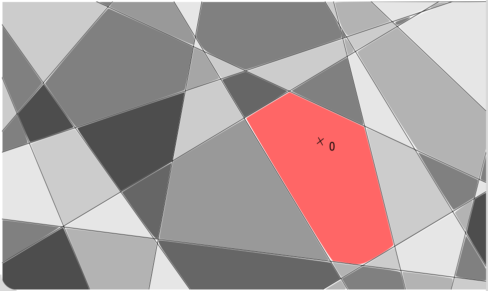
Random tessellations - some high-dimensional aspects

Daniel Hug | June 13, 2011

CORTONA

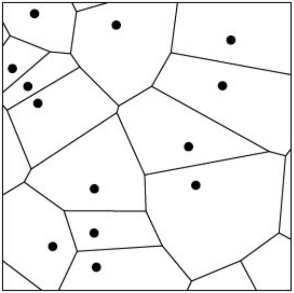


Voronoi tessellation

Let $\widetilde{X} \subset \mathbb{R}^n$ denote a point set in general position. For $x \in \widetilde{X}$,

$$C(\widetilde{X}, x) := \{y \in \mathbb{R}^n : d(y, x) \leq \mathrm{dist}(y, \widetilde{X})\}$$

is the Voronoi cell of $\widetilde{X}$ with centre (nucleus) x. The collection X of all these cells constitutes a Voronoi tessellation.



Poisson Voronoi tessellation

Let $\widetilde{X}$ be a stationary Poisson point process in $\mathbb{R}^n$ with intensity λ.

This is a random collection of points in space such that, for $A \subset \mathbb{R}^n$, the random variable $\widetilde{X}(A) := |\widetilde{X} \cap A|$ follows a Poisson distribution with Poisson parameter $\lambda \cdot V_d(A) = \mathbb{E}\widetilde{X}(A)$.

The constant $\lambda \geq 0$ is the intensity of $\widetilde{X}$.

The induced random Voronoi tessellation $X := \{C(\widetilde{X}, x) : x \in \widetilde{X}\}$ is called **Poisson Voronoi tessellation** (PVT). It also has intensity λ.

Typical Poisson Voronoi cell

Let X be a stationary PVT with intensity λ.

> *A 'uniform random selection' of one cell Z from the collection of infinitely many cells of X, after translation of the cell so that its nucleus is at the origin, is called typical cell of X.*

Let $B \subset \mathbb{R}^n$ with $\lambda^n(B) = 1$. The distribution of the **typical cell** Z of X is

$$\mathbb{P}\{Z \in \cdot\} := \frac{1}{\lambda} \cdot \mathbb{E} \sum_{x \in \widetilde{X}} \mathbf{1}\{C(\widetilde{X}, x) - x \in \cdot\} \mathbf{1}_B(x).$$

A characteristic property of Poisson processes (due to J. Mecke) and the translation invariance of $\widetilde{X}$ yield

$$\mathbb{P}\{Z \in \cdot\} = \mathbb{P}\{C(\widetilde{X} \cup \{o\}, o) \in \cdot\},$$

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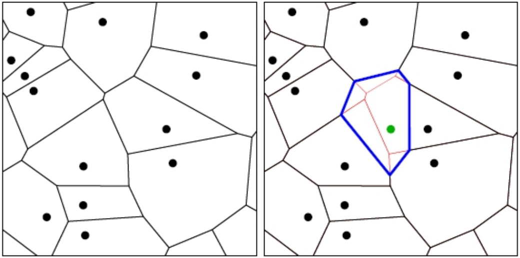
$$\mathbb{P}\{Z \in \cdot\} = \mathbb{P}\{C(\widetilde{X} \cup \{o\}, o) \in \cdot\},$$

that is,

$$Z = C(\widetilde{X} \cup \{o\}, o) = Z_o(Y),$$

where Y is an isotropic but instationary Poisson hyperplane process:

$$Y = \left\{ H\left(\frac{x}{\|x\|}, \frac{1}{2}\|x\| \right) : x \in \widetilde{X} \setminus \{o\} \right\} :$$



Shape of large cells: Kendall's problem

To estimate the **size** of the typical cell Z of X, we can use e.g. **intrinsic volumes**

$$V_1, \dots, V_n$$

or the **centred inradius**

$$R_m.$$

The **deviation from spherical shape** is measured by

$$\vartheta := \frac{R_M - R_m}{R_M + R_m},$$

R_M is the **centred circumradius**.

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Hug, Reitzner, Schneider '04

Let X be a stationary PVT with intensity λ in $\mathbb{R}^n$ and $k \in \{1, \ldots, n\}$. There is a constant $c_0 = c_0(n)$ such that the following is true:

If $\epsilon \in (0, 1)$ and $a \geq 1$, then

$$\mathbb{P}\{\vartheta(Z) \geq \epsilon \mid V_k(Z) \geq a\} \leq c \exp\left\{-c_0 \epsilon^{(n+3)/2} a^{n/k} \lambda\right\}$$

and

$$\mathbb{P}\{\vartheta(Z) \geq \epsilon \mid R_m(Z) \geq a\} \leq c \exp\left\{-c_0 \epsilon^{(n+1)/2} a^{n} \lambda\right\},$$

where $c = c(n, \epsilon)$.

Hug & Schneider '11

Let $\rho \geq 1$, and choose α with

$$0 < \alpha < \tfrac{n-1}{n+1}, \quad \text{so that} \quad \beta := \tfrac{n-1}{2} - \alpha \tfrac{n+1}{2} > 0.$$

Then there exist $c_1 = c_1(n, \gamma)$ and $c_2 = c_2(n)$ such that

$$\mathbb{P}\left\{R_M(Z) \leq \rho + \rho^{-\alpha} \mid R_m(Z) \geq \rho\right\} \geq 1 - c_1 \exp\left\{-c_2 \lambda \rho^{\beta}\right\}.$$

- If $\rho B^n \subset Z$, then $B^n \subset \rho^{-1} Z \subset (1 + \rho^{-1-\alpha}) B^n$ w.h.p.
- $n = 2$: Calka '02

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Random mosaics in high dimensions

(joint with Julia Hörrmann)

K. Alishahi, M. Sharifitabar '08

Let Z denote the typical cell of a Poisson Voronoi tessellation in $\mathbb{R}^n$ with intensity λ. Then, for all $n \in \mathbb{N}$,

$$\mathbb{E}[V_n(Z)] = \frac{1}{\lambda}$$

and

$$c \cdot \frac{1}{\sqrt{n}} \left(\frac{4}{3\sqrt{3}} \right)^n \leq \mathrm{Var}[V_n(Z)] \leq C \cdot \frac{1}{\sqrt{n}} \left(\frac{4}{3\sqrt{3}} \right)^n .$$

Newman, Rinott'85: Convergence in distribution via convergence of moments

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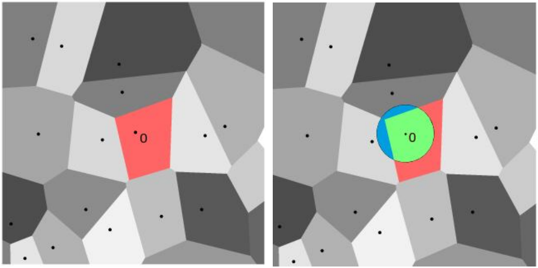
Newman, Rinott'85: Convergence in distribution via convergence of moments

What can we say about the shape of the typcial cell Z?

Idea: consider $V_n(Z \cap B_u^n)$ as a function of the parameter $u \geq 0$



B_u^n: ball centred at o with n-dimensional volume u

K. Alishahi, M. Sharifitabar ’08

$$\mathbb{E}[V_n(Z \cap B_u^n)] = \frac{1}{\lambda}(1 - e^{-\lambda u}), \qquad \text{for } u \in (0, \infty) \text{ and all } n,$$

$$\mathrm{Var}(V_n(Z \cap B_u^n)) \to 0, \qquad \text{as } n \to \infty.$$

Consequence:

$$\mathbb{E}\Delta_s(Z, B_u^n) \geq \ln(2)/\lambda$$

and

$$\Delta_s(Z, B_u^n) - \mathbb{E}\Delta_s(Z, B_u^n) \to 0 \quad \text{in } L^2 \text{ as } n \to \infty.$$

Covariogram of random sets

For a set $Z \subset \mathbb{R}^n$ (random or not), let

$$g_Z(v) := V_n(Z \cap (Z + v)), \qquad v \in \mathbb{R}^n,$$

be the geometric covariogram of Z.

A & S '08: for the typical cell Z of a stationary PVT, $s \geq 0$, $v_n \in \mathbb{S}^{n-1}$,

$$\mathbb{E}[g_Z(sv_n)] \to c(\lambda, s) \qquad \text{as } n \to \infty,$$

where $c(\lambda, s)$ is given by an integral (rather explicitly).

Yao '10:

$$\text{Var}(g_Z(v)) \leq 4 \cdot \text{Var}(V_n(Z)), \qquad v \in \mathbb{R}^n.$$

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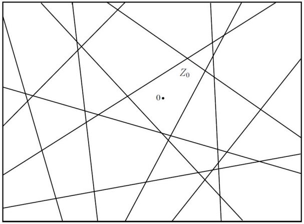
Are there analogous results for the volume of the zero cell of Poisson hyperplane tessellations?

Recall: Random hyperplane systems induce random tessellations. Specifically, we consider a stationary **Poisson hyperplane process** X in $\mathbb{R}^n$ of intensity γ and the induced PHT.



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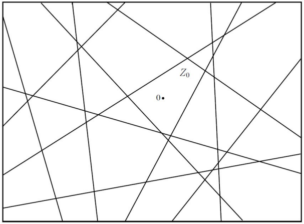

Let Z_o denote the cell containing the origin.

A hyperplane process X in $\mathbb{R}^n$ is a Poisson process if

$$\mathbb{P}\{X(A) = k\} = \frac{\Theta(A)^k}{k!} \cdot e^{-\Theta(A)},$$

for all measurable $A \subset \mathcal{H}$ and $k \in \mathbb{N}_0$ and a locally finite measure Θ on the space $\mathcal{H}$ of hyperplanes.

- In particular, $\mathbb{E}X(A) = \Theta(A)$.
- If X is stationary, then Θ is a translation invariant measure.
- If X is isotropic, then Θ is a rotation invariant measure.
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If X is a stationary (Poisson) hyperplane process in $\mathbb{R}^n$, then its translation invariant **intensity measure** $\mathbb{E}X(\cdot)$ is of the form

$$\mathbb{E}X = 2\gamma \int_0^\infty \int_{\mathbb{S}^{n-1}} \mathbf{1}\{u^\perp + tu \in \cdot\}\, \varphi(du)\, dt$$

with some even probability measure φ on $\mathbb{S}^{n-1}$ and $\gamma \geq 0$.

Terminology:

- γ: intensity of X
- φ: direction distribution of X

Special case:

- φ normalized spherical Lebesgue measure

Are there analogous results for the volume of the zero cell of Poisson hyperplane tessellations in high dimensions?

Let Z_o be the zero cell of a stationary and isotropic Poisson hyperplane tessellation of intensity γ in $\mathbb{R}^n$. Then

$$\mathbb{E} V_n(Z_o) \to \infty \qquad \text{and} \qquad \mathrm{Var}(V_n(Z_o)) \to \infty,$$

as $n \to \infty$.

Modification(s)?

A natural first attempt is to adjust the intensity $\gamma = \gamma(n, \lambda)$ in such a way that $\mathbb{E} V_n(Z_o) = \lambda^{-1}$.

However, then the **variance is still divergent**.

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What kind of change is happening if we pass from Poisson Voronoi to Poisson hyperplane tessellations?

A parametric model was suggested in H, Schneider '07:

Let X be a PHP with intensity measure of the form

$$\Theta(\mathcal{A}) = \frac{2\gamma}{n\kappa_n} \int_{\mathbb{S}^{n-1}} \int_0^\infty \mathbf{1}\{H(u,t) \in A\} t^{r-1}\, dt\, \sigma(du)$$

for $A \subset \mathcal{H}$, with intensity $\gamma > 0$ and distance exponent $r \in [1, \infty)$.

- X is isotropic, but stationary only for $r = 1$.
- Voronoi-case: $\gamma_{\text{Voronoi}} = n\kappa_n 2^{n-1}\lambda$ and $r = n$.

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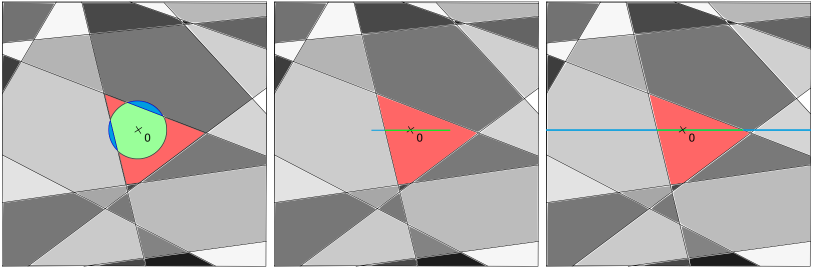
Further objects of investigation



Volume of the section of Z_o with

- n-dim. ball: $V_n(Z_o \cap B^n_u)$, for $u \in (0, \infty)$
- m-dim. ball: $V_m(Z_o \cap B^m_u)$, for $u \in (0, \infty), m \leq n$
- subspace through o: $V_m(Z_o \cap L)$, for $L \in G(n, m), m \leq n$.

Distance exponent r proportional to n:

	$V_n(Z_o)$	$V_m(Z_o \cap B_u^m)$	$V_{n-l}(Z_o \cap B_u^{n-l})$	$V_{n-l}(Z_o \cap L)$
$r = an$, $a > 0$ γ constant	$\mathbb{E} \to 0$ Var $\to 0$	$\mathbb{E} \to V_m(B_{R(a)}^m \cap B_u^m)$ Var $\to 0$	$\mathbb{E} \to 0$ Var $\to 0$	-
$r = an$, $a > 0$ $\gamma(a, n, \lambda)$, $\lambda > 0$	$\mathbb{E} \to \frac{1}{\lambda}$ Var $\to 0$	$\mathbb{E} \to u$ Var $\to 0$	$\mathbb{E} \to I(a, u, l, \lambda)$ Var $\to 0$	$\mathbb{E} \to \frac{e^{\frac{l}{2}}}{\lambda}$ Var $\to 0$

$m, l \in \mathbb{N}$ constant, $L \in G(n, n-l)$

$\gamma(1, n, \lambda) = \gamma_{Voronoi}$!

Slicing problem

Let $K \subset \mathbb{R}^n$ be a convex body with $V_n(K) = 1$. $\exists\, H \in \mathcal{H}^n$ such that

$$V_{n-1}(K \cap H) \geq c?$$

For a convex body $K \subset \mathbb{R}^n$, the isotropic constant L_K of K is defined by

$$n \cdot L_K^2 := \min_T \frac{1}{V_n(TK)^{1+\frac{2}{n}}} \int_{TK} \|x\|^2 \, dx.$$

Is there a universal constant C such that

$$L_K \leq C$$

for all convex bodies $K \subset \mathbb{R}^n$ and all $n \in \mathbb{N}$?

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Some known results

- $L_K \leq c \cdot n^{1/4} \log(n)$ Bourgain (1991)
- $L_K \leq c \cdot n^{1/4}$ Klartag (2006)
- Conjecture holds for special classes of bodies (zonoids, ...)
- $L_P \leq C \cdot (f_0(P)/n)^{1/2}$ Alonso-Gutiérrez, Bastero, Bernués, Wolff (2010)
- The isotropic constant of certain r.p. is bounded with high probability:
  - Gaussian polytopes Klartag, Kozma (2008)
  - Random polytopes whose vertices have independent coordinates
  - Random polytopes spanned by r.p. from $\mathbb{S}^{n-1}$ Alonso-Gutiérrez (2008)
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Yet another partial ...

Let $\overline{Z}_o := \big(V_n(Z_o)\big)^{-\frac{1}{n}} Z_o$ be the normalized zero cell of a PHP in $\mathbb{R}^n$ with distance exponent $r = an$ and intensity $\gamma(a, n, \lambda)$. Then, for any $L \in G(n, n-1)$,

$$\mathbb{P}\left\{ V_{n-1}(\overline{Z}_o \cap L) > \frac{\sqrt{e}}{2} \right\} \geq 1 - C\left(\frac{1}{\sqrt{n}} \left(\frac{2}{\sqrt{5}}\right)^{an} + \left(\frac{2}{\sqrt{5}}\right)^{n} \right)$$

for a universal constant $C > 0$.

$$\mathbb{E} f_0(Z_o) \geq c \cdot n^{(n-2)/2}.$$

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Some explicit formulas – based on integral geometry

$$\mathbb{E}[V_n(Z_o)] \quad = \Gamma\left(\frac{n}{r}+1\right)\kappa_n\left(\frac{n\kappa_n r}{2\gamma c(n,r)}\right)^{\frac{n}{r}}$$

$$\mathbb{E}[\big(V_n(Z_o)\big)^2] = b_{n,2}4\pi\int_0^\pi 2\int_0^\infty\int_0^1 \exp\Big[-\frac{2\gamma c(n,r)}{n\kappa_n r}s^r\frac{\Gamma(\frac{r}{2}+1)}{\sqrt{\pi}\Gamma(\frac{r+1}{2})}$$

$$\times\Big(\int_{-\frac{\pi}{2}}^{\alpha(\varphi,t)}(\cos\theta)^r d\theta\, t^r + \int_{\alpha(\varphi,t)-\varphi}^{\frac{\pi}{2}}(\cos\theta)^r d\theta\Big)\Big]s^{2n-1}t^{n-1}(\sin\varphi)^{n-2}dt\,ds\,d\varphi$$