A measure of orthogonality in isotropic convex bodies

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A result of Lutwak, Yang and Zhang

Lutwak, Yang and Zhang, *Moment-entropy inequalities*, Annals of Prob., (2004)

For every $q \geq 1$, the minimum of the quantity

$$Y_q(K, M) := \left(\int_K \int_M |\langle x, y \rangle|^q dy dx \right)^{1/q},$$

with K, M compact sets in $\mathbb{R}^n$ of Lebesgue measure 1, is attained when $K = M = D_n$.

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* Note that $Y_q(D_n, D_n) \simeq \sqrt{qn}$ for all $1 \leq q \leq n$.

Our aim is to study $Y_q(K) := Y_q(K,K)$ and $Y_q(K,M)$ when K and M are isotropic convex bodies.

Notation and definitions

- Let $K \subset \mathbb{R}^n$ be a convex body of volume 1. Associated with K is a family of symmetric, convex bodies $Z_q(K)$, $q \geq 1$, whose support function $h_{Z_q(K)}$ is defined as follows:

$$h_{Z_q(K)}(\theta) := \left(\int_K |\langle x, \theta \rangle|^q dx \right)^{1/q}, \quad \theta \in S^{n-1}.$$

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- K is isotropic iff it has Lebesgue measure 1, it is centered and there exists a constant L_K so that

$$Z_2(K) = L_K B_2^n.$$

Notation and definitions (cont.)

- We denote by $I_q(K)$ the integral

$$\left(\int_K \|x\|_2^q dx\right)^{1/q}, \quad -n < q < +\infty,\ q \neq 0.$$

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- In general, for any **symmetric** convex body C we write
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- In general, for any **symmetric** convex body C we write

$$I_q(K, C) := \left(\int_K \|x\|_C^q dx\right)^{1/q}.$$

- Finally, we denote by $R(K)$ the radius of K, i.e.

$$R(K) := \max_{x \in K} \|x\|_2 = \max_{\theta \in S^{n-1}} h_K(\theta).$$

  Obviously, $K \subseteq R(K) B_2^n$.

Obvious upper and lower bounds

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- $Y_q(K) \leq R(Z_q(K)) \cdot \left(\int_K \|x\|_2^q dx \right)^{1/q} = R(Z_q(K)) \cdot I_q(K)$
- $$\begin{aligned} Y_q(K) &= \left(\int_K \max_{z \in Z_q(K)} |\langle x, z \rangle|^q dx \right)^{1/q} \\ &\geq \max_{z \in Z_q(K)} \left(\int_K |\langle x, z \rangle|^q dx \right)^{1/q} \\ &= \max_{z \in Z_q(K)} h_{Z_q(K)}(z) = [R(Z_q(K))]^2. \end{aligned}$$

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($q_*(K)$ being at least $\geq c\sqrt{n}$ for all isotropic bodies K).

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Thus, $Y_q(K) \simeq [R(Z_q(K))]^2$ for all $q \geq q_*$.

What happens for the other q?

How does $Y_q(K)$ behave when $q \leq q_*$?

Example of a ψ_1-body close to the Euclidean ball:

Theorem (Paouris, *On the* Ψ_2 *behavior of linear functionals on isotropic convex bodies*, Studia Math., 2005)

For every $n \geq 1$, there exist $a_n, R_n \simeq \sqrt{n}$ and $b_n \simeq 1/\sqrt{n}$ such that the convex body of revolution

$$K = \{y = (x, u) : |u| \leq R_n, \|x\|_2 \leq a_n - b_n|u|\}$$

is isotropic.

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is isotropic. Moreover, one can show that $d_G(K, B_2^n) \leq C$ and that $\|\langle \cdot, e_n\rangle\|_{L^q(K)} \simeq \min\{q, \sqrt{n}\}$.

How does $Y_q(K)$ behave when $q \leq q_*$? (cont.)

- For this particular body, which is both ψ_1 and close to the Euclidean ball, we can compute that

$$Y_q(K) \simeq \min\big\{n, \max\{\sqrt{qn}, q^2\}\big\}$$

for all $1 \leq q \leq n$.

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- Similar behaviour on all unconditional convex bodies.
- Other instances of "ψ_2−behaviour on average":

1. $h_{Z_q(K)}(\theta) \ \mapsto \ w(Z_q(K)) := \int_{S^{n-1}} h_{Z_q(K)}(\theta) d\sigma(\theta)$

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- Other instances of "ψ_2-behaviour on average":
  1. $h_{Z_q(K)}(\theta) \ \mapsto \ w(Z_q(K)) := \int_{S^{n-1}} h_{Z_q(K)}(\theta) d\sigma(\theta)$
  2. $R(Z_q(K)) \gg qL_K$ in some cases, however $[\mathrm{vol}(Z_q(K))]^{1/n} \ll \sqrt{q}L_K/\sqrt{n}$ always!

The quantity $Y_q(K, UK)$

$$Y_q(K, UK) = \left(\int_K \int_{UK} |\langle x, y \rangle|^q dy dx \right)^{1/q}$$

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We can show that

$$\| Y_q(K, UK) \|_{L^q(O(n))} \simeq \sqrt{\frac{q}{n}} [I_q(K)]^2 \simeq \max \left\{ \sqrt{qn} L_K^2, \sqrt{q/n} [R(Z_q(K))]^2 \right\}.$$

The quantity $Y_q(K, UK)$ (cont.)

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Given that $R(Z_q(K)) \ll qL_K$, we deduce that:

Corollary

The quantity $Y_q(K, UK)$ (cont.)

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Given that $R(Z_q(K)) \ll qL_K$, we deduce that:

Corollary

For every $1 \leq q \leq \sqrt{n}$, there exists a subset $A_q \subseteq O(n)$ of measure $> 1 - e^{-q}$ having the property that

$$Y_q(K,UK) \leq C\sqrt{qn}L_K^2 \quad \text{for all } U \in A_q.$$

A question of Latala and Wojtaszczyk

Reminder. For any convex body K of volume 1 and any symmetric convex body B we write

$$I_q(K,B) := \left(\int_K \|x\|_B^q dx \right)^{1/q}.$$

A question of Latala and Wojtaszczyk (cont.)

In their paper *On the infimum convolution inequality* (Studia Math., (2008)), Latala and Wojtaszczyk posed the following

Question

Is it true that there exist absolute constants $c_1, c_2 > 0$ (independent of n) such that for every convex body $K \subset \mathbb{R}^n$ of volume 1 and every symmetric convex body $B \subset \mathbb{R}^n$,

$$\begin{aligned} I_q(K,B) &\leq c_1 I_1(K,B) + c_2 \sup_{y \in B^\circ} \left(\int_K |\langle x, y \rangle|^q dx \right)^{1/q} \\ &= c_1 I_1(K,B) + c_2 \sup_{y \in B^\circ} h_{Z_q(K)}(y) \end{aligned}$$

for all $1 < q < +\infty$?

A result in this direction

Latala has recently proven a slightly weaker version of their question (*Weak and strong moments of random vectors*, preprint, (2010)), which applies to all **unconditional** isotropic convex bodies K.

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Let μ be the product exponential measure in $\mathbb{R}^n$, with density

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Theorem

Let μ be the product exponential measure in $\mathbb{R}^n$, with density

$$d\mu(x) := 2^{-n/2} \exp(-\sqrt{2}\|x\|_1) dx.$$

Then, $I_q(K, B) \leq c_1 \int_{\mathbb{R}^n} \|x\|_B d\mu(x) + c_2 \sup_{y \in B^\circ} h_{Z_q(K)}(y)$

for all unconditional isotropic bodies K and all symmetric bodies B.

Connection with $Y_q(K)$

Since

$$Y_q(K) = \left(\int_K h^q_{Z_q(K)} dx \right)^{1/q} = I_q(K, Z_q^\circ(K)),$$

if Latala and Wojtaszczyk's question is answered in the affirmative, we will have for all $1 < q < +\infty$,

$$\begin{aligned} Y_q(K) &\leq c_1 I_1(K, Z_q^\circ(K)) + c_2 \sup_{y \in Z_q(K)} h_{Z_q(K)}(y) \\ &= c_1 I_1(K, Z_q^\circ(K)) + c_2 [R(Z_q(K))]^2. \end{aligned}$$

$Y_q(K, M)$ when K, M are unconditional

- When both K and M are unconditional and isotropic,

$$Y_q(K, M) \leq c_1 \int_{\mathbb{R}^n} h_{Z_q(M)}(x) d\mu(x) + c_2 \sup_{y \in Z_q(M)} h_{Z_q(K)}(y)$$

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where we have also used the estimate
$h_{Z_q(M)}(x) \ll \sqrt{qn} \|x\|_\infty$ by Bobkov and Nazarov.

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  where we have also used the estimate $h_{Z_q(M)}(x) \ll \sqrt{qn}\|x\|_\infty$ by Bobkov and Nazarov.
- Apart from the logarithmic term, this is the best we can hope for:

  e.g. $Y_q(\bar{B}_1^n) \geq c \max\{\sqrt{qn}, q^2\}$ for all $1 \leq q \leq n$.

The quantity $I_1(K, Z_q^\circ(K))$

- $I_1(K, Z_q^\circ(K)) = \int_K h_{Z_q(K)}(x)dx \leq \sqrt{n}L_K R(Z_q(K))$
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A few more lower bounds for $I_1(K, Z_q^\circ(K))$

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In the case of unconditional convex bodies:

$$c_1\sqrt{qn} \leq I_1(K, Z_q^\circ(K)) \leq c_2\sqrt{qn}\log n.$$

Connection with the slicing problem

Definition. Let $\kappa, \tau > 0$ be absolute constants. We say that a convex body $K \subset \mathbb{R}^n$ is $(\kappa, \tau)-$regular if

$$* \quad \log N(K, tB_2^n) \leq \frac{\kappa n^2 \log^2 n}{t^2} \quad \text{for all } t \geq \tau\sqrt{n \log n}.$$

Connection with the slicing problem (cont.)

Theorem (GPV)

There exists an absolute constant $\rho \in (0,1)$ with the following property: suppose we are given $\kappa, \tau \geq 1$, a sufficiently large integer $n \geq n_0(\tau)$, and an isotropic convex body $K \subset \mathbb{R}^n$ which is $(\kappa, \tau)-$regular.

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$$2 \leq q \leq \rho^2 n \text{ and } I_1(K, Z_q^\circ(K)) \leq \rho n L_K^2$$

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$$2 \leq q \leq \rho^2 n \text{ and } I_1(K, Z_q^{\circ}(K)) \leq \rho n L_K^2$$

we have:

$$L_K^2 \leq C\kappa \frac{\sqrt{n}\log^2 n}{\sqrt{q}} \max\left\{1, \frac{I_1(K, Z_q^{\circ}(K))}{\sqrt{qn}L_K^2}\right\}.$$

Obtaining a general upper bound

Let $L_n := \sup_{K \in \mathcal{IK}_{[n]}} L_K$.

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Dafnis, Paouris, *Small ball probability estimates, ψ_2-behavior and the hyperplane conjecture*, J. of Funct. Anal., (2010)

For some explicit absolute constants $\kappa, \tau \geq 1$ and $\delta < 1$ and for every integer $n \geq 1$, we can find isotropic convex bodies $K \subset \mathbb{R}^{2n}$ which are $(\kappa, \tau)-$regular and also satisfy

$$L_K \geq \delta L_{2n}.$$

Obtaining a general upper bound (cont.)

That gives us for any admissible q:

$$L_n^2 \leq C\delta^{-2}\kappa \frac{\sqrt{n}\log^2 n}{\sqrt{q}} \max\left\{1, \inf_K \frac{I_1(K, Z_q^\circ(K))}{\sqrt{qn}L_K^2}\right\}$$

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where the infimum is taken over all isotropic bodies K whose existence was established in the previous theorem.

Conclusion

To put it more simply,

$$L_n \leq C' \frac{\sqrt[4]{n}\log n}{\sqrt[4]{q}} \cdot \sup_{K\in\mathcal{IK}_{[n]}} \sqrt{\frac{I_1(K, Z_q^\circ(K))}{\sqrt{qn}L_K^2}}$$

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$$\leq C'' \frac{\sqrt[4]{n}\log n}{q^{(1-s)/2}}$$

for all admissible q and every $s \in [\frac{1}{2}, 1]$ such that the inequality

$$I_1(K, Z_q^\circ(K)) \ll q^s \sqrt{n} L_K^2$$

holds true for all isotropic bodies K.

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holds true for all isotropic bodies K. Grazie mille!