Some known results and open problems related to the covariogram

Gabriele Bianchi

Università di Firenze

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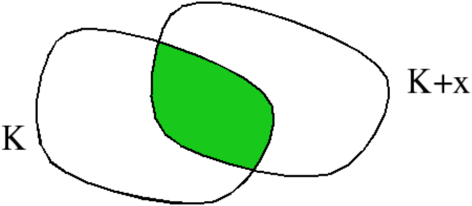
Definition of covariogram

$K \subset \mathbb{R}^n$ compact set, with $K = \mathrm{cl}\,(\mathrm{int}\,K)$
covariogram (or set covariance) of K = function $g_K : \mathbb{R}^n \to \mathbb{R}$ defined as

$$g_K(x) := \mathrm{vol}\,(K \cap (K + x)), \quad x \in \mathbb{R}^n$$



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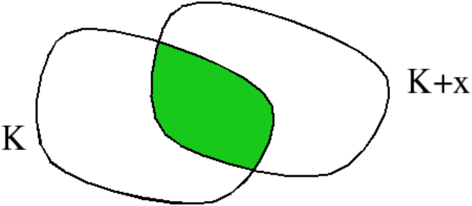


- g_K is invariant with respect to translations and reflections of K;
- the covariogram is the autocorrelation of 1_K,

$$g_K(x) = 1_K * 1_{(-K)}$$

that is, passing to Fourier transforms,

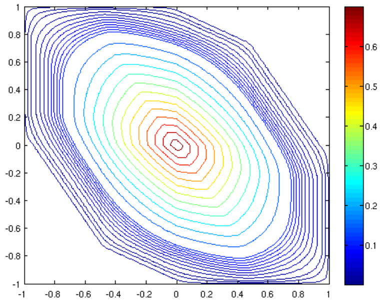
$$\widehat{g_K}(x) = |\widehat{1_K}(x)|^2 \quad \forall x \in \mathbb{R}^n.$$

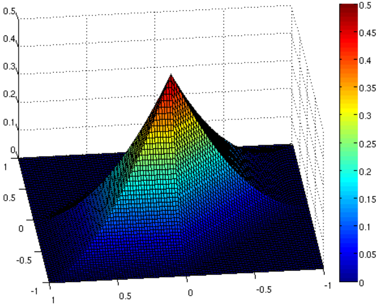
Structure of talk

1. Does g_K determine the set K? (“Covariogram problem”)
2. Which functions are covariograms?
3. Which geometric properties of K can be explicitly read in g_K?
4. A view at Question 1 from the Fourier transform side, in complex variables.
   - Zero sets of Fourier transforms of 1_K

Properties of covariogram: general

- support of g_K is $K + (-K)$;
  - $\Longrightarrow$ covariogram gives the width of K in every direction;
- $g_K(0) = \mathrm{vol}\,(K)$;
- g_K is even;
- when K is convex:
  - $(g_K)^{1/n}$ is concave (Brunn-Minkowski inequality).





Properties of covariogram: regularity and derivatives

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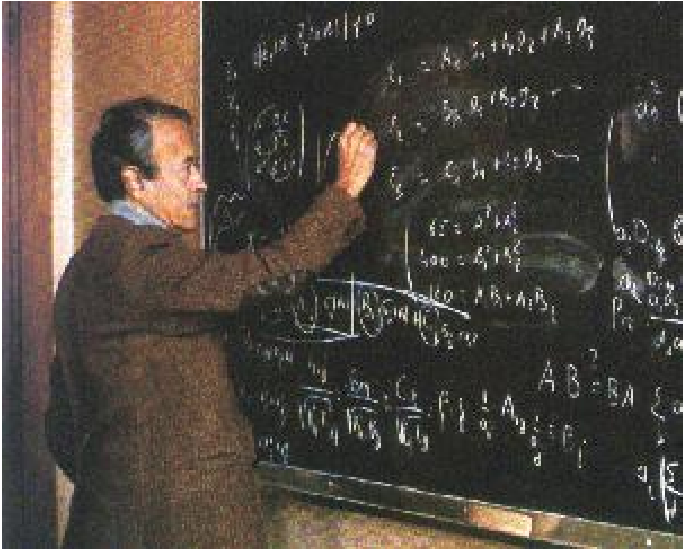
$$-\frac{\partial g_K}{\partial u}(0) = \mathrm{vol}\,(K|u^\perp) \qquad \forall u \in S^{n-1}$$

  - If K is strictly convex $\Longrightarrow g_K$ is C^1 in $\mathrm{int}\,(\mathrm{supp}\, g_K) \setminus \{0\}$;
  - If K is $C^1 \Longrightarrow g_K$ is C^2 in $\mathrm{int}\,(\mathrm{supp}\, g_K) \setminus \{0\}$;
  - explicit formulae for ∇g_K and $D^2 g_K$

  Meyer, Reisner and Schmuckenslager, Mathematika (1993).

G. Matheron and mathematical morphology



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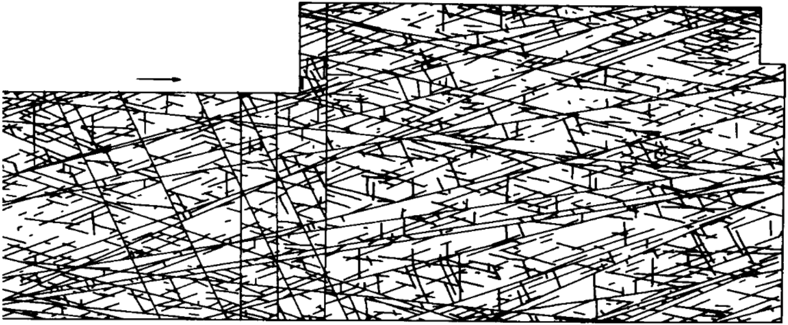

Figure IX.4. Simulation of geological faults.

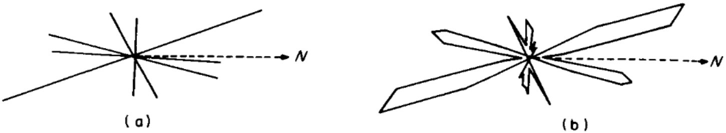

Figure IX.5. Rose of directions of the simulation on Fig. IX-4(a), and its estimation (b).

Section 1

Covariogram problem

Does g_K determine the set K?

G. Matheron, Le covariogramme géometrique des compacts convexes de $\mathbb{R}^2$ (1986).

R. Adler and R. J. Pyke, Inst. Math. Statistics Bull. (1991).

Results for Covariogram Problem I

Non-convex sets

In general NO

Rosenblatt and Seymour (1982); Gardner, Gronchi and Zhong (2005).

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Convex sets in $\mathbb{R}^2$

YES

- G. Averkov and G. Bianchi, J. Eur. Math. Soc. (2009).
- Other contributions from W. Nagel, G. Bianchi, F. Segala, A. Volčič.

Results for Covariogram Problem II

Convex Polytopes in any dimension

- YES for any generic polytope in $\mathbb{R}^n$, for any n

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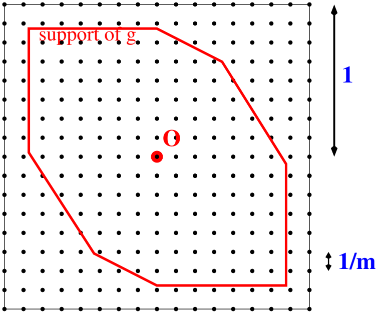
- YES in $\mathbb{R}^3$

  G. Bianchi, Adv. Math. (2009).

Covariogram problem: Algorithm for reconstruction

G. Bianchi, R. J. Gardner and M. Kiderlen, Journal of the American Mathematical Society (2011).

INPUT knowledge, affected by a random error, of g_K at all points of a grid containing $\operatorname{supp} g_K$



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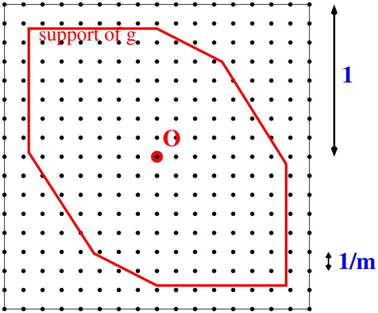

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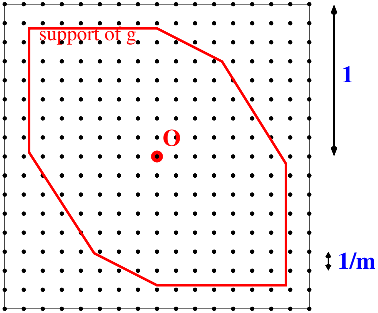

Theorem

Assume that $K \subset \mathbb{R}^n$ is a convex body determined by its covariogram. Then, as $m \to \infty$ (i.e. the grid gets finer and finer), almost surely,

$$P_m \to K \quad \text{or} \quad P_m \to -K.$$

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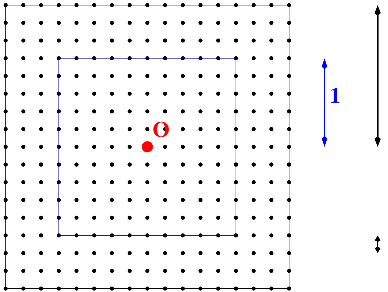

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Some open problems:

- Polytopes: fill the gap (other counterexamples and positive results)
- Explicit stability estimates of the map $g_K \to K$ (when it is injective)
  - estimates needed to obtain converge rates in the previous algorithm
- C^2_+ convex bodies: open even in dimension 3

Case of C^2_+ convex bodies in $\mathbb{R}^n$, $n \geq 3$

For each $u \in S^{n-1}$ let us consider the "Hessian matrix" of h_K:

$$A_K(u) := \nabla^2 h_K(u) + h_K(u) I.$$

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The covariogram gives for each u:

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- asymptotic behav. of g_K near point of $\mathrm{supp}\, g_K$ with outer normal u:

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Therefore we know:

1. $\Longrightarrow A_K(u) + A_K(-u)$
2. $\Longrightarrow \det A_K(u) + \det A_K(-u)$
3. $\Longrightarrow \det\left(A_K(u)^{-1} + A_K(-u)^{-1}\right)$

C^2_+ convex: information easily readable

The information that is easily readable in g_K is thus:

1. $A_K(u) + A_K(-u)$ (i.e. the even part of h_K)
2. the non-ordered set $\{\mathrm{Gauss}_K(u), \mathrm{Gauss}_K(-u)\}$

Ambiguity: Which is $\mathrm{Gauss}_K(u)$ and which is $\mathrm{Gauss}_K(-u)$?

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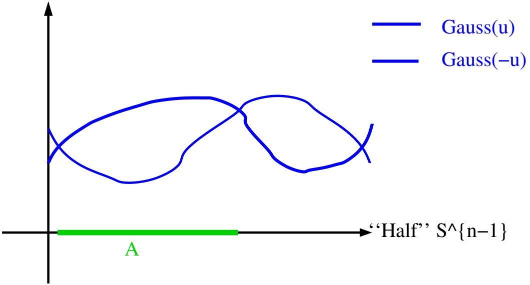

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- Not enough information to fully determine both matrices $A(u)$ and $A(-u)$ (In $\mathbb{R}^3$ we have "6 unknowns and only 5 conditions");
- For particular subclasses of C^2_+ convex bodies this difficulty can be overcome (bodies with analytic boundaries; bodies of revolution)

Section 2

Which functions are covariograms

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- A. Fish, D. Ryabogin and A. Zvavitch

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  - any $n \geq 2$ and $K \in C^2_+$ (it is a consequence of some formula presented in the previous slides:

    $$\text{the set} \quad \{\mathrm{Gauss}_K(u), \mathrm{Gauss}_K(-u)\} \quad \text{does not depend on } u \in S^{n-1}$$

    and continuity implies that Gauss_K is constant).

Proof of “radial g_K implies radial K”

Theorem

Assume $f \in L^2(\mathbb{R}^n)$, $n \geq 2$, real valued and with compact support. Assume that

$$|\widehat{f}|(x) \quad \text{is a radial function for } x \in \mathbb{R}^n.$$

Then f is a translation of a radial function.
Moreover f is determined in $L^2(\mathbb{R}^n)$ by $|\widehat{f}|$, up to translations.

W. Lawton, J. Opt. Soc. Am. (1981)

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- Let $\widehat{f}(z)$ be Fourier transform defined for $z \in \mathbb{C}^n$. Then

$$\widehat{f(z)} = c \left(\sum_{j=1}^{n} z_j^2 \right)^m e^{2\pi i\; x_0 \cdot z} \prod_{i=1}^{\infty} \left(1 - \frac{\sum_{j=1}^{n} z_j^2}{\lambda_i^2} \right) \left(1 - \frac{\sum_{j=1}^{n} z_j^2}{\overline{\lambda_i}^2} \right)$$

where $m \in \mathbb{N}$, $c \in \mathbb{C}$, $x_0 \in \mathbb{R}^n$, (λ_i) is a sequence in $\mathbb{C}$.

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- theorem false if f is complex valued

Which radial functions are covariograms?

Consequences:

- a radial function h is the covariogram of a convex set if and only if

$$h(x) = R^n g_{B(o,1)}\left(\frac{x}{R}\right) \quad \text{for } R = \frac{1}{2}\left(\frac{\mathrm{vol}\,(\mathrm{supp}\, h)}{\mathrm{vol}\,(B(o,1))}\right)^{\frac{1}{n}}.$$

- reduction of the corresponding problem for general set to a one-dimensional problem

Section 3

Which geometric properties of K can be explicitly read in g_K?

Determine whether the set is convex or not

Define two classes of sets:

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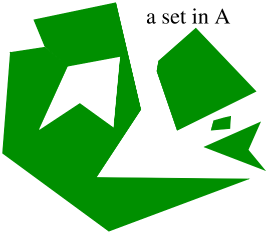

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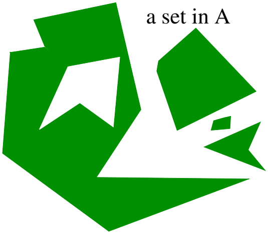

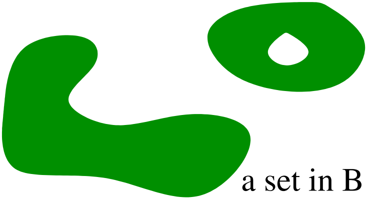

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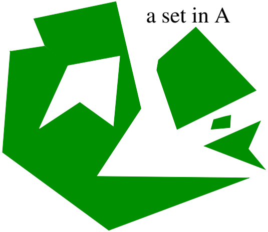

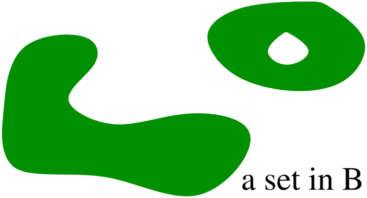

Theorem

Assume a-priori that g_K is the covariogram of a set in $\mathcal{A} \cup \mathcal{B}$.
Then there are explicitly computable properties of g_K that determine whether K is convex or not.

Benassi, Bianchi and D'Ercole, Mathematika (2010).

Determine whether the set is convex or not: test

First test (for class $\mathcal{B}$) : Check whether, for each $u \in S^1$, it is

$$-\frac{\partial}{\partial u} g_K(o) = \frac{1}{2}\mathrm{width}(\mathrm{supp}\, g_K, \text{``}u + \pi/2\text{''})$$

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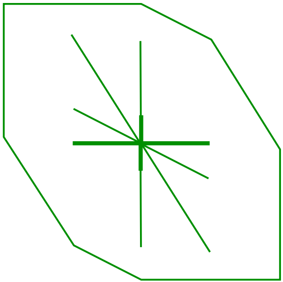

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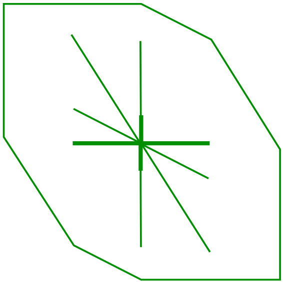
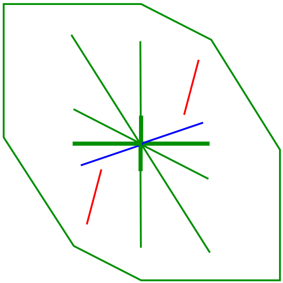
Second test (for class $\mathcal{A}$) : Check certain properties of the set of discontinuities of ∇g_K



Convex



Non-convex

Recognize other properties

Theorem. Central symmetry

Let $K \subset \mathbb{R}^n$ be a regular compact set.
The set K is convex and centrally symmetric if and only if

$$g_K(o) = \frac{1}{2}\left(\mathrm{vol}\left(\mathrm{supp}\, g_K\right)\right)^{\frac{1}{n}}$$

When this holds $K = (1/2)\mathrm{supp}\, g_K$.

- A consequence of Brunn-Minkowski inequality and its equality cases.

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- A consequence of Brunn-Minkowski inequality and its equality cases.

Theorem. Homothetic level lines of g_K

Let $K \subset \mathbb{R}^n$ be a centrally symmetric convex body. Assume that the level lines $\{x \ : \ g_K(x) = t\}$ are homothetic to each other for all $t > 0$ small.
Then K is an ellipsoid.

Meyer, Reisner and Schmuckenslager, Mathematika (1993).

- Connections with floating bodies.

Some open problems

Convexity or not It is known that when $K \subset \mathbb{R}^2$ is convex then $\sqrt{g_K}$ is concave on its support.
Does this property characterize the convexity of K in the class of planar regular compact set (or in some subclass)?

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Central symmetry Assume K non-convex. Is it possible to read in g_K whether K is centrally symmetric or not?

Homothetic level lines (MRS result) Is the central symmetry of K necessary?
Is it sufficient to assume that only two level lines are homothetic?

Section 4

A view at the covariogram problem from the Fourier transform side, in complex variables.

- J. L. C. Sanz and T. S. Huang, J. Math. Anal. Appl. (1984).
- N. E. Hurt, Phase retrieval and zero crossing (1989).
- T. Kobayashi, J. Fac. Sci. Univ. Tokyo Sect. IA Math. (1989).

Phase Retrieval?

Phase Retrieval in complex variables

Fourier transform in $\mathbb{C}^n$

Let $f \in L^2(\mathbb{R}^n)$ with compact support. Define, for $z \in \mathbb{C}^n$,

$$\widehat{f}(z) = \int_{\mathbb{R}^n} e^{ix \cdot z} f(x)\, dx.$$

FACT: $\widehat{f}$ is an holomorphic function *of exponential type*.

- *"of exponential type"* means that $|\widehat{f}(z)| \leq ae^{b\|z\|}$ for some $a, b \in \mathbb{C}$ and all $z \in \mathbb{C}^n$

Phase Retrieval in $\mathbb{C}^n$: role of factorization

Theorem. J. Sanz and T. Huang, J. Math. Anal. Appl. (1984)

Let $f \in L^2(\mathbb{R}^n)$ with compact support. If $\widehat{f}$ is irreducible in $\mathbb{C}^n$ then f is determined by the knowledge of $|\widehat{f}(x)|$ for $x \in \mathbb{R}^n$.

($\widehat{f}$ is irreducible in $\mathbb{C}^n$ if it cannot be written as $\widehat{f} = f_1\ f_2$, with f_1 and f_2 holomorphic entire functions, both with non-empty zero sets.)

Theorem. R. Barakat and G. Newsam, J. Math. Phys. (1984)

Let $f, g \in L^2(\mathbb{R}^2)$ with compact support. If $f \neq g$ and $|\widehat{f}(x)| = |\widehat{g}(x)|$ for each $x \in \mathbb{R}^2$, then, for all $z \in \mathbb{C}^2$,

$$\widehat{f}(z) = f_1(z)\ f_2(z) \quad \text{and} \quad \widehat{g}(z) = e^{ic\cdot z} f_1(z)\ \overline{f_2(\overline{z})},$$

for suitable holomorphic functions f_1 and f_2, both with non-empty zero set.

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for suitable holomorphic functions f_1 and f_2, both with non-empty zero set.

- The only known example of different convex sets with equal covariogram are $K_1 \times K_2$ and $K_1 \times (-K_2) \subset \mathbb{R}^n \times \mathbb{R}^m$. In Fourier space this corresponds to the phenomenon described in the Theorem.

Phase Retrieval in $\mathbb{C}^n$: zero set of $\widehat{1_K}(z)$

Theorem

Any $f \in L^2(\mathbb{R}^n)$ with compact support is determined, up to a translation, by the knowledge of

1) *the modulus of its FT, $|\widehat{f}(x)|$, for $x \in \mathbb{R}^n$;*
2) *the zero set of its FT, $\{z \in \mathbb{C}^n : \widehat{f}(z) = 0\}$.*

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Regarding the covariogram problem, what do we know about

$$\{z \in \mathbb{C}^n : \widehat{1_K}(z) = 0\}?$$

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Regarding the covariogram problem, what do we know about

$$\{z \in \mathbb{C}^n : \widehat{1_K}(z) = 0\}?$$

- The identity $g_K = 1_K * 1_{-K}$ seen in $\mathbb{C}^n$ becomes

$$\widehat{g_K}(z) = \widehat{1_K}(z)\,\overline{\left(\widehat{1_K}(\overline{z})\right)} \qquad \forall z \in \mathbb{C}^n. \tag{1}$$

Thus we have:

$$\text{zero set of } \widehat{g_K} = (\text{zero set of } \widehat{1_K}) \bigcup \overline{(\text{zero set of } \widehat{1_K})}$$

summarizing...

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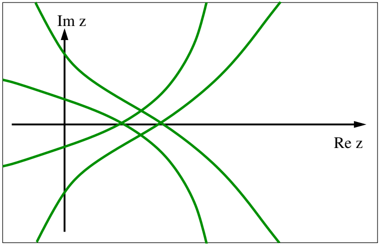
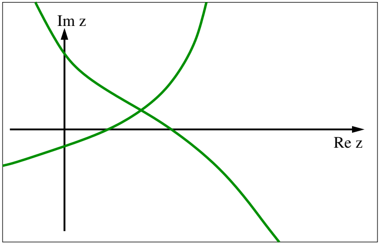


zero set of $\widehat{g_K}$



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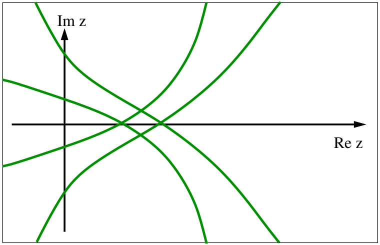

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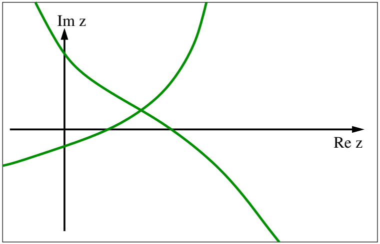

zero set of $\widehat{1_K}$

To determine 1_K it suffices:

- modulus of $\widehat{1_K}(x)$ for x real
- zero set of $\widehat{1_K}$ in $\mathbb{C}^n$

The knowledge of g_K gives us

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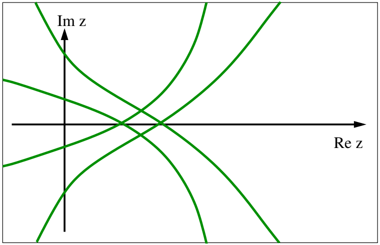

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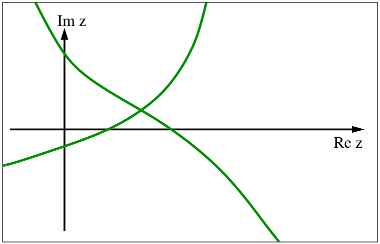

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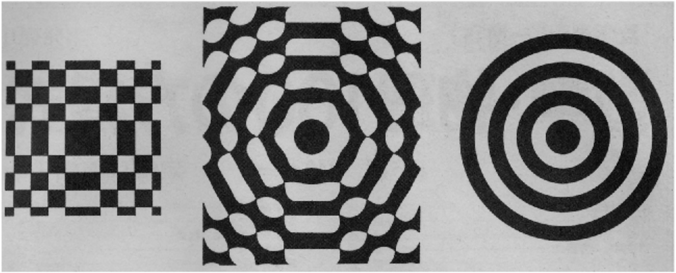
- (zero set of $\widehat{1_K}$ in $\mathbb{C}^n$) $\bigcup$ (its conjugate)

- Passing from $K_1 \times K_2$ to $K_1 \times (-K_2) \Longrightarrow$ substituting zeros of $\widehat{1_{K_2}}$ with their conjugates and leaving others zeros unchanged.

Kobayashi study of the zero set of $\widehat{1_K}$, for K convex

T. Kobayashi, J. Fac. Sci. Univ. Tokyo Sect. IA Math. (1989).



イ）Ω は正方形　　ロ）Ω は正六角形　　ハ）Ω は円板

- Image represents the restriction to $\mathbb{R}^2$ (i.e.$\{\mathrm{Im}\, z = 0\}$) of $\widehat{1_K}$ (which is real valued because K is o-symmetric);
- zero set = boundary between white and black regions;
- different behaviours:
  - Square : Zero set is non-compact and connected, it factors;
  - Ball: Zero set has countably many connected components.

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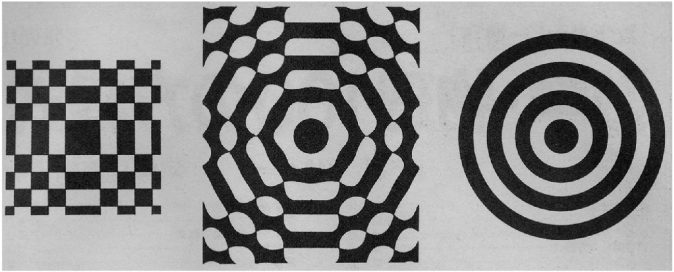

K=Square Hexagon Circle

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zero set of $\widehat{1_K}$, for K convex, smooth, with $\mathrm{Gauss}_K > 0$

Kobayashi proves that when K is convex, smooth enough and with $\mathrm{Gauss}_K > 0$ the asymptotic behaviour is similar to that of a ball.

Define a subset S of $\mathbb{C}^n$:

$$S = \text{“}S^{n-1} \times \mathbb{C}\text{”} = \{z \in \mathbb{C}^n : z = \lambda u + i\mu u, \text{ with } \lambda, \mu \in \mathbb{R}, u \in S^{n-1}\}$$

Theorem

$$\exists m_0 \in \mathbb{N} : \qquad (\textit{zero set of } \widehat{1_K}) \cap S = (\textit{compact set}) \bigcup \left(\bigcup_{j=m_0}^{\infty} \mathcal{Z}_j \right)$$

and each $\mathcal{Z}_j$ is a regular submanifold in S ($\subset \mathbb{C}^n$) diffeomorphic to S^{n-1}. Moreover

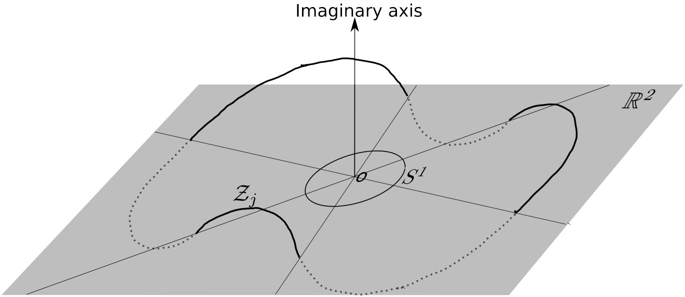
$$\mathcal{Z}_j = \{F_j(u)u \ : \ u \in S^{n-1}\}$$

with, as $j \to \infty$,

$$F_j(u) = \frac{\pi(2j + n - 1)}{\mathrm{width}_K(u)} + \mathbf{i} \log \left(\frac{\mathrm{Gauss}_K(u)}{\mathrm{Gauss}_K(-u)} \right) + \mathrm{O}\left(\frac{1}{j}\right).$$

shape of the components at ∞ of zero set of $\widehat{1_K}$

$$\mathcal{Z}_j : u \in S^{n-1} \to \left(\frac{\pi(2j+n-1)}{\mathrm{width}_K(u)} + \mathbf{i} \log\left(\frac{\mathrm{Gauss}_K(u)}{\mathrm{Gauss}_K(-u)} \right) + \mathrm{O}\left(\frac{1}{j}\right)\right) u.$$



- ∞ similar copies; each of them carries the same information;
- $\mathcal{Z}_j$ intersects $\mathbb{R}^n$ (i.e. $\mathrm{Im}\, z = 0$) for u such that $\mathrm{Gauss}_K(u) = \mathrm{Gauss}_K(-u)$;
- g_K gives us $\mathcal{Z}_j \bigcup \overline{\mathcal{Z}_j}$, and we need from this to determine $\mathcal{Z}_j$
  - the ambiguity is where $\mathcal{Z}_j$ and $\overline{\mathcal{Z}_j}$ meet: that is in $\mathcal{Z}_j \cap \mathbb{R}^n$ (again u such that $\mathrm{Gauss}_K(u) = \mathrm{Gauss}_K(-u)$!!!!!)

Does the zero set alone determine K?

- The paper by Kobayashi tries to understand whether the (asymptotics of the) zero set alone does determine K.
- His formula proves that this zero set determines, for each $u \in S^{n-1}$,

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Theorem (Kobayashi)

Assume K planar "smooth enough" convex body with $\mathrm{Gauss}_K > 0$. The asymptotic behaviour of the set

$$(\textit{zero set of } \widehat{1_K}) \cap S$$

determines K, up to translations.

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Problem

What about dimension $n \geq 3$?