Analytic and discrete aspects of the covariogram problem

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Outline

- Introduction to the covariogram problems.
- Detection of central symmetry.
- Reconstruction of lattice-convex sets.

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- We are given an (unknown) object A located in space.
- The diffraction information of A is available.
- How do we reconstruct A?
- This is a common reconstruction problem in physics.

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A general analytic formulation

- Let f be a distribution on $\mathbb{R}^d$ (with compact support).
- How do we reconstruct f from $|\widehat{f}|$?
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Analytic geometric version

- Let $K \subseteq \mathbb{R}^d$ be nonempty and compact with $K = \mathrm{cl}(\mathrm{int}(K))$.
- Then the function $x \mapsto g_K(x)$ on $\mathbb{R}^d$ defined by

$$g_K(x) = \mathrm{vol}(K \cap (K + x))$$

  is said to be the *covariogram* of K.
- The function g_K provides the same data as $|\widehat{\mathbf{1}_K}|$.
- Thus, reconstruction of K from g_K is a special case of the phase retrieval problem.
- The reconstruction is not unique, since $g_K(x)$ does not change
  - with respect to translations of K and
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- These are the trivial ambiguities.
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- E.g., considers finite sets $S, T \subseteq \mathbb{R}^d$ such that the sum of S and T is direct. Then the sum of S and $-T$ is also direct and...
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Results on uniqueness

- Within centrally symmetric objects the reconstruction is unique up to translations (no additional assumptions are required).
- Within planar convex bodies $K \subseteq \mathbb{R}^2$ the reconstruction of K from g_K is unique, up to translations and reflections (A. & Bianchi, 2009).
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Detecting central symmetry

- Can we detect from the diffraction data that the underlying object is centrally symmetric?
- In certain cases, yes.
- E.g., if K, H are convex bodies in $\mathbb{R}^d$, K is centrally symmetric and $g_K = g_H$. Then H is a translate of K. (Consequence of the Brunn-Minkowski inequality).
- Other cases?

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Detecting central symmetry for finite sets

Theorem 1 (A. 2009)

Let $A, B \subseteq \mathbb{R}^d$ be finite, let A be centrally symmetric and $g_A = g_B$. Then B is a translate of A.

- Proof idea:
- The case $d = 1$ is settled by induction.
- The case of general d is reduced to the case $d = 1$ by inductive argument...
- using some folkore results due to Renyi, Heppes et al.

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Detecting central symmetry in the discretized analytic case

Corollary 2

Let $K = A + [0,1]^d$ and $H = B + [0,1]^d$ where $A, B \subseteq \mathbb{Z}^d$ are finite. Let K be centrally symmetric and $g_K = g_H$. Then H is a translate of K.

- Proof idea (borrowed from Gardner, Gronchi and Zong):
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Covariogram problem for lattice convex sets

- A finite subset K of $\mathbb{Z}^d$ is said to be *lattice-convex* if K is the intersection of $\mathbb{Z}^d$ with a convex set.
- Problem: reconstruction of K from g_K in the class of lattice-convex sets.
- The problem was posed by Daurat, Gérard, Nivat (2005) and Gardner, Gronchi, Zong (2005).

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- Covariograms are the same.

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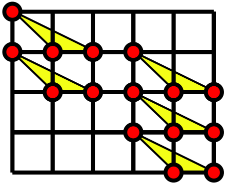
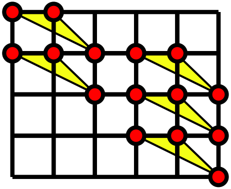
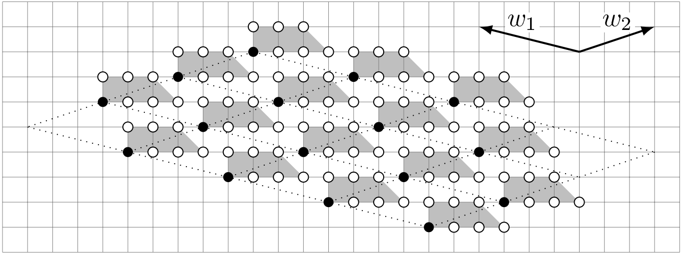
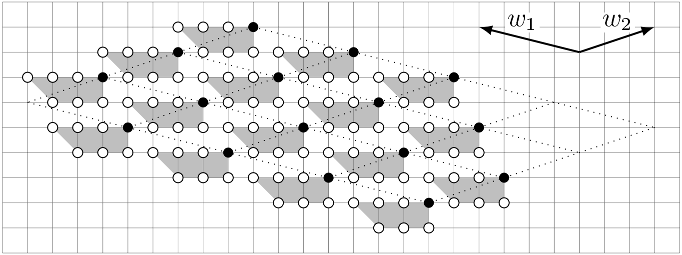




- This is the reason!

An infinite family of counterexamples





Direct sums are rarely lattice-convex

Theorem 3 (A. & Langfeld, 2011)

Let k, ℓ be integers with $k > \ell \geq 0$. We define

- $T := (\{0, \ldots, k\} \times \{0\}) \cup (\{0, \ldots, \ell\} \times \{1\})$ *(a set of lattice width one),*
- $w_1 := (-k-1, 1)$, $w_2 := (\ell+1, 1)$,
- *the lattice* $\mathbb{L} := \mathbb{Z}w_1 + \mathbb{Z}w_2$.

Let S be a set with $o \in S \subseteq \mathbb{Z}^2$. Then the following conditions are equivalent:

(i) *The sum of S and T is direct and lattice-convex.*

(ii) *S is lattice-convex with respect to $\mathbb{L}$ and $\operatorname{conv} S$ is a polygon in $\mathbb{R}^2$ such that*

- *every edge of $\operatorname{conv} S$ is parallel to w_1 or w_2 (in the case $k > \ell + 1$),*
- *every edge of $\operatorname{conv} S$ is parallel to w_1, w_2, or $w_1 + w_2$ (in the case $k = \ell + 1$).*

Direct lattice-convex summands of lattice-convex sets

- The situation that a lattice-convex set has a direct lattice-convex summand is very uncommon (work in progress).

Notation for the discrete uniqueness result

- Let K be a finite lattice-convex set in $\mathbb{R}^2$ such that $\operatorname{conv} K$ is two-dimensional.
- The *support set* of K in direction $u \in \mathbb{R}^d$ is defined by

$$F(K,u) := \{x \in K \,:\, \langle x, u\rangle = h(K,u)\}\,.$$

- The set of outer edge normals:

$$U(K) := \Big\{ u \in \mathbb{Z}^2\{o\} \,: \\ u \text{ is an outer normal to an edge of } \operatorname{conv} K \text{ and } \mathrm{gcd(u)=1} \Big\}.$$

- To measure the number of lattice points on the edges and the difference of parallel edges of K we introduce

$$\begin{aligned} m'(K) &:= \min\{\#F(K,u) \,:\, u \in U(K)\}\,, \\ m''(K) &:= \min\Big\{\#F(K,u) - \#F(K,-u) + 1 \,: \\ &\qquad u \in \mathbb{Z}^2 \setminus \{o\} \wedge \#F(K,u) > \#F(K,-u) > 1\Big\} \\ m(K) &:= \min\{m'(K), m''(K)\}, \end{aligned}$$

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Further notation

- For a finite set U of vectors in $\mathbb{R}^2$ linearly spanning $\mathbb{R}^2$ let

$$D(U) := \{|\det(u_1, u_2)| \ : \ u_1, u_2 \in U\} \setminus \{0\}$$

- We call

$$\delta(U) := \frac{\max D(U)}{\min D(U)}$$

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Positive result

Theorem 4

Let $K, L \subseteq \mathbb{Z}^2$ *be bounded and lattice-convex Then*

I. $m'(K)$, $m''(K)$, $m(K)$, $U(K) \cup U(-K)$ *and* $\delta(K)$ *are determined by* g_K.

II. *If*

$$m(K) \geq \delta(K)^2 + \delta(K) + 1$$

and

$$g_K = g_L,$$

then K *and* L *coincide up to translations and reflections.*

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