Intersection bodies and some generalizations of the Busemann's Theorem.

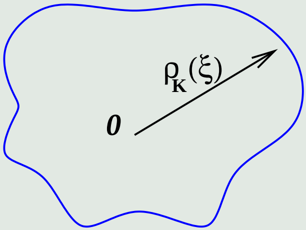
Artem Zvavitch
(with a BIG help from my friends)

Kent State University

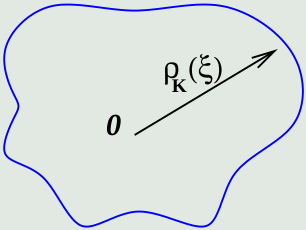
Convex Geometry - Analytic Aspects,
Cortona, Italy, June 12-18, 2011.

Standard definition.

Radial function: $\rho_K(\xi) = \sup\{a : a\xi \in K\}$, for $\xi \in \mathbb{S}^{n-1}$.

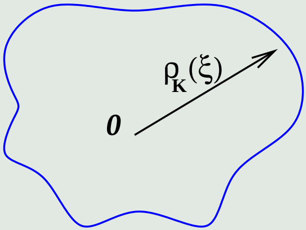


Radial function: $\rho_K(\xi) = \sup\{a : a\xi \in K\}$, for $\xi \in \mathbb{S}^{n-1}$.



Also $\rho_K(\xi) = \|\xi\|_K^{-1}$, where $\|\xi\|_K^{-1}$ is a Minkowski functional, or, in convex symmetric case, just a norm for which K is a unit ball.

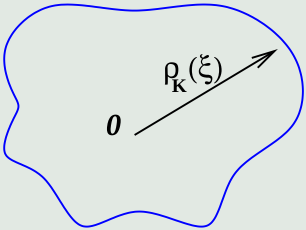
Radial function: $\rho_K(\xi) = \sup\{a : a\xi \in K\}$, for $\xi \in \mathbb{S}^{n-1}$.



Also $\rho_K(\xi) = \|\xi\|_K^{-1}$, where $\|\xi\|_K^{-1}$ is a Minkowski functional, or, in convex symmetric case, just a norm for which K is a unit ball.

- K is a star body if $\rho_K(\xi)$ is positive and continuous function on S^{n-1}.

Radial function: $\rho_K(\xi) = \sup\{a : a\xi \in K\}$, for $\xi \in \mathbb{S}^{n-1}$.

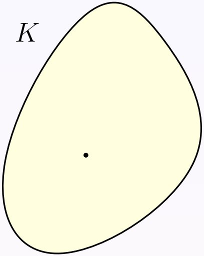


Also $\rho_K(\xi) = \|\xi\|_K^{-1}$, where $\|\xi\|_K^{-1}$ is a Minkowski functional, or, in convex symmetric case, just a norm for which K is a unit ball.

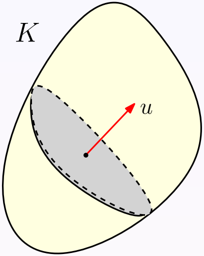
- K is a star body if $\rho_K(\xi)$ is positive and continuous function on S^{n-1}.
- $\xi^\perp = \{x \in \mathbb{R}^n : x \cdot \xi = 0\}$.

E. Lutwak: Intersection body, of a body K

E. Lutwak: Intersection body, of a body K

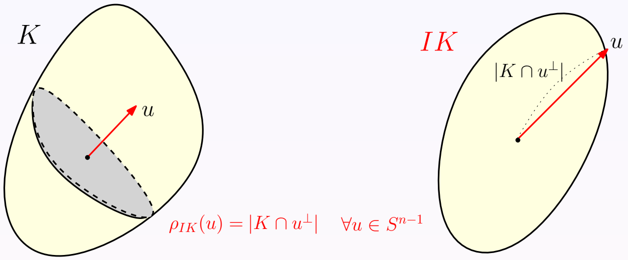


E. Lutwak: Intersection body, of a body K

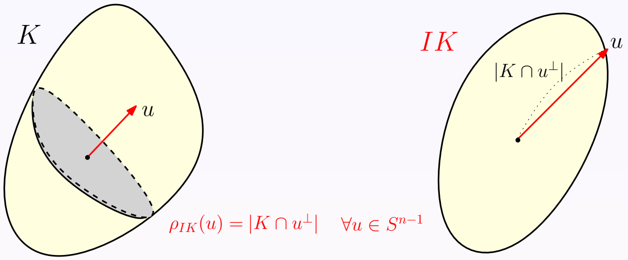


$|K \cap u^{\perp}| \quad \forall u \in S^{n-1}$

E. Lutwak: Intersection body, of a body K

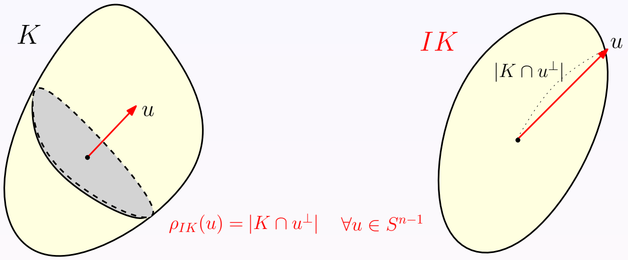


E. Lutwak: Intersection body, of a body K



R. Gardner, G. Zhang: More general definition: L is intersection body if it is limit in radial metric of $\mathrm{I}K$.

E. Lutwak: Intersection body, of a body K

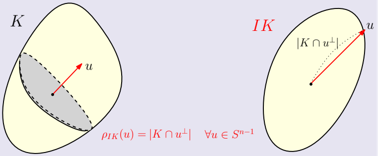


R. Gardner, G. Zhang: More general definition: L is intersection body if it is limit in radial metric of IK.

Why do we need them?

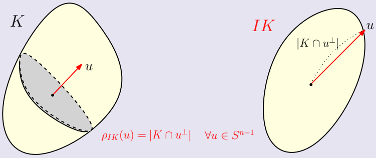
Solution of Busemann-Petty problem. Definition of L_{-1}. Very nice questions in Harmonic Analysis & just for fun.

E. Lutwak: Intersection body, of a body K



R. Gardner, G. Zhang: More general definition: L is intersection body if it is limit in radial metric of $\mathrm{I}K$.

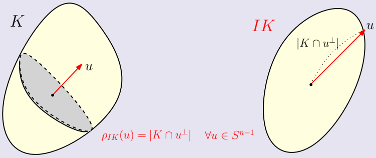
E. Lutwak: Intersection body, of a body K



R. Gardner, G. Zhang: More general definition: L is intersection body if it is limit in radial metric of $\mathrm{I}K$.

- $K \subset \mathbb{R}^2$, symmetric, then $\mathrm{I}K$ is just a rotation of $2K$ by $\pi/2$.

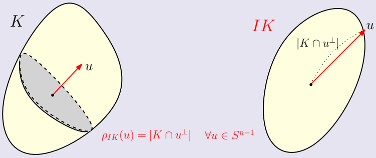
E. Lutwak: Intersection body, of a body K



R. Gardner, G. Zhang: More general definition: L is intersection body if it is limit in radial metric of $\mathrm{I}K$.

- $K \subset \mathbb{R}^2$, symmetric, then $\mathrm{I}K$ is just a rotation of $2K$ by $\pi/2$.
- $B_2^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$, then

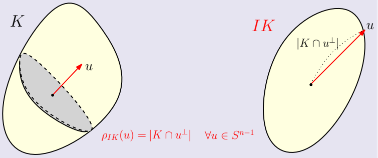
E. Lutwak: Intersection body, of a body K



R. Gardner, G. Zhang: More general definition: L is intersection body if it is limit in radial metric of $\mathrm{I}K$.

- $K \subset \mathbb{R}^2$, symmetric, then $\mathrm{I}K$ is just a rotation of $2K$ by $\pi/2$.
- $B_2^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$, then $\mathrm{I}B_2^n = |B_2^{n-1}| B_2^n = c_n B_2^n$.

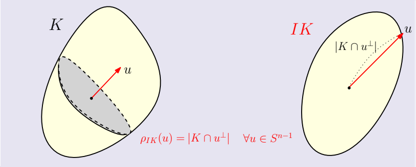
E. Lutwak: Intersection body, of a body K



R. Gardner, G. Zhang: More general definition: L is intersection body if it is limit in radial metric of $\mathrm{I}K$.

- $K \subset \mathbb{R}^2$, symmetric, then $\mathrm{I}K$ is just a rotation of $2K$ by $\pi/2$.
- $B_2^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$, then $\mathrm{I}B_2^n = |B_2^{n-1}|B_2^n = c_n B_2^n$.
- R. Gardner, A. Koldobsky, T. Schlumprecht: All convex symmetric bodies are intersection bodies in $\mathbb{R}^n$, $n \leq 4$. NOT true for $n \geq 5$.

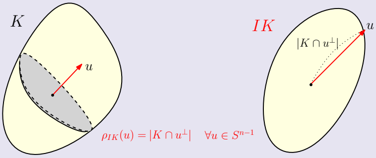
E. Lutwak: Intersection body, of a body K



R. Gardner, G. Zhang: More general definition: L is intersection body if it is limit in radial metric of $\mathrm{I}K$.

- $K \subset \mathbb{R}^2$, symmetric, then $\mathrm{I}K$ is just a rotation of $2K$ by $\pi/2$.
- $B_2^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$, then $\mathrm{I}B_2^n = |B_2^{n-1}|B_2^n = c_n B_2^n$.
- R. Gardner, A. Koldobsky, T. Schlumprecht: All convex symmetric bodies are intersection bodies in $\mathbb{R}^n$, $n \leq 4$. NOT true for $n \geq 5$.
- A. Koldobsky: B_p^n - intersection body for $p \in (0,2]$; NOT intersection body for $p > 2$, $n \geq 5$.

E. Lutwak: Intersection body, of a body K



R. Gardner, G. Zhang: More general definition: L is intersection body if it is limit in radial metric of $\mathrm{I}K$.

- $K \subset \mathbb{R}^2$, symmetric, then $\mathrm{I}K$ is just a rotation of $2K$ by $\pi/2$.
- $B_2^n = \{x \in \mathbb{R}^n : |x| \le 1\}$, then $\mathrm{I}B_2^n = |B_2^{n-1}| B_2^n = c_n B_2^n$.
- R. Gardner, A. Koldobsky, T. Schlumprecht: All convex symmetric bodies are intersection bodies in $\mathbb{R}^n$, $n \le 4$. NOT true for $n \ge 5$.
- A. Koldobsky: B_p^n - intersection body for $p \in (0, 2]$; NOT intersection body for $p > 2$, $n \ge 5$.
- Books: Gardner; Koldobsky; Koldobsky & Yaskin. Papers: Lutwak, Gardner, Zhang, Koldobsky, Goodey, Weil, Nazarov, Ludwig, Campi, Ryabogin, Berck, Yaskin, Grinberg, E. Milman, Kalton, Fish, Haberl, Paouris, Alfonseca, Kim, Zymonopoulou, Yaskina, Rubin, ...

Spherical coordinates in $\xi^\perp$

$$\rho_{IK}(\xi) = |K \cap \xi^\perp| = \frac{1}{n-1} \int\limits_{\mathbb{S}^{n-1}\cap\xi^\perp} \rho_K^{n-1}(\theta)d\theta = \frac{1}{n-1} R\rho_K^{n-1}(\xi).$$

Connection to Spherical Radon Transform

Spherical coordinates in $\xi^\perp$

$$\rho_{\mathrm{I}K}(\xi) = |K \cap \xi^\perp| = \frac{1}{n-1} \int\limits_{\mathbb{S}^{n-1}\cap\xi^\perp} \rho_K^{n-1}(\theta)d\theta = \frac{1}{n-1} R\rho_K^{n-1}(\xi).$$

Spherical Radon Transform:

$$\mathcal{R}f(\xi) = \int\limits_{\mathbb{S}^{n-1}\cap\xi^\perp} f(\theta)d\theta$$

Many geometric questions about intersection bodies can be rewritten as questions about $\mathcal{R}$.

Spherical coordinates in $\xi^\perp$

$$\rho_{\mathrm{I}K}(\xi) = |K \cap \xi^\perp| = \frac{1}{n-1} \int\limits_{\mathbb{S}^{n-1}\cap\xi^\perp} \rho_K^{n-1}(\theta)d\theta = \frac{1}{n-1} R\rho_K^{n-1}(\xi).$$

Spherical Radon Transform:

$$\mathcal{R}f(\xi) = \int\limits_{\mathbb{S}^{n-1}\cap\xi^\perp} f(\theta)d\theta$$

Many geometric questions about intersection bodies can be rewritten as questions about $\mathcal{R}$.

More general definition of Intersection Body (C^∞-case).

A symmetric star body L is an intersection body if $\mathcal{R}^{-1}\rho_L \geq 0$.

Intersection Bodies: Fix $\varepsilon \in (0, 1/10)$

Consider body K such that for every $u \in \mathbb{S}^{n-1}$ there exits an intersection body K_u, which coincide with K on a ε-neighborhood of u. Is it true that K must be an intersection body itself?

Example (of something that we can do): Local/Equatorial characterization.

Intersection Bodies: Fix $\varepsilon \in (0, 1/10)$

Consider body K such that for every $u \in \mathbb{S}^{n-1}$ there exits an intersection body K_u, which coincide with K on a ε-neighborhood of u. Is it true that K must be an intersection body itself?

Radon Transform: Fix $\varepsilon \in (0, 1/10)$

Consider a symmetric function f on $\mathbb{S}^{n-1}$, such that for every $u \in \mathbb{S}^{n-1}$ there exits a function f_u, which is equal to f on a ε-neighborhood of u and $\mathcal{R}^{-1} f_u > 0$. Is it true that $\mathcal{R}^{-1} f > 0$?

Example (of something that we can do): Local/Equatorial characterization.

Intersection Bodies: Fix $\varepsilon \in (0, 1/10)$

Consider body K such that for every $u \in \mathbb{S}^{n-1}$ there exits an intersection body K_u, which coincide with K on a ε-neighborhood of u. Is it true that K must be an intersection body itself?

Radon Transform: Fix $\varepsilon \in (0, 1/10)$

Consider a symmetric function f on $\mathbb{S}^{n-1}$, such that for every $u \in \mathbb{S}^{n-1}$ there exits a function f_u, which is equal to f on a ε-neighborhood of u and $\mathcal{R}^{-1} f_u > 0$. Is it true that $\mathcal{R}^{-1} f > 0$?

F. Nazarov, D. Ryabogin, A. Z., 2008:

- NO!

Example (of something that we can do): Local/Equatorial characterization.

Intersection Bodies: Fix $\varepsilon \in (0, 1/10)$

Consider body K such that for every $u \in \mathbb{S}^{n-1}$ there exits an intersection body K_u, which coincide with K on a ε-neighborhood of u. Is it true that K must be an intersection body itself?

Radon Transform: Fix $\varepsilon \in (0, 1/10)$

Consider a symmetric function f on $\mathbb{S}^{n-1}$, such that for every $u \in \mathbb{S}^{n-1}$ there exits a function f_u, which is equal to f on a ε-neighborhood of u and $\mathcal{R}^{-1} f_u > 0$. Is it true that $\mathcal{R}^{-1} f > 0$?

F. Nazarov, D. Ryabogin, A. Z., 2008:

- NO!
- If we instead of regular neighborhoods around points would take neighborhood around equators then YES for even n and NO for odd n!!!

Example (of something that we can do): Local/Equatorial characterization.

Intersection Bodies: Fix $\varepsilon \in (0, 1/10)$

Consider body K such that for every $u \in \mathbb{S}^{n-1}$ there exits an intersection body K_u, which coincide with K on a ε-neighborhood of u. Is it true that K must be an intersection body itself?

Radon Transform: Fix $\varepsilon \in (0, 1/10)$

Consider a symmetric function f on $\mathbb{S}^{n-1}$, such that for every $u \in \mathbb{S}^{n-1}$ there exits a function f_u, which is equal to f on a ε-neighborhood of u and $\mathcal{R}^{-1} f_u > 0$. Is it true that $\mathcal{R}^{-1} f > 0$?

F. Nazarov, D. Ryabogin, A. Z., 2008:

- NO!
- If we instead of regular neighborhoods around points would take neighborhood around equators then YES for even n and NO for odd n!!!

Original Dual problem for Zonoids: The same answer: Local - W. Weil; Local equatorial: G. Panina; W. Weil and P. Goodey – even dimensions; F. Nazarov, D. Ryabogin, A.Z. – odd dimensions. J. Schlaerth - generalizations of subspaces of L_p. W. Weil and P. Goodey - other generalizations.

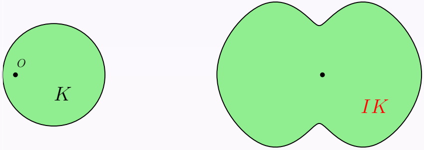
H. Busemann, 1949.

Let K be a symmetric convex body in $\mathbb{R}^n$. Then its intersection body $\mathrm{I}K$ is convex.

H. Busemann, 1949.

Let K be a symmetric convex body in $\mathbb{R}^n$. Then its intersection body $\mathrm{I}K$ is convex.

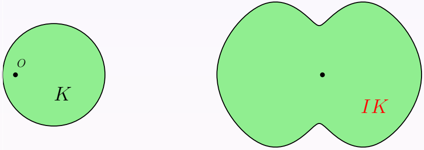
- Not true without symmetry assumption



H. Busemann, 1949.

Let K be a symmetric convex body in $\mathbb{R}^n$. Then its intersection body $\mathrm{I}K$ is convex.

- Not true without symmetry assumption

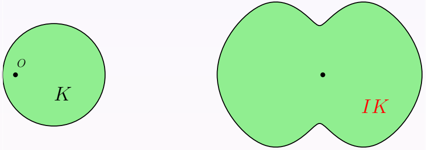


- Not true without convexity assumption (easy examples, but we will talk about "not so easy" example in a couple of slides).

H. Busemann, 1949.

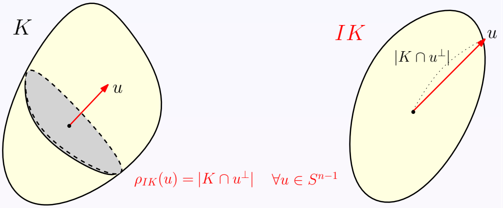
Let K be a symmetric convex body in $\mathbb{R}^n$. Then its intersection body $\mathrm{I}K$ is convex.

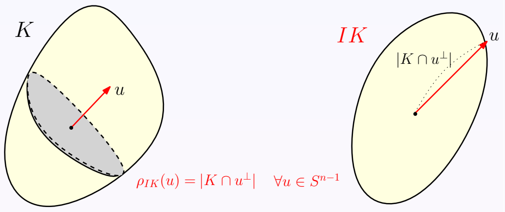
- Not true without symmetry assumption



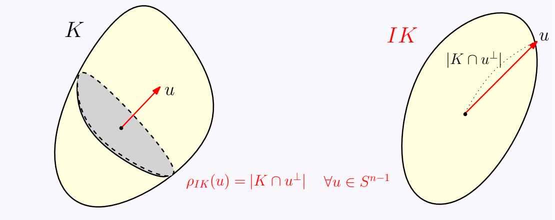
- Not true without convexity assumption (easy examples, but we will talk about "not so easy" example in a couple of slides).
- There are a lot of "nice" intersection bodies which are convex, but not an intersection body of a convex body (B_p^n, $p \in [1,2)$, n-big, we will explain it in a funny way soon). So what we should assume about K to guarantee that $\mathrm{I}K$ is convex?

Intersection bodies and Linear Transformations.

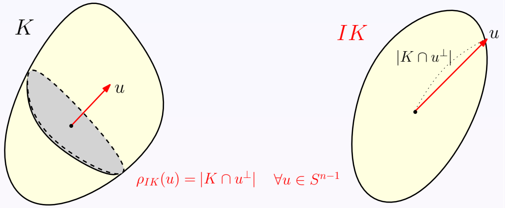




- Take $T \in GL(n)$, then $\mathrm{I}(TK) = |\det T|(T^*)^{-1}\mathrm{I}K$.

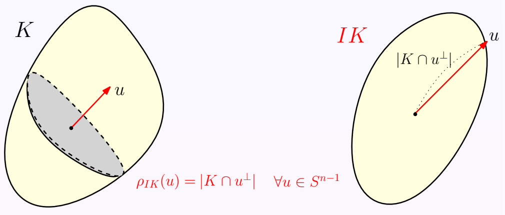


- Take $T \in GL(n)$, then $\mathrm{I}(TK) = |\det T|(T^*)^{-1}\mathrm{I}K$.
- $E = TB_2^n$ - Ellipsoid. Then $\mathrm{I}E$ is an Ellipsoid!



- Take $T \in GL(n)$, then $\mathrm{I}(TK) = |\det T|(T^*)^{-1}\mathrm{I}K$.
- $E = TB_2^n$ - Ellipsoid. Then $\mathrm{I}E$ is an Ellipsoid!

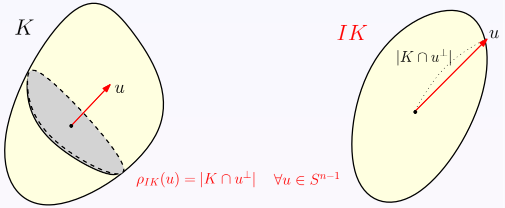
Banach-Mazur distance: $d_{BM}(K, L) = \inf\{b/a : \exists T \in GL(n) : aK \subset TL \subset bK\}$.



- Take $T \in GL(n)$, then $\mathrm{I}(TK) = |\det T|(T^*)^{-1}\mathrm{I}K$.
- $E = TB_2^n$ - Ellipsoid. Then $\mathrm{I}E$ is an Ellipsoid!

Banach-Mazur distance: $d_{BM}(K,L) = \inf\{b/a : \exists T \in GL(n) : aK \subset TL \subset bK\}$.

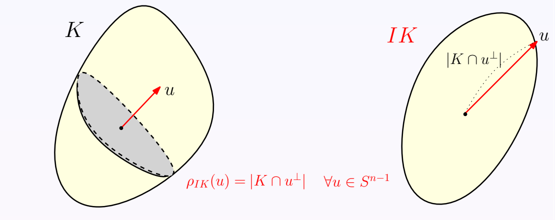
- $d_{BM}(\mathrm{I}T_1K, \mathrm{I}T_2L) = d_{BM}(\mathrm{I}K, \mathrm{I}L)$, where $T_1, T_2 \in GL(n)$.



- Take $T \in GL(n)$, then $\mathrm{I}(TK) = |\det T|(T^*)^{-1}\mathrm{I}K$.
- $E = TB_2^n$ - Ellipsoid. Then $\mathrm{I}E$ is an Ellipsoid!

Banach-Mazur distance: $d_{BM}(K,L) = \inf\{b/a : \exists T \in GL(n) : aK \subset TL \subset bK\}$.

- $d_{BM}(\mathrm{I}T_1K, \mathrm{I}T_2L) = d_{BM}(\mathrm{I}K, \mathrm{I}L)$, where $T_1, T_2 \in GL(n)$.
- $d_{BM}(B_2^n, \mathrm{I}B_2^n) = 1$ and
- $d_{BM}(E, \mathrm{I}E) = 1$.



- Take $T \in GL(n)$, then $\mathrm{I}(TK) = |\det T|(T^*)^{-1}\mathrm{I}K$.
- $E = TB_2^n$ - Ellipsoid. Then $\mathrm{I}E$ is an Ellipsoid!

Banach-Mazur distance: $d_{BM}(K,L) = \inf\{b/a : \exists T \in GL(n) : aK \subset TL \subset bK\}$.

- $d_{BM}(\mathrm{I}T_1K, \mathrm{I}T_2L) = d_{BM}(\mathrm{I}K, \mathrm{I}L)$, where $T_1, T_2 \in GL(n)$.
- $d_{BM}(B_2^n, \mathrm{I}B_2^n) = 1$ and
- $d_{BM}(E, \mathrm{I}E) = 1$.
- So Banach-Mazur distance is logical to measure the "difference" between intersection bodies.

D. Hensley

There are absolute positive constants c and C such that for every convex symmetric body $K \subset \mathbb{R}^n$, there exists a $T \in GL(n)$ such that

$$c \leq \frac{|TK \cap u^{\perp}|}{|TK \cap v^{\perp}|} \leq C, \quad \forall u, v \in \mathbb{S}^{n-1}.$$

D. Hensley

There are absolute positive constants c and C such that for every convex symmetric body $K \subset \mathbb{R}^n$, there exists a $T \in GL(n)$ such that

$$c \leq \frac{|TK \cap u^{\perp}|}{|TK \cap v^{\perp}|} \leq C, \quad \forall u, v \in \mathbb{S}^{n-1}.$$

This is an amazing and very useful fact (Ball/Bourgain/Milman & Pajor ......).

D. Hensley

There are absolute positive constants c and C such that for every convex symmetric body $K \subset \mathbb{R}^n$, there exists a $T \in GL(n)$ such that

$$c \leq \frac{|TK \cap u^{\perp}|}{|TK \cap v^{\perp}|} \leq C, \quad \forall u, v \in \mathbb{S}^{n-1}.$$

This is an amazing and very useful fact (Ball/Bourgain/Milman & Pajor ......).

So what is it for intersection bodies?

D. Hensley

There are absolute positive constants c and C such that for every convex symmetric body $K \subset \mathbb{R}^n$, there exists a $T \in GL(n)$ such that

$$c \leq \frac{|TK \cap u^{\perp}|}{|TK \cap v^{\perp}|} \leq C, \quad \forall u, v \in \mathbb{S}^{n-1}.$$

This is an amazing and very useful fact (Ball/Bourgain/Milman & Pajor ......).

So what is it for intersection bodies?

- There is an absolute positive constant C such that for every convex symmetric body $K \subset \mathbb{R}^n$: $d_{BM}(IK, B_2^n) \leq C$.

D. Hensley

There are absolute positive constants c and C such that for every convex symmetric body $K \subset \mathbb{R}^n$, there exists a $T \in GL(n)$ such that

$$c \le \frac{|TK \cap u^\perp|}{|TK \cap v^\perp|} \le C, \quad \forall u, v \in \mathbb{S}^{n-1}.$$

This is an amazing and very useful fact (Ball/Bourgain/Milman & Pajor ......).

So what is it for intersection bodies?

- There is an absolute positive constant C such that for every convex symmetric body $K \subset \mathbb{R}^n$: $d_{BM}(IK, B_2^n) \le C$. (thus, b.t.w. B_p^n is not an intersection body of a convex body for n-large)!

D. Hensley

There are absolute positive constants c and C such that for every convex symmetric body $K \subset \mathbb{R}^n$, there exists a $T \in GL(n)$ such that

$$c \leq \frac{|TK \cap u^{\perp}|}{|TK \cap v^{\perp}|} \leq C, \quad \forall u, v \in \mathbb{S}^{n-1}.$$

This is an amazing and very useful fact (Ball/Bourgain/Milman & Pajor ......).

So what is it for intersection bodies?

- There is an absolute positive constant C such that for every convex symmetric body $K \subset \mathbb{R}^n$: $d_{BM}(IK, B_2^n) \leq C$. (thus, b.t.w. B_p^n is not an intersection body of a convex body for n-large)!
- This is very cool! Do not forget that there are convex, symmetric $K \subset \mathbb{R}^n$ such that $d_{BM}(K, B_2^n) = \sqrt{n}$.

Intersection bodies, fixed points and convergence:

- $d_{BM}(E, \mathrm{I}E) = 1$.
- $d_{BM}(K, \mathrm{I}K) = 1$, $K \subset \mathbb{R}^2$, K-symmetric.

- $d_{BM}(E, \mathrm{I}E) = 1$.
- $d_{BM}(K, \mathrm{I}K) = 1$, $K \subset \mathbb{R}^2$, K-symmetric.

E. Lutwak:

Do there exists other fixed points (with respect to d_{BM}) of I in $\mathbb{R}^n$, $n \geq 3$?

- $d_{BM}(E, \mathrm{I}E) = 1$.
- $d_{BM}(K, \mathrm{I}K) = 1$, $K \subset \mathbb{R}^2$, K-symmetric.

E. Lutwak:

Do there exists other fixed points (with respect to d_{BM}) of I in $\mathbb{R}^n$, $n \geq 3$?

- $d_{BM}(\mathrm{I}K, B_2^n) \leq C$, for all convex, symmetric bodies $K \subset \mathbb{R}^n$.

- $d_{BM}(E, \mathrm{I}E) = 1$.
- $d_{BM}(K, \mathrm{I}K) = 1$, $K \subset \mathbb{R}^2$, K-symmetric.

E. Lutwak:

Do there exists other fixed points (with respect to d_{BM}) of I in $\mathbb{R}^n$, $n \geq 3$?

- $d_{BM}(\mathrm{I}K, B_2^n) \leq C$, for all convex, symmetric bodies $K \subset \mathbb{R}^n$.

A. Fish, F. Nazarov, D. Ryabogin, A.Z.:

Consider a star body $K \subset \mathbb{R}^n$, $n \geq 3$, is it true that

$$d_{BM}(\mathrm{I}^m K, B_2^n) \to 1, \text{ as } m \to \infty?$$

- $d_{BM}(E, \mathrm{I}E) = 1.$
- $d_{BM}(K, \mathrm{I}K) = 1$, $K \subset \mathbb{R}^2$, K-symmetric.

E. Lutwak:

Do there exists other fixed points (with respect to d_{BM}) of I in $\mathbb{R}^n$, $n \geq 3$?

- $d_{BM}(\mathrm{I}K, B_2^n) \leq C$, for all convex, symmetric bodies $K \subset \mathbb{R}^n$.

A. Fish, F. Nazarov, D. Ryabogin, A.Z.:

Consider a star body $K \subset \mathbb{R}^n$, $n \geq 3$, is it true that

$$d_{BM}(\mathrm{I}^m K, B_2^n) \to 1, \text{ as } m \to \infty?$$

Or even simpler.....

Consider a star body $K \subset \mathbb{R}^n$, is it true that

$$d_{BM}(\mathrm{I}K, B_2^n) \leq d_{BM}(K, B_2^n)?$$

Is it true that $d_{BM}(\mathrm{I}^m K, B_2^n) \to 1$, as $m \to \infty$?

Is it true that $d_{BM}(\mathrm{I}^m K, B_2^n) \to 1$, as $m \to \infty$?

A. Fish, F. Nazarov, D. Ryabogin, A.Z., (2009)

$\exists \varepsilon_n > 0$ such that $\forall K \subset \mathbb{R}^n$ such that K-start body, $d_{BM}(K, B_2^n) < 1 + \varepsilon_n$, we get

$$d_{BM}(\mathrm{I}^m K, B_2^n) \to 1, \text{ as } m \to \infty.$$

Is it true that $d_{BM}(\mathrm{I}^m K, B_2^n) \to 1$, as $m \to \infty$?

A. Fish, F. Nazarov, D. Ryabogin, A.Z., (2009)

$\exists \varepsilon_n > 0$ such that $\forall K \subset \mathbb{R}^n$ such that K-start body, $d_{BM}(K, B_2^n) < 1 + \varepsilon_n$, we get

$$d_{BM}(\mathrm{I}^m K, B_2^n) \to 1, \text{ as } m \to \infty.$$

Remarks:

Is it true that $d_{BM}(\mathrm{I}^m K, B_2^n) \to 1$, as $m \to \infty$?

A. Fish, F. Nazarov, D. Ryabogin, A.Z., (2009)

$\exists \varepsilon_n > 0$ such that $\forall K \subset \mathbb{R}^n$ such that K-start body, $d_{BM}(K, B_2^n) < 1 + \varepsilon_n$, we get

$$d_{BM}(\mathrm{I}^m K, B_2^n) \to 1, \text{ as } m \to \infty.$$

Remarks:

- We do not assume convexity of K. Such an assumption will much simplify the proofs, through Busemann's theorem.

Is it true that $d_{BM}(\mathrm{I}^m K, B_2^n) \to 1$, as $m \to \infty$?

A. Fish, F. Nazarov, D. Ryabogin, A.Z., (2009)

$\exists \varepsilon_n > 0$ such that $\forall K \subset \mathbb{R}^n$ such that K-start body, $d_{BM}(K, B_2^n) < 1 + \varepsilon_n$, we get

$$d_{BM}(\mathrm{I}^m K, B_2^n) \to 1, \text{ as } m \to \infty.$$

Remarks:

- We do not assume convexity of K. Such an assumption will much simplify the proofs, through Busemann's theorem.
- Even if K is convex symmetric, then $d_{BM}(K, B_2^n) \leq \sqrt{n}$, which is very far from ε_n.

Is it true that $d_{BM}(\mathrm{I}^m K, B_2^n) \to 1$, as $m \to \infty$?

A. Fish, F. Nazarov, D. Ryabogin, A.Z., (2009)

$\exists \varepsilon_n > 0$ such that $\forall K \subset \mathbb{R}^n$ such that K-start body, $d_{BM}(K, B_2^n) < 1 + \varepsilon_n$, we get

$$d_{BM}(\mathrm{I}^m K, B_2^n) \to 1, \text{ as } m \to \infty.$$

Remarks:

- We do not assume convexity of K. Such an assumption will much simplify the proofs, through Busemann's theorem.
- Even if K is convex symmetric, then $d_{BM}(K, B_2^n) \le \sqrt{n}$, which is very far from ε_n.
- Yes, yes ... we may say by Hensley's theorem after one step of iteration , $d_{BM}(K, B_2^n) \le C$, but this is still very, very far from ε_n.

A. Fish, F. Nazarov, D. Ryabogin, A.Z., (2009)

$\exists \varepsilon_n > 0$ such that $\forall K \subset \mathbb{R}^n$ such that K-start body, $d_{BM}(K, B_2^n) < 1 + \varepsilon_n$, we get

$$d_{BM}(\mathrm{I}^m K, B_2^n) \to 1, \text{ as } m \to \infty.$$

Remarks:

- We do not assume convexity of K. Such an assumption will much simplify the proofs, through Busemann's theorem.
- Even if K is convex symmetric, then $d_{BM}(K, B_2^n) \leq \sqrt{n}$, which is very far from ε_n.
- Yes, yes ... we may say by Hensley's theorem after one step of iteration , $d_{BM}(K, B_2^n) \leq C$, but this is still very, very far from ε_n.
- So, we do not use Hensley's theorem or any theorem of this type! We have No idea how to start using it for this question!

A. Fish, F. Nazarov, D. Ryabogin, A.Z., (2009)

$\exists \varepsilon_n > 0$ such that $\forall K \subset \mathbb{R}^n$ such that K-start body, $d_{BM}(K, B_2^n) < 1 + \varepsilon_n$, we get

$$d_{BM}(\mathrm{I}^m K, B_2^n) \to 1, \text{ as } m \to \infty.$$

Remarks:

- We do not assume convexity of K. Such an assumption will much simplify the proofs, through Busemann's theorem.
- Even if K is convex symmetric, then $d_{BM}(K, B_2^n) \leq \sqrt{n}$, which is very far from ε_n.
- Yes, yes ... we may say by Hensley's theorem after one step of iteration , $d_{BM}(K, B_2^n) \leq C$, but this is still very, very far from ε_n.
- So, we do not use Hensley's theorem or any theorem of this type! We have No idea how to start using it for this question!
- We do NOT show $d_{BM}(\mathrm{I}K, B_2^n) \leq d_{BM}(K, B_2^n)$. We really DO need a lot of iterations to make $\mathrm{I}^m K$ better, before computing the distance to B_2^n.

Main Idea: Spherical Radon Transform (even functions).

(Normalized) Spherical Radon Transform:

$$\mathcal{R}f(\xi) = \frac{1}{|S^{n-1}|} \int_{S^{n-1}\cap\xi^\perp} f(\theta)d\theta$$

Question: $(n \geq 3)$

Consider even function $f : S^{n-1} \to \mathbb{R}^+$: $f = \mathcal{R}f^{n-1}$, is it true that then $f = 1$?

Main Idea: Spherical Radon Transform (even functions).

(Normalized) Spherical Radon Transform:

$$\mathcal{R}f(\xi) = \frac{1}{|S^{n-1}|} \int\limits_{S^{n-1}\cap\xi^{\perp}} f(\theta)d\theta$$

Question: $(n \geq 3)$

Consider even function $f : S^{n-1} \to \mathbb{R}^{+}$: $f = \mathcal{R}f^{n-1}$, is it true that then $f = 1$?

$\mathcal{H}_k-$ space of Spherical Harmonics of degree k.

(Normalized) Spherical Radon Transform:

$$\mathcal{R}f(\xi) = \frac{1}{|S^{n-1}|} \int_{S^{n-1}\cap\xi^{\perp}} f(\theta)d\theta$$

Question: $(n \geq 3)$

Consider even function $f : S^{n-1} \to \mathbb{R}^+$: $f = \mathcal{R}f^{n-1}$, is it true that then $f = 1$?

$\mathcal{H}_k$– space of Spherical Harmonics of degree k.
H_k^f the projection of f to $\mathcal{H}_k$, so

(Normalized) Spherical Radon Transform:

$$\mathcal{R}f(\xi) = \frac{1}{|S^{n-1}|} \int_{S^{n-1}\cap\xi^{\perp}} f(\theta)d\theta$$

Question: $(n \geq 3)$

Consider even function $f : S^{n-1} \to \mathbb{R}^{+}$: $f = \mathcal{R}f^{n-1}$, is it true that then $f = 1$?

$\mathcal{H}_k$– space of Spherical Harmonics of degree k.
H_k^f the projection of f to $\mathcal{H}_k$, so $f \sim \sum_{k\geq 0} H_k^f$

Main Idea: Spherical Radon Transform (even functions).

(Normalized) Spherical Radon Transform:

$$\mathcal{R}f(\xi) = \frac{1}{|S^{n-1}|} \int_{S^{n-1}\cap\xi^{\perp}} f(\theta)d\theta$$

Question: $(n \geq 3)$

Consider even function $f : S^{n-1} \to \mathbb{R}^+$: $f = \mathcal{R}f^{n-1}$, is it true that then $f = 1$?

$\mathcal{H}_k$– space of Spherical Harmonics of degree k.
H_k^f the projection of f to $\mathcal{H}_k$, so $f \sim \sum_{k\geq 0} H_k^f$

Assume that $n \geq 3$. If $H_k \in \mathcal{H}_k$, k-even, then

$$\mathcal{R}H_k(\xi) = v_{n,k}H_k(\xi), \text{ for all } \xi \in S^{n-1},$$

where $v_{n,0} = 1$ and

(Normalized) Spherical Radon Transform:

$$\mathcal{R}f(\xi) = \frac{1}{|S^{n-1}|} \int_{S^{n-1}\cap\xi^{\perp}} f(\theta)d\theta$$

Question: $(n \geq 3)$

Consider even function $f : S^{n-1} \to \mathbb{R}^+$: $f = \mathcal{R}f^{n-1}$, is it true that then $f = 1$?

$\mathcal{H}_k$– space of Spherical Harmonics of degree k.
H_k^f the projection of f to $\mathcal{H}_k$, so $f \sim \sum_{k\geq 0} H_k^f$

Assume that $n \geq 3$. If $H_k \in \mathcal{H}_k$, k-even, then

$$\mathcal{R}H_k(\xi) = v_{n,k}H_k(\xi), \text{ for all } \xi \in S^{n-1},$$

where $v_{n,0} = 1$ and $v_{n,2} = \frac{1}{n-1}$ and $v_{n,k} \approx k^{-n-2}$.

(Normalized) Spherical Radon Transform:

$$\mathcal{R}f(\xi) = \frac{1}{|S^{n-1}|} \int\limits_{S^{n-1}\cap\xi^{\perp}} f(\theta)d\theta$$

Question: $(n \geq 3)$

Consider even function $f : S^{n-1} \to \mathbb{R}^+$: $f = \mathcal{R}f^{n-1}$, is it true that then $f = 1$?

$\mathcal{H}_k$– space of Spherical Harmonics of degree k.
H_k^f the projection of f to $\mathcal{H}_k$, so $f \sim \sum_{k\geq 0} H_k^f$

Assume that $n \geq 3$. If $H_k \in \mathcal{H}_k$, k-even, then

$$\mathcal{R}H_k(\xi) = v_{n,k}H_k(\xi), \text{ for all } \xi \in S^{n-1},$$

where $v_{n,0} = 1$ and $v_{n,2} = \frac{1}{n-1}$ and $v_{n,k} \approx k^{-n-2}$.

- $\mathcal{R}f = \mathcal{R}g$, then $f = g$.

Main Idea: Spherical Radon Transform (even functions).

(Normalized) Spherical Radon Transform:

$$\mathcal{R}f(\xi) = \frac{1}{|S^{n-1}|} \int\limits_{S^{n-1}\cap\xi^{\perp}} f(\theta)d\theta$$

Question: $(n \geq 3)$

Consider even function $f : S^{n-1} \to \mathbb{R}^+$: $f = \mathcal{R}f^{n-1}$, is it true that then $f = 1$?

$\mathcal{H}_k$– space of Spherical Harmonics of degree k.
H_k^f the projection of f to $\mathcal{H}_k$, so $f \sim \sum_{k\geq 0} H_k^f$

Assume that $n \geq 3$. If $H_k \in \mathcal{H}_k$, k-even, then

$$\mathcal{R}H_k(\xi) = v_{n,k}H_k(\xi), \text{ for all } \xi \in S^{n-1},$$

where $v_{n,0} = 1$ and $v_{n,2} = \frac{1}{n-1}$ and $v_{n,k} \approx k^{-n-2}$.

- $\mathcal{R}f = \mathcal{R}g$, then $f = g$.
- $\mathcal{R}f = f$, then $f = 1$

(Normalized) Spherical Radon Transform:

$$\mathcal{R}f(\xi) = \frac{1}{|S^{n-1}|} \int\limits_{S^{n-1}\cap\xi^{\perp}} f(\theta)d\theta$$

Question: $(n \geq 3)$

Consider even function $f : S^{n-1} \to \mathbb{R}^+$: $f = \mathcal{R}f^{n-1}$, is it true that then $f = 1$?

$\mathcal{H}_k$– space of Spherical Harmonics of degree k.
H_k^f the projection of f to $\mathcal{H}_k$, so $f \sim \sum_{k\geq 0} H_k^f$

Assume that $n \geq 3$. If $H_k \in \mathcal{H}_k$, k-even, then

$$\mathcal{R}H_k(\xi) = v_{n,k}H_k(\xi), \text{ for all } \xi \in S^{n-1},$$

where $v_{n,0} = 1$ and $v_{n,2} = \frac{1}{n-1}$ and $v_{n,k} \approx k^{-n-2}$.

- $\mathcal{R}f = \mathcal{R}g$, then $f = g$.
- $\mathcal{R}f = f$, then $f = 1$ (o.k. $f = const$).

$$f \sim \sum_{k \geq 0} H_k^f \Rightarrow$$

THE MAIN PROBLEM:

$$f \sim \sum_{k \geq 0} H_k^f \Rightarrow$$

$$f^{n-1} \sim \ ????$$

Back to our result: $f \approx 1$

$f = 1 + \phi$, where ϕ is even with small L_∞ norm, $\int_{S^{n-1}} \phi = 0$.

Back to our result: $f \approx 1$

$f = 1+\phi$, where ϕ is even with small L_∞ norm, $\int_{S^{n-1}} \phi = 0$.
$\mathcal{R}f^{n-1} = 1+(n-1)\mathcal{R}\phi+\mathcal{R}O(\phi^2)$

Back to our result: $f \approx 1$

$f = 1 + \phi$, where ϕ is even with small L_∞ norm, $\int_{S^{n-1}} \phi = 0$.
$\mathcal{R} f^{n-1} = 1 + (n-1)\mathcal{R}\phi + \mathcal{R}O(\phi^2)$
So our main goal is to show that $(n-1)\mathcal{R}\phi + \mathcal{R}O(\phi^2)$ is "very small".

Back to our result: $f \approx 1$

$f = 1 + \phi$, where ϕ is even with small L_∞ norm, $\int_{S^{n-1}} \phi = 0$.
$\mathcal{R} f^{n-1} = 1 + (n-1)\mathcal{R}\phi + \mathcal{R}O(\phi^2)$
So our main goal is to show that $(n-1)\mathcal{R}\phi + \mathcal{R}O(\phi^2)$ is "very small".

Problems:

1) Working with Spherical Harmonics we need to talk about L_2 norm! If we assume convexity, then those are "almost" equivalent. Much more work required to "prepare" the function to be ready for the L_2, L_∞ game in non-convex case.

$f = 1 + \phi$, where ϕ is even with small L_∞ norm, $\int_{S^{n-1}} \phi = 0$.
$\mathcal{R} f^{n-1} = 1 + (n-1)\mathcal{R}\phi + \mathcal{R}O(\phi^2)$
So our main goal is to show that $(n-1)\mathcal{R}\phi + \mathcal{R}O(\phi^2)$ is "very small".

Problems:

1) Working with Spherical Harmonics we need to talk about L_2 norm! If we assume convexity, then those are "almost" equivalent. Much more work required to "prepare" the function to be ready for the L_2, L_∞ game in non-convex case.
2) The crucial step is to show that

$$\|(n-1)\mathcal{R}\phi\|_{L_2} \leq \lambda \|\phi\|_{L_2}, \text{ for some } \lambda < 1.$$

Back to our result: $f \approx 1$

$f = 1 + \phi$, where ϕ is even with small L_∞ norm, $\int_{S^{n-1}} \phi = 0$.
$\mathcal{R} f^{n-1} = 1 + (n-1)\mathcal{R}\phi + \mathcal{R}O(\phi^2)$
So our main goal is to show that $(n-1)\mathcal{R}\phi + \mathcal{R}O(\phi^2)$ is "very small".

Problems:

1) Working with Spherical Harmonics we need to talk about L_2 norm! If we assume convexity, then those are "almost" equivalent. Much more work required to "prepare" the function to be ready for the L_2, L_∞ game in non-convex case.
2) The crucial step is to show that

$$\|(n-1)\mathcal{R}\phi\|_{L_2} \leq \lambda \|\phi\|_{L_2}, \text{ for some } \lambda < 1.$$

Indeed, then $\|\mathcal{R}\phi^2\|_{L_2} \leq \|\phi\|_{L_\infty} \|\phi\|_{L_2}$ (do not forget $\|\mathcal{R}\|_{L_2 \to L_2} \leq 1$).

$f = 1 + \phi$, where ϕ is even with small L_∞ norm, $\int_{S^{n-1}} \phi = 0$.
$\mathcal{R} f^{n-1} = 1 + (n-1)\mathcal{R}\phi + \mathcal{R}O(\phi^2)$
So our main goal is to show that $(n-1)\mathcal{R}\phi + \mathcal{R}O(\phi^2)$ is "very small".

Problems:

1) Working with Spherical Harmonics we need to talk about L_2 norm! If we assume convexity, then those are "almost" equivalent. Much more work required to "prepare" the function to be ready for the L_2, L_∞ game in non-convex case.
2) The crucial step is to show that

$$\|(n-1)\mathcal{R}\phi\|_{L_2} \le \lambda \|\phi\|_{L_2}, \text{ for some } \lambda < 1.$$

Indeed, then $\|\mathcal{R}\phi^2\|_{L_2} \le \|\phi\|_{L_\infty} \|\phi\|_{L_2}$ (do not forget $\|\mathcal{R}\|_{L_2 \to L_2} \le 1$). Write

$$\phi \sim \sum H_{2k}^{\phi}$$

$f = 1 + \phi$, where ϕ is even with small L_∞ norm, $\int_{S^{n-1}} \phi = 0$.
$\mathcal{R} f^{n-1} = 1 + (n-1)\mathcal{R}\phi + \mathcal{R}O(\phi^2)$
So our main goal is to show that $(n-1)\mathcal{R}\phi + \mathcal{R}O(\phi^2)$ is "very small".

Problems:

1) Working with Spherical Harmonics we need to talk about L_2 norm! If we assume convexity, then those are "almost" equivalent. Much more work required to "prepare" the function to be ready for the L_2, L_∞ game in non-convex case.
2) The crucial step is to show that

$$\|(n-1)\mathcal{R}\phi\|_{L_2} \le \lambda \|\phi\|_{L_2}, \text{ for some } \lambda < 1.$$

Indeed, then $\|\mathcal{R}\phi^2\|_{L_2} \le \|\phi\|_{L_\infty} \|\phi\|_{L_2}$ (do not forget $\|\mathcal{R}\|_{L_2 \to L_2} \le 1$). Write

$$\phi \sim \sum H_{2k}^{\phi} \qquad \text{then} \qquad (n-1)\mathcal{R}\phi \sim \sum (n-1) v_{n,2k} H_{2k}^{\phi}.$$

$f = 1 + \phi$, where ϕ is even with small L_∞ norm, $\int_{S^{n-1}} \phi = 0$.
$\mathcal{R} f^{n-1} = 1 + (n-1)\mathcal{R}\phi + \mathcal{R}O(\phi^2)$
So our main goal is to show that $(n-1)\mathcal{R}\phi + \mathcal{R}O(\phi^2)$ is "very small".

Problems:

1) Working with Spherical Harmonics we need to talk about L_2 norm! If we assume convexity, then those are "almost" equivalent. Much more work required to "prepare" the function to be ready for the L_2, L_∞ game in non-convex case.
2) The crucial step is to show that

$$\|(n-1)\mathcal{R}\phi\|_{L_2} \le \lambda \|\phi\|_{L_2}, \text{ for some } \lambda < 1.$$

Indeed, then $\|\mathcal{R}\phi^2\|_{L_2} \le \|\phi\|_{L_\infty} \|\phi\|_{L_2}$ (do not forget $\|\mathcal{R}\|_{L_2 \to L_2} \le 1$). Write

$$\phi \sim \sum H_{2k}^{\phi} \qquad \text{then} \qquad (n-1)\mathcal{R}\phi \sim \sum (n-1) v_{n,2k} H_{2k}^{\phi}.$$

If $(n-1)v_{n,2k}$ are small then we are DONE! Unfortunately this is NOT the case $(n-1)v_{n,2} = 1$ (but $(n-1)v_{n,2k} \le 3/4$ for all $k > 1$).

Back to our result: $f \approx 1$

$f = 1 + \phi$, where ϕ is even with small L_∞ norm, $\int_{S^{n-1}} \phi = 0$.
$\mathcal{R}f^{n-1} = 1 + (n-1)\mathcal{R}\phi + \mathcal{R}O(\phi^2)$
So our main goal is to show that $(n-1)\mathcal{R}\phi + \mathcal{R}O(\phi^2)$ is "very small".

Problems:

1) Working with Spherical Harmonics we need to talk about L_2 norm! If we assume convexity, then those are "almost" equivalent. Much more work required to "prepare" the function to be ready for the L_2, L_∞ game in non-convex case.
2) The crucial step is to show that

$$\|(n-1)\mathcal{R}\phi\|_{L_2} \le \lambda \|\phi\|_{L_2}, \text{ for some } \lambda < 1.$$

Indeed, then $\|\mathcal{R}\phi^2\|_{L_2} \le \|\phi\|_{L_\infty}\|\phi\|_{L_2}$ (do not forget $\|\mathcal{R}\|_{L_2 \to L_2} \le 1$). Write

$$\phi \sim \sum H_{2k}^{\phi} \qquad \text{then} \qquad (n-1)\mathcal{R}\phi \sim \sum (n-1) v_{n,2k} H_{2k}^{\phi}.$$

If $(n-1)v_{n,2k}$ are small then we are DONE! Unfortunately this is NOT the case $(n-1)v_{n,2} = 1$ (but $(n-1)v_{n,2k} \le 3/4$ for all $k > 1$).
Thus we need to KILL H_2^{ϕ}.

Back to our result: $f \approx 1$

$f = 1+\phi$, where ϕ is even with small L_∞ norm, $\int_{S^{n-1}} \phi = 0$.
$\mathcal{R} f^{n-1} = 1 + (n-1)\mathcal{R}\phi + \mathcal{R}O(\phi^2)$
So our main goal is to show that $(n-1)\mathcal{R}\phi + \mathcal{R}O(\phi^2)$ is "very small".

1) Working with Spherical Harmonics we need to talk about L_2 norm! If we assume convexity, then those are "almost" equivalent. Much more work required to "prepare" the function to be ready for the L_2, L_∞ game.
2) The crucial step is to show that

$$\|(n-1)\mathcal{R}\phi\|_{L_2} \le \lambda \|\phi\|_{L_2}, \text{ for some } \lambda < 1.$$

Indeed, then $\|\mathcal{R}\phi^2\|_{L_2} \le \|\phi\|_{L_\infty}\|\phi\|_{L_2}$ (do not forget $\|\mathcal{R}\|_{L_2\to L_2} \le 1$). Write

$$\phi \sim \sum H_{2k}^{\phi} \qquad \text{then} \qquad (n-1)\mathcal{R}\phi \sim \sum (n-1) v_{n,2k} H_{2k}^{\phi}.$$

If $(n-1)v_{n,2k}$ are small then we are DONE! Unfortunately this is NOT the case $(n-1)v_{n,2} = 1$ (but $(n-1)v_{n,2k} \le 3/4$ for all $k > 1$).
Thus we need to KILL H_2^{ϕ}. HOW ? Main idea – in the end of the day, H_2^{ϕ} is just quadratic polynomial make it constant on S^{n-1}, using linear transformation. YES, "like" isotropic position, BUT in Fourier coordinates.

Quasi-convexity:

K be a star body in $\mathbb{R}^n$, $0 < q \le 1$. K is called *q-**convex*** if

$$t^{\frac{1}{q}}x + (1-t)^{\frac{1}{q}}y \in K \quad \text{whenever } x, y \in K, t \in [0,1]$$

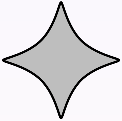
or, equivalently, $\|x+y\|_K^q \le \|x\|_K^q + \|y\|_K^q$.

K be a star body in $\mathbb{R}^n$, $0 < q \leq 1$. K is called q-**convex** if

$$t^{\frac{1}{q}} x + (1-t)^{\frac{1}{q}} y \in K \quad \text{whenever } x, y \in K, t \in [0,1]$$

or, equivalently, $\|x+y\|_K^q \leq \|x\|_K^q + \|y\|_K^q$.

$$|x|^{1/2} + |y|^{1/2} \leq 1$$



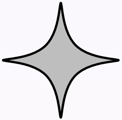
$q = \frac{1}{2}$

K be a star body in $\mathbb{R}^n$, $0 < q \le 1$. K is called ***q*-convex** if

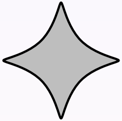
$$t^{\frac{1}{q}} x + (1-t)^{\frac{1}{q}} y \in K \quad \text{whenever } x, y \in K, t \in [0,1]$$

or, equivalently, $\|x+y\|_K^q \le \|x\|_K^q + \|y\|_K^q$.

$$|x|^{1/2} + |y|^{1/2} \le 1$$



$q = \frac{1}{4}$



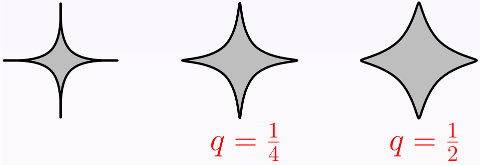
$q = \frac{1}{2}$

K be a star body in $\mathbb{R}^n$, $0 < q \le 1$. K is called ***q*-convex** if

$$t^{\frac{1}{q}} x + (1-t)^{\frac{1}{q}} y \in K \quad \text{whenever } x, y \in K, t \in [0,1]$$

or, equivalently, $\|x+y\|_K^q \le \|x\|_K^q + \|y\|_K^q$.

$$|x|^{1/2} + |y|^{1/2} \le 1$$

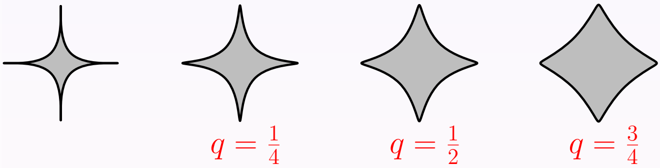


K be a star body in $\mathbb{R}^n$, $0 < q \le 1$. K is called q-**convex** if

$$t^{\frac{1}{q}} x + (1-t)^{\frac{1}{q}} y \in K \quad \text{whenever } x, y \in K, t \in [0,1]$$

or, equivalently, $\|x+y\|_K^q \le \|x\|_K^q + \|y\|_K^q$.

$$|x|^{1/2} + |y|^{1/2} \le 1$$

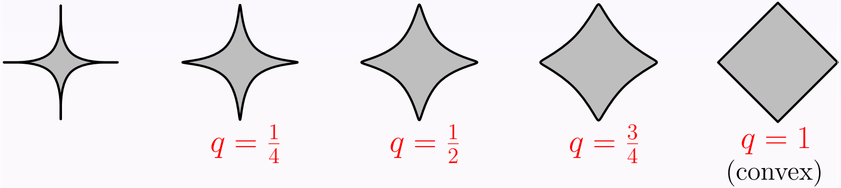


K be a star body in $\mathbb{R}^n$, $0 < q \leq 1$. K is called q-**convex** if

$$t^{\frac{1}{q}}x + (1-t)^{\frac{1}{q}}y \in K \quad \text{whenever } x, y \in K, t \in [0,1]$$

or, equivalently, $\|x+y\|_K^q \leq \|x\|_K^q + \|y\|_K^q$.

$$|x|^{1/2} + |y|^{1/2} \leq 1$$



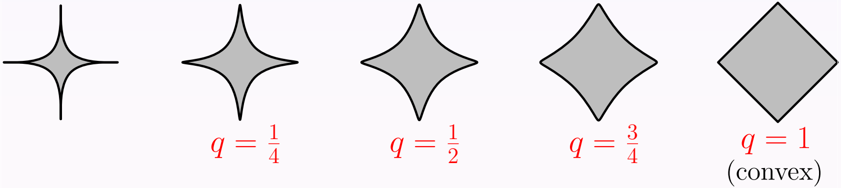
Quasi-convexity:

K be a star body in $\mathbb{R}^n$, $0 < q \le 1$. K is called *q*-convex if

$$t^{\frac{1}{q}}x + (1-t)^{\frac{1}{q}}y \in K \quad \text{whenever } x, y \in K, t \in [0,1]$$

or, equivalently, $\|x+y\|_K^q \le \|x\|_K^q + \|y\|_K^q$.

$$|x|^{1/2} + |y|^{1/2} \le 1$$



I learned it from works of:

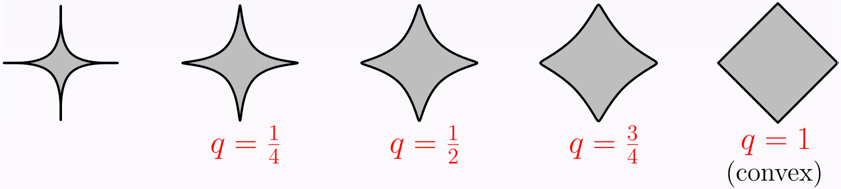
Aoki, Bastero, Bernues, Peña, Dilworth, Gordon, Kalton, Koldobsky, Guedon, Litvak, Peck, Rolewicz, Roberts, Tam, Milman, Schechtman, Pajor, . . .

K be a star body in $\mathbb{R}^n$, $0 < q \le 1$. K is called q-**convex** if

$$t^{\frac{1}{q}}x + (1-t)^{\frac{1}{q}}y \in K \quad \text{whenever } x, y \in K, t \in [0,1]$$

or, equivalently, $\|x+y\|_K^q \le \|x\|_K^q + \|y\|_K^q$.

$$|x|^{1/2} + |y|^{1/2} \le 1$$



Questions and Dreams:

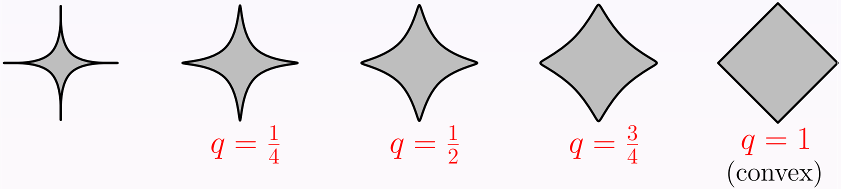
If K is q-convex, for which q' the intersection body IK is q'-convex?

K be a star body in $\mathbb{R}^n$, $0 < q \le 1$. K is called *q*-convex if

$$t^{\frac{1}{q}}x + (1-t)^{\frac{1}{q}}y \in K \quad \text{whenever } x, y \in K, t \in [0,1]$$

or, equivalently, $\|x+y\|_K^q \le \|x\|_K^q + \|y\|_K^q$.

$$|x|^{1/2} + |y|^{1/2} \le 1$$



Questions and Dreams:

If K is q-convex, for which q' the intersection body IK is q'-convex?

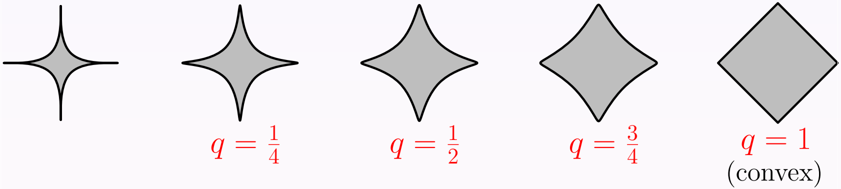
- Is it true that $q' > q$.

K be a star body in $\mathbb{R}^n$, $0 < q \le 1$. K is called q-convex if

$$t^{\frac{1}{q}}x + (1-t)^{\frac{1}{q}}y \in K \quad \text{whenever } x, y \in K, t \in [0,1]$$

or, equivalently, $\|x+y\|_K^q \le \|x\|_K^q + \|y\|_K^q$.

$$|x|^{1/2} + |y|^{1/2} \le 1$$



Questions and Dreams:

If K is q-convex, for which q' the intersection body IK is q'-convex?

- Is it true that $q' > q$.
- Does there exists q for which $q' = 1$ (i.e. IK is convex)?

Quasi-convexity for Intersection bodies

J. Kim, V. Yaskin, A. Z., 2010

Let K be an origin-symmetric q-convex body in $\mathbb{R}^n$, $q \in (0,1]$, and E a $(k-1)$-dimensional subspace of $\mathbb{R}^n$ for $1 \le k \le n$. Then the map

$$u \longmapsto \frac{|u|}{\left|K \cap \mathrm{span}(u,E)\right|_k}, \quad u \in E^{\perp}$$

defines the Minkowski functional of a q'-convex body in $E^{\perp}$ with $q' = [(1/q-1)\,k+1]^{-1}$.

J. Kim, V. Yaskin, A. Z., 2010

Let K be an origin-symmetric q-convex body in $\mathbb{R}^n$, $q \in (0,1]$, and E a $(k-1)$-dimensional subspace of $\mathbb{R}^n$ for $1 \le k \le n$. Then the map

$$u \longmapsto \frac{|u|}{\big|K \cap \mathrm{span}(u,E)\big|_k}, \quad u \in E^\perp$$

defines the Minkowski functional of a q'-convex body in $E^\perp$ with $q' = [(1/q-1)\,k+1]^{-1}$.

Let K be a symmetric star body in $\mathbb{R}^n$ and $0 < q \le 1$. Then, if K is q-convex, $\mathrm{I}K$ is q'-convex where $q' = [(1/q-1)\,(n-1)+1]^{-1}$.

Quasi-convexity for Intersection bodies

J. Kim, V. Yaskin, A. Z., 2010

Let K be an origin-symmetric q-convex body in $\mathbb{R}^n$, $q \in (0,1]$, and E a $(k-1)$-dimensional subspace of $\mathbb{R}^n$ for $1 \le k \le n$. Then the map

$$u \longmapsto \frac{|u|}{\left|K \cap \mathrm{span}(u,E)\right|_k}, \quad u \in E^{\perp}$$

defines the Minkowski functional of a q'-convex body in $E^{\perp}$ with $q' = [(1/q-1)\,k+1]^{-1}$.

Let K be a symmetric star body in $\mathbb{R}^n$ and $0 < q \le 1$. Then, if K is q-convex, IK is q'-convex where $q' = [(1/q-1)\,(n-1)+1]^{-1}$.

- Similar to Busemann's original proof.

Quasi-convexity for Intersection bodies

J. Kim, V. Yaskin, A. Z., 2010

Let K be an origin-symmetric q-convex body in $\mathbb{R}^n$, $q \in (0,1]$, and E a $(k-1)$-dimensional subspace of $\mathbb{R}^n$ for $1 \le k \le n$. Then the map

$$u \longmapsto \frac{|u|}{\left|K \cap \mathrm{span}(u,E)\right|_k}, \quad u \in E^\perp$$

defines the Minkowski functional of a q'-convex body in $E^\perp$ with $q' = [(1/q-1)\,k+1]^{-1}$.

Let K be a symmetric star body in $\mathbb{R}^n$ and $0 < q \le 1$. Then, if K is q-convex, $\mathrm{I}K$ is q'-convex where $q' = [(1/q-1)\,(n-1)+1]^{-1}$.

- Similar to Busemann's original proof.
- Yes, q' does not look nice!

J. Kim, V. Yaskin, A. Z., 2010

Let K be an origin-symmetric q-convex body in $\mathbb{R}^n$, $q \in (0,1]$, and E a $(k-1)$-dimensional subspace of $\mathbb{R}^n$ for $1 \le k \le n$. Then the map

$$u \longmapsto \frac{|u|}{\left|K \cap \mathrm{span}(u, E)\right|_k}, \quad u \in E^\perp$$

defines the Minkowski functional of a q'-convex body in $E^\perp$ with $q' = [(1/q-1)\,k+1]^{-1}$.

Let K be a symmetric star body in $\mathbb{R}^n$ and $0 < q \le 1$. Then, if K is q-convex, $\mathrm{I}K$ is q'-convex where $q' = [(1/q-1)\,(n-1)+1]^{-1}$.

- Similar to Busemann's original proof.
- Yes, q' does not look nice!
- Is q' optimal?

J. Kim, V. Yaskin, A. Z., 2010

Let K be an origin-symmetric q-convex body in $\mathbb{R}^n$, $q \in (0,1]$, and E a $(k-1)$-dimensional subspace of $\mathbb{R}^n$ for $1 \le k \le n$. Then the map

$$u \longmapsto \frac{|u|}{\left|K \cap \mathrm{span}(u,E)\right|_k}, \quad u \in E^\perp$$

defines the Minkowski functional of a q'-convex body in $E^\perp$ with $q' = [(1/q-1)\,k+1]^{-1}$.

Let K be a symmetric star body in $\mathbb{R}^n$ and $0 < q \le 1$. Then, if K is q-convex, $\mathrm{I}K$ is q'-convex where $q' = [(1/q-1)\,(n-1)+1]^{-1}$.

- Similar to Busemann's original proof.
- Yes, q' does not look nice!
- Is q' optimal? **It is sharp asymptotically** (next slide), thus q-convexity alone can not work as a condition for $\mathrm{I}K$ to be convex.

J. Kim, V. Yaskin, A. Z., 2010

Let K be an origin-symmetric q-convex body in $\mathbb{R}^n$, $q \in (0,1]$, and E a $(k-1)$-dimensional subspace of $\mathbb{R}^n$ for $1 \le k \le n$. Then the map

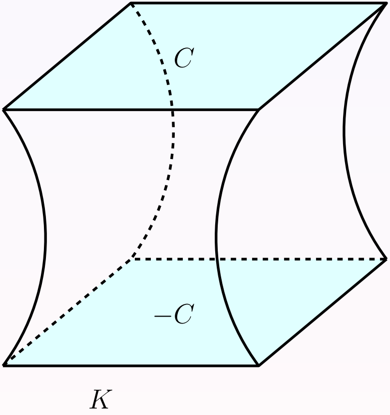
$$u \longmapsto \frac{|u|}{\left|K \cap \operatorname{span}(u,E)\right|_k}, \quad u \in E^\perp$$

defines the Minkowski functional of a q'-convex body in $E^\perp$ with $q' = [(1/q-1)\,k+1]^{-1}$.

Let K be a symmetric star body in $\mathbb{R}^n$ and $0 < q \le 1$. Then, if K is q-convex, IK is q'-convex where $q' = [(1/q-1)\,(n-1)+1]^{-1}$.

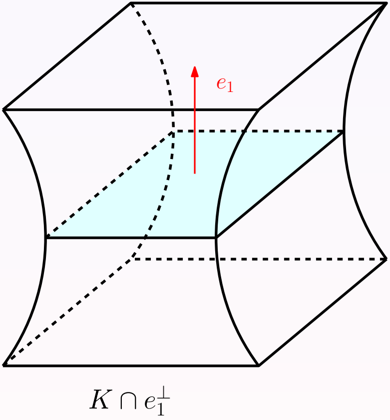
- Similar to Busemann's original proof.
- Yes, q' does not look nice!
- Is q' optimal? **It is sharp asymptotically** (next slide), thus q-convexity alone can not work as a condition for IK to be convex.
- By K. Ball's theorem, the classical Busemann's theorem can be generalized to **log-concave measure**. The same is true for q-convex case, but requires more work then just direct generalization of K. Ball's result!

$$K=\left\{t^{\frac{1}{q}}x+(1-t)^{\frac{1}{q}}y \,\middle|\, x\in C, y\in -C, 0\le t\le 1\right\},\quad C=\{1\}\times[-1,1]^{n-1}$$



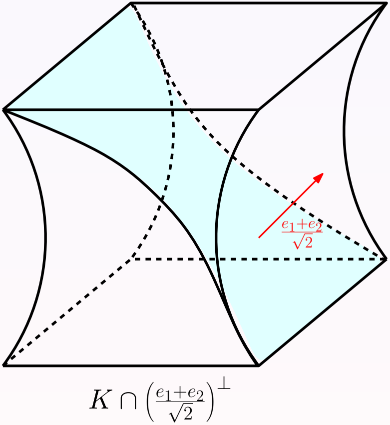
- K is q-convex

$$K = \left\{ t^{\frac{1}{q}} x + (1-t)^{\frac{1}{q}} y \,\middle|\, x \in C, y \in -C, 0 \le t \le 1 \right\}, \quad C = \{1\} \times [-1,1]^{n-1}$$



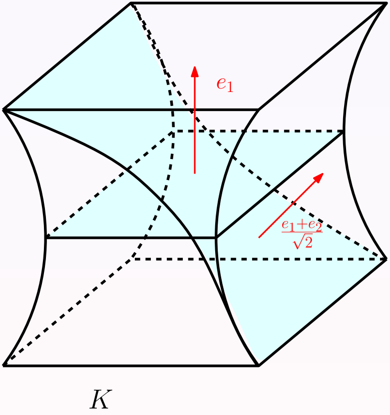
- K is q-convex
- $\left| K \cap e_1^{\perp} \right| = 2^{(2-\frac{1}{q})(n-1)}$

$$K=\left\{t^{\frac{1}{q}}x+(1-t)^{\frac{1}{q}}y \,\middle|\, x\in C, y\in -C, 0\le t\le 1\right\},\quad C=\{1\}\times[-1,1]^{n-1}$$



- K is q-convex
- $\left|K\cap e_1^{\perp}\right| = 2^{(2-\frac{1}{q})(n-1)}$
- $\left|K\cap\left(\frac{e_1+e_2}{\sqrt{2}}\right)^{\perp}\right| \ge 2^{n-\frac{1}{2}-\log_2 n}$

$$K = \left\{ t^{\frac{1}{q}} x + (1-t)^{\frac{1}{q}} y \,\middle|\, x \in C, y \in -C, 0 \le t \le 1 \right\}, \quad C = \{1\} \times [-1,1]^{n-1}$$



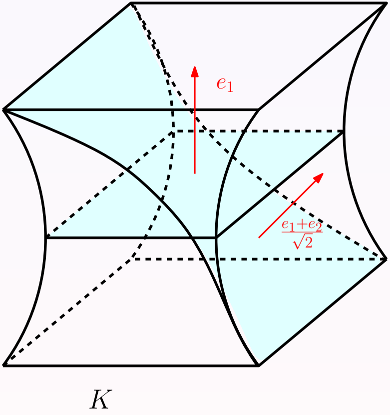
- K is q-convex
- $\left| K \cap e_1^{\perp} \right| = 2^{(2-\frac{1}{q})(n-1)}$
- $\left| K \cap \left(\frac{e_1+e_2}{\sqrt{2}} \right)^{\perp} \right| \ge 2^{n-\frac{1}{2}-\log_2 n}$

It means

$\|e_1\|_{IK} = 2^{(\frac{1}{q}-2)(n-1)}$,

$\|\frac{e_1+e_2}{2}\|_{IK} \le 2^{1-n+\log_2 n}$

$$K=\left\{t^{\frac{1}{q}}x+(1-t)^{\frac{1}{q}}y \,\middle|\, x\in C, y\in -C, 0\le t\le 1\right\},\quad C=\{1\}\times[-1,1]^{n-1}$$



- K is q-convex
- $\left|K\cap e_1^{\perp}\right| = 2^{(2-\frac{1}{q})(n-1)}$
- $\left|K\cap\left(\frac{e_1+e_2}{\sqrt{2}}\right)^{\perp}\right| \ge 2^{n-\frac{1}{2}-\log_2 n}$

It means
$\|e_1\|_{IK} = 2^{(\frac{1}{q}-2)(n-1)}$,
$\|\frac{e_1+e_2}{2}\|_{IK} \le 2^{1-n+\log_2 n}$

From $\|e_1\|_{IK}^{q'} \le \|\frac{e_1+e_2}{2}\|_{IK}^{q'} + \|\frac{e_1-e_2}{2}\|_{IK}^{q'}$,

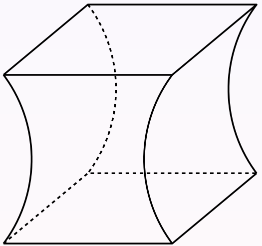
$$q' \le [(1/q-1)(n-1)+1-\log_2 n]^{-1} \approx [(1/q-1)(n-1)+1]^{-1}$$

Question: $d_{BM}(\mathrm{I}K, B_2^n) \le d_{BM}(K, B_2^n)$?

- Not known for symmetric convex case.
- VERY Not true without convexity!

Question: $d_{BM}(\mathrm{I}K, B_2^n) \le d_{BM}(K, B_2^n)$?

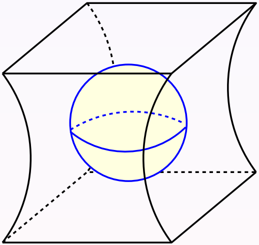
- Not known for symmetric convex case.
- VERY Not true without convexity!



- $\|e_1\|_{\mathrm{I}K} = 2^{(\frac{1}{p}-2)(n-1)}$
- $\|\frac{e_1+e_2}{2}\|_{IK} \le 2^{1-n+\log_2 n}$

Question: $d_{BM}(\mathrm{I}K, B_2^n) \le d_{BM}(K, B_2^n)$?

- Not known for symmetric convex case.
- VERY Not true without convexity!

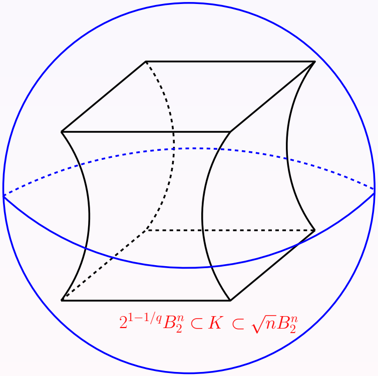


- From $2^{1-1/q}B_2^n \subset K \subset \sqrt{n}B_2^n$,

- $\|e_1\|_{\mathrm{I}K} = 2^{(\frac{1}{p}-2)(n-1)}$
- $\|\frac{e_1+e_2}{2}\|_{IK} \le 2^{1-n+\log_2 n}$

Question: $d_{BM}(\mathrm{I}K, B_2^n) \le d_{BM}(K, B_2^n)$?

- Not known for symmetric convex case.
- VERY Not true without convexity!



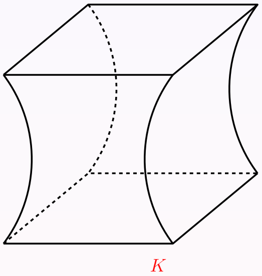
- From $2^{1-1/q}B_2^n \subset K \subset \sqrt{n}B_2^n$,

$$d_{BM}(K, B_2^n) \le 2^{1/q-1}\sqrt{n}$$

- $\|e_1\|_{\mathrm{I}K} = 2^{(\frac{1}{p}-2)(n-1)}$
- $\|\frac{e_1+e_2}{2}\|_{IK} \le 2^{1-n+\log_2 n}$

Question: $d_{BM}(\mathrm{I}K, B_2^n) \le d_{BM}(K, B_2^n)$?

- Not known for symmetric convex case.
- VERY Not true without convexity!

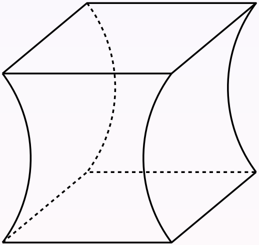


- $\|e_1\|_{\mathrm{I}K} = 2^{(\frac{1}{p}-2)(n-1)}$
- $\|\frac{e_1+e_2}{2}\|_{IK} \le 2^{1-n+\log_2 n}$

- From $2^{1-1/q} B_2^n \subset K \subset \sqrt{n} B_2^n$,
$$d_{BM}(K, B_2^n) \le 2^{1/q-1}\sqrt{n}$$
- Let $d = d_{BM}(IK, B_2^n)$, i.e, $\mathrm{I}K \subset TB_2^n \subset d\,\mathrm{I}K$. So we have $\mathrm{I}K \subset conv(\mathrm{I}K) \subset d\,\mathrm{I}K$.
$$d_{BM}(\mathrm{I}K, B_2^n) = d \ge \frac{\|e_1\|_{IK}}{\|e_1\|_{conv(IK)}}$$
$$\ge \frac{\|e_1\|_{IK}}{\|\frac{e_1+e_2}{2}\|_{conv(IK)} + \|\frac{e_1-e_2}{2}\|_{conv(IK)}} \ge \frac{\|e_1\|_{IK}}{\|e_1+e_2\|_{IK}}$$
$$\ge 2^{(\frac{1}{q}-1)(n-1)-1-\log_2 n}$$

Question: $d_{BM}(\mathrm{I}K, B_2^n) \le d_{BM}(K, B_2^n)$?

- Not known for symmetric convex case.
- VERY Not true without convexity!



K

- $\|e_1\|_{\mathrm{I}K} = 2^{(\frac{1}{p}-2)(n-1)}$
- $\|\frac{e_1+e_2}{2}\|_{IK} \le 2^{1-n+\log_2 n}$

- From $2^{1-1/q} B_2^n \subset K \subset \sqrt{n} B_2^n$,
$$d_{BM}(K, B_2^n) \le 2^{1/q-1}\sqrt{n}$$
- Let $d = d_{BM}(IK, B_2^n)$, i.e, $\mathrm{I}K \subset TB_2^n \subset d\,\mathrm{I}K$. So we have $\mathrm{I}K \subset conv(\mathrm{I}K) \subset d\,\mathrm{I}K$.
$$d_{BM}(\mathrm{I}K, B_2^n) = d \ge \frac{\|e_1\|_{IK}}{\|e_1\|_{conv(IK)}}$$
$$\ge \frac{\|e_1\|_{IK}}{\|\frac{e_1+e_2}{2}\|_{conv(IK)} + \|\frac{e_1-e_2}{2}\|_{conv(IK)}} \ge \frac{\|e_1\|_{IK}}{\|e_1+e_2\|_{IK}}$$
$$\ge 2^{(\frac{1}{q}-1)(n-1)-1-\log_2 n}$$
- $d_{BM}(IK, B_2^n) >> d_{BM}(K, B_2^n)$

Questions:

Assume $\mathrm{I}K$ is convex, what can we say about K?

Do there exists other then $c_n B_2^n$ fixed points of I in $\mathbb{R}^n$, $n \geq 3$?

Consider a star body $K \subset \mathbb{R}^n$, $n \geq 3$, is it true that

$$d_{BM}(\mathrm{I}^m K, B_2^n) \to 1, \text{ as } m \to \infty?$$

Consider a **convex** body $K \subset \mathbb{R}^n$, is it true that

$$d_{BM}(\mathrm{I}K, B_2^n) \leq d_{BM}(K, B_2^n),$$

with equality iff K is an Ellipsoid.

Do not like intersection bodies? Want to do harmonic analysis?

Consider an even function $f : \mathbb{S}^{n-1} \to \mathbb{R}^+$, such that $f = \mathcal{R} f^{n-1}$, is it true that then f is a constant?