exp\left(C\frac{\sqrt{nk}}{t} + C\frac{k}{t^2}\log(k/t^2) \right),
\end{aligned}
$$

where the second term appears only if $k \geq ct^2$.

Sketch of the proof (contd.)

- We apply this to the sets $A_i = P_F\left(\frac{Z_{2^i}(K)}{2^{i/2}L_K}\right)$.

Then, for every $t \geq 1$ we can write

$$
\begin{aligned}
N(P_F(T), 2tB_F) &\leq (c_2\sqrt{n})^{\lfloor \log_2 n \rfloor} \left[\prod_{i=1}^{\lfloor \log_2 n \rfloor} N\left(P_F\left(\frac{Z_{2^i}(K)}{2^{i/2}L_K}\right), tB_F\right)\right] \\
&\leq e^{c_3 \log^2 n} \exp\left(C \sum_{i=1}^{\lfloor \log_2 n \rfloor} \frac{2^{i/2}\sqrt{k}}{t} + C \sum_{t^2 \leq 2^i \leq k} \frac{k}{t^2}\right) \\
&\leq e^{c_3 \log^2 n} \exp\left(C\frac{\sqrt{nk}}{t} + C\frac{k}{t^2}\log(k/t^2)\right),
\end{aligned}
$$

where the second term appears only if $k \geq ct^2$. **Choose** $t_0 = \sqrt{n/k}$.

Question (distribution)

$\psi_K(t)$

We introduce the function

$$\psi_K(t) := \sigma\left(\{\theta \in S^{n-1} : h_{\Psi_2(K)}(\theta) \leq ct\sqrt{\log n}L_K\}\right).$$

The problem is to give lower bounds for $\psi_K(t)$, $t \geq 1$.

Question (distribution)

$\psi_K(t)$

We introduce the function

$$\psi_K(t) := \sigma\left(\{\theta \in S^{n-1} : h_{\Psi_2(K)}(\theta) \leq ct\sqrt{\log n}L_K\}\right).$$

The problem is to give lower bounds for $\psi_K(t)$, $t \geq 1$.

A measure estimate (G-Paouris-Valettas)

Let K be an isotropic convex body in $\mathbb{R}^n$. For every $t \geq 1$ we have

$$\psi_K(t) \geq \exp(-cn/t^2),$$

where $c > 0$ is an absolute constant.

Sketch of the proof

Recall that for every $F \in G_{n,k}$ there exists $\theta \in S_F$ such that

$$\|\langle \cdot, \theta \rangle\|_{\psi_2} \leq C\sqrt{n/k}\, L_K.$$

Sketch of the proof

Recall that for every $F \in G_{n,k}$ there exists $\theta \in S_F$ such that

$$\|\langle \cdot, \theta \rangle\|_{\psi_2} \leq C \sqrt{n/k} \, L_K.$$

Given $t \geq 1$, we solve the equation $t = \sqrt{n/k}$ in k.

Sketch of the proof

Recall that for every $F \in G_{n,k}$ there exists $\theta \in S_F$ such that

$$\|\langle \cdot, \theta \rangle\|_{\psi_2} \leq C \sqrt{n/k}\, L_K.$$

Given $t \geq 1$, we solve the equation $t = \sqrt{n/k}$ in k.
Then, we know that the set

$$A = \{\theta \in S^{n-1} : h_{\Psi_2(K)}(\theta) \leq ct\sqrt{\log n} L_K\}$$

intersects S_F for every $F \in G_{n,k}$.

Sketch of the proof (contd.)

Simple lemma

Let $1 \le k \le n$ and let A be a subset of S^{n-1} which satisfies $A \cap F \neq \emptyset$ for every $F \in G_{n,k}$. Then, for every $\varepsilon > 0$ we have

$$\sigma(A_\varepsilon) \ge \frac{1}{2}\left(\frac{\varepsilon}{2}\right)^{k-1},$$

where $A_\varepsilon = \left\{ y \in S^{n-1} : \inf\{\|y-\theta\|_2 : \theta \in A\} \le \varepsilon \right\}$.

Sketch of the proof (contd.)

Simple lemma

Let $1 \leq k \leq n$ and let A be a subset of S^{n-1} which satisfies $A \cap F \neq \emptyset$ for every $F \in G_{n,k}$. Then, for every $\varepsilon > 0$ we have

$$\sigma(A_\varepsilon) \geq \frac{1}{2}\left(\frac{\varepsilon}{2}\right)^{k-1},$$

where $A_\varepsilon = \left\{y \in S^{n-1} : \inf\{\|y-\theta\|_2 : \theta \in A\} \leq \varepsilon\right\}$.

We apply the Lemma with $\varepsilon = \frac{1}{\sqrt{k}}$. If $y \in A_\varepsilon$, then

$$\|\langle \cdot, y\rangle\|_{\psi_2} \leq C\sqrt{n/k}\, L_K \simeq tL_K.$$

Simple lemma

Let $1 \leq k \leq n$ and let A be a subset of S^{n-1} which satisfies $A \cap F \neq \emptyset$ for every $F \in G_{n,k}$. Then, for every $\varepsilon > 0$ we have

$$\sigma(A_\varepsilon) \geq \frac{1}{2}\left(\frac{\varepsilon}{2}\right)^{k-1},$$

where $A_\varepsilon = \left\{y \in S^{n-1} : \inf\{\|y-\theta\|_2 : \theta \in A\} \leq \varepsilon\right\}$.

We apply the Lemma with $\varepsilon = \frac{1}{\sqrt{k}}$. If $y \in A_\varepsilon$, then

$$\|\langle \cdot, y\rangle\|_{\psi_2} \leq C\sqrt{n/k}\, L_K \simeq tL_K.$$

Note that

$$\sigma(A_\varepsilon) \geq e^{-c_1 k \log k} \geq e^{-c_2 n/t^2}.$$

Sketch of the proof (contd.)

Simple lemma

Let $1 \leq k \leq n$ and let A be a subset of S^{n-1} which satisfies $A \cap F \neq \emptyset$ for every $F \in G_{n,k}$. Then, for every $\varepsilon > 0$ we have

$$\sigma(A_\varepsilon) \geq \frac{1}{2}\left(\frac{\varepsilon}{2}\right)^{k-1},$$

where $A_\varepsilon = \left\{y \in S^{n-1} : \inf\{\|y-\theta\|_2 : \theta \in A\} \leq \varepsilon\right\}$.

We apply the Lemma with $\varepsilon = \frac{1}{\sqrt{k}}$. If $y \in A_\varepsilon$, then

$$\|\langle \cdot, y\rangle\|_{\psi_2} \leq C\sqrt{n/k}\, L_K \simeq tL_K.$$

Note that

$$\sigma(A_\varepsilon) \geq e^{-c_1 k \log k} \geq e^{-c_2 n/t^2}.$$

It follows that

$$\psi_K(t) \geq \sigma(A_\varepsilon) \geq e^{-c_2 n/t^2}.$$

Large values of t

Theorem (G-Paouris-Valettas)

Let K be an isotropic convex body in $\mathbb{R}^n$. For every $t \geq c_1 \sqrt[4]{n}/\sqrt{\log n}$ one has

$$\psi_K(t) \geq 1 - e^{-c_2 t^2 \log n},$$

where $c_1, c_2 > 0$ are absolute constants.

Large values of t

Theorem (G-Paouris-Valettas)

Let K be an isotropic convex body in $\mathbb{R}^n$. For every $t \geq c_1 \sqrt[4]{n}/\sqrt{\log n}$ one has

$$\psi_K(t) \geq 1 - e^{-c_2 t^2 \log n},$$

where $c_1, c_2 > 0$ are absolute constants.

This follows from the estimate

$$w(\Psi_2(K)) \leq c\sqrt[4]{n} L_K.$$

Since $h_{\Psi_2(K)}$ is $\sqrt{n}L_K$-Lipschitz, we have that

$$\sigma\left(\{\theta \in S^{n-1} : h_{\Psi_2(K)}(\theta) - w(\Psi_2(K)) \geq s w(\Psi_2(K))\}\right) \leq e^{-cns^2\left(\frac{w(\Psi_2(K))}{\sqrt{n}L_K}\right)^2}.$$

Mean width I

Deeper understanding of the function $\psi_K(t)$ would have important applications.

Problem

To give an upper bound for the mean width of an isotropic convex body K in $\mathbb{R}^n$.

- From the inclusion $K \subseteq [(n+1)L_K]B_2^n$, one has the obvious bound $w(K) \leq cnL_K$.

Mean width I

Deeper understanding of the function $\psi_K(t)$ would have important applications.

Problem

To give an upper bound for the mean width of an isotropic convex body K in $\mathbb{R}^n$.

- From the inclusion $K \subseteq [(n+1)L_K]B_2^n$, one has the obvious bound $w(K) \leq cnL_K$.
- It is known that

$$w(K) \leq cn^{3/4}L_K.$$

Several approaches: Hartzoulaki, Pivovarov, "Z_q-bound".

Mean width I

Deeper understanding of the function $\psi_K(t)$ would have important applications.

Problem

To give an upper bound for the mean width of an isotropic convex body K in $\mathbb{R}^n$.

- From the inclusion $K \subseteq [(n+1)L_K]B_2^n$, one has the obvious bound $w(K) \leq cnL_K$.
- It is known that

$$w(K) \leq cn^{3/4}L_K.$$

  Several approaches: Hartzoulaki, Pivovarov, "Z_q-bound".
- None of them leads to an exponent smaller than $3/4$.

Connection with $\psi_K(t)$

We are interested in

$$w(K) = \int_{S^{n-1}} h_K(\theta)\, d\sigma(\theta).$$

Connection with $\psi_K(t)$

We are interested in

$$w(K) = \int_{S^{n-1}} h_K(\theta)\, d\sigma(\theta).$$

Observe that

$$h_K(\theta) \simeq \|\langle \cdot, \theta \rangle\|_{L_n(K)} \leq c\sqrt{n}\, \|\langle \cdot, \theta \rangle\|_{\psi_2}.$$

Connection with $\psi_K(t)$

We are interested in

$$w(K) = \int_{S^{n-1}} h_K(\theta)\, d\sigma(\theta).$$

Observe that

$$h_K(\theta) \simeq \|\langle \cdot, \theta\rangle\|_{L_n(K)} \leq c\sqrt{n}\, \|\langle \cdot, \theta\rangle\|_{\psi_2}.$$

The question is to solve the equation $\psi_K(t) = 1 - \frac{1}{n}$ in t.

Connection with $\psi_K(t)$

We are interested in

$$w(K) = \int_{S^{n-1}} h_K(\theta)\, d\sigma(\theta).$$

Observe that

$$h_K(\theta) \simeq \|\langle \cdot, \theta\rangle\|_{L_n(K)} \leq c\sqrt{n}\,\|\langle \cdot, \theta\rangle\|_{\psi_2}.$$

The question is to solve the equation $\psi_K(t) = 1 - \frac{1}{n}$ in t.
Our estimates:

- $\psi_K(t) \geq \exp(-cn/t^2)$ for all $t \geq 1$.
- $\psi_K(t) \geq 1 - e^{-c_2 t^2 \log n}$ for all $t \geq c_1 \sqrt[4]{n}/\sqrt{\log n}$.

Mean width II

Minimal surface area position

- A convex body K of volume 1 has *minimal surface area* if $\partial(K) \leq \partial(TK)$ for every $T \in SL(n)$.

Mean width II

Minimal surface area position

- A convex body K of volume 1 has *minimal surface area* if $\partial(K) \leq \partial(TK)$ for every $T \in SL(n)$.
- Petty: K has minimal surface area if and only if σ_K is isotropic.

Mean width II

Minimal surface area position

- A convex body K of volume 1 has *minimal surface area* if $\partial(K) \leq \partial(TK)$ for every $T \in SL(n)$.
- Petty: K has minimal surface area if and only if σ_K is isotropic.
- Equivalently, if

$$\partial(K) = n \int_{S^{n-1}} \langle u, \theta \rangle^2 \sigma_K(du)$$

for every $\theta \in S^{n-1}$.

Mean width II

Minimal surface area position

- A convex body K of volume 1 has *minimal surface area* if $\partial(K) \leq \partial(TK)$ for every $T \in SL(n)$.
- Petty: K has minimal surface area if and only if σ_K is isotropic.
- Equivalently, if

$$\partial(K) = n \int_{S^{n-1}} \langle u, \theta \rangle^2 \sigma_K(du)$$

for every $\theta \in S^{n-1}$.

Question

How large can $w(K)$ be when K has volume 1 and minimal surface area?

Answer in the symmetric case

Theorem (Markessinis-Paouris-Saroglou)

Let K be a symmetric convex body of volume 1 in $\mathbb{R}^n$ which has minimal surface area. Then,

$$w(K) \leq C\frac{n^{3/2}}{\partial_K}.$$

Answer in the symmetric case

Theorem (Markessinis-Paouris-Saroglou)

Let K be a symmetric convex body of volume 1 in $\mathbb{R}^n$ which has minimal surface area. Then,

$$w(K) \leq C\frac{n^{3/2}}{\partial_K}.$$

Since $\partial_K \geq c\sqrt{n}$, we get $w(K) \leq Cn$.

Answer in the symmetric case

Theorem (Markessinis-Paouris-Saroglou)

Let K be a symmetric convex body of volume 1 in $\mathbb{R}^n$ which has minimal surface area. Then,

$$w(K) \leq C \frac{n^{3/2}}{\partial_K}.$$

Since $\partial_K \geq c\sqrt{n}$, we get $w(K) \leq Cn$.

The result is sharp

There exists an unconditional convex body Q of volume 1 in $\mathbb{R}^n$ which has minimal surface area and satisfies

$$w(Q) \geq \frac{cn}{\log n}.$$

Answer in the symmetric case

Theorem (Markessinis-Paouris-Saroglou)

Let K be a symmetric convex body of volume 1 in $\mathbb{R}^n$ which has minimal surface area. Then,

$$w(K) \leq C\frac{n^{3/2}}{\partial_K}.$$

Since $\partial_K \geq c\sqrt{n}$, we get $w(K) \leq Cn$.

The result is sharp

There exists an unconditional convex body Q of volume 1 in $\mathbb{R}^n$ which has minimal surface area and satisfies

$$w(Q) \geq \frac{cn}{\log n}.$$

Actually, $Q := a\overline{B}_1^k \times bC_m$ for suitable $a, b > 0$.