# Successive radii of convex bodies

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(joint work with Bernardo González)

Universidad de Murcia

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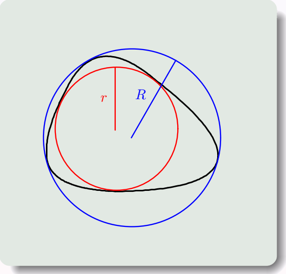

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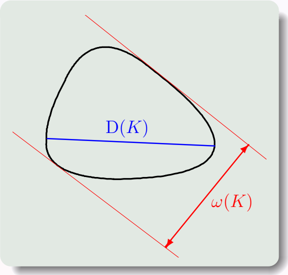

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- $L^\perp$ = the orthogonal complement of $L \in \mathcal{L}_i^n$.
- $K|L$ = the orthogonal projection of $K \in \mathcal{K}^n$ onto $L \in \mathcal{L}_i^n$.

# Successive inner and outer radii (projections/sections)

**Definition: Successive outer and inner radii**

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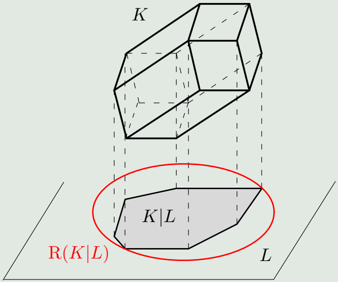

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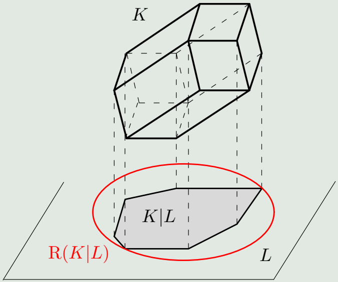


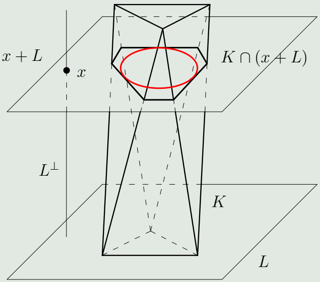

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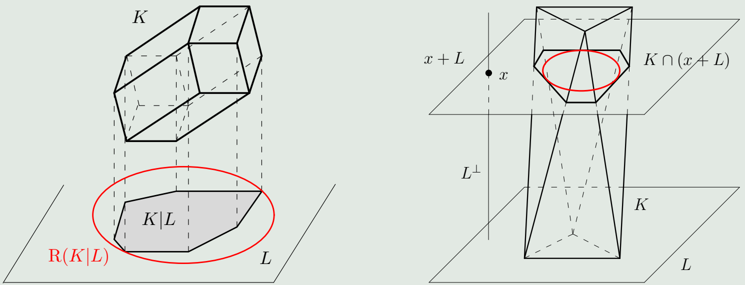

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$$\frac{\omega(K)}{2} = \mathrm{R}_1(K) \leq \cdots \leq \mathrm{R}_n(K) = \mathrm{R}(K)$$

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# Successive outer and inner radii: geometrical interpretation

$$\mathrm{R}_i(K) = \min_{L \in \mathcal{L}_i^n} \mathrm{R}(K|L) \qquad \mathrm{r}_i(K) = \max_{L \in \mathcal{L}_i^n} \max_{x \in L^\perp} \mathrm{r}\big(K \cap (x+L); x+L\big)$$

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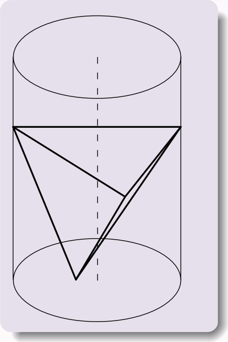

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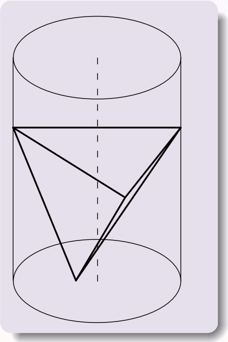
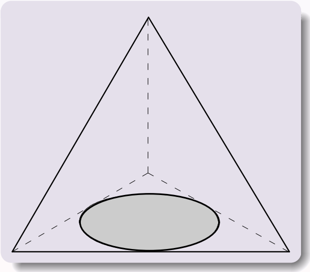
- $\mathrm{r}_i(K)$ = radius of the greatest $i$-dimensional ball contained in $K$.



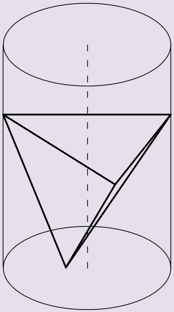


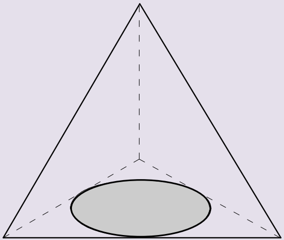
# Successive outer and inner radii: geometrical interpretation

Ball, K.: Ellipsoids of maximal volume in convex bodies, *Geom. Dedicata* **41** (1992)

**Theorem (Ball, 1992)**

The regular $n$-simplex $S_n$ circumscribing $B_n$ contains an $i$-dimensional ball of radius $\mathrm{r}_i(S_n) = \sqrt{n(n+1)/i(i+1)}$ *in each of its $i$-faces.*





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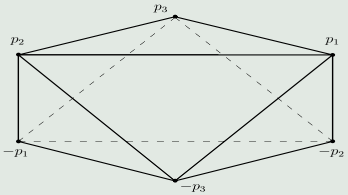

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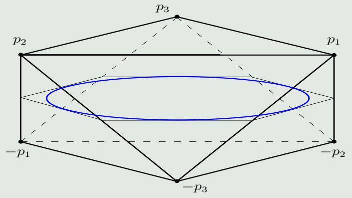


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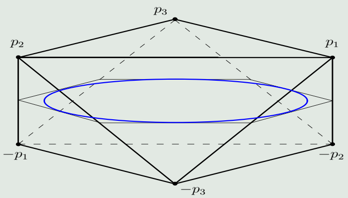


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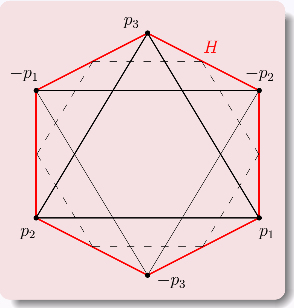


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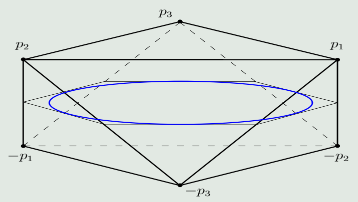


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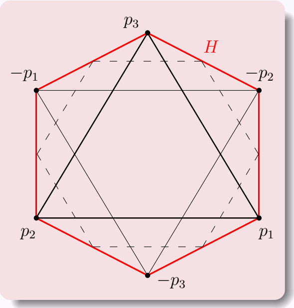


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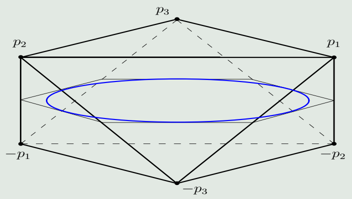


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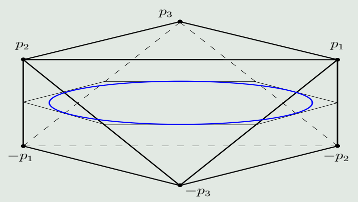


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- $\mathrm{r}_i : \{K \in \mathcal{K}^n : \dim K = n\} \longrightarrow \mathbb{R}_{\geq 0}$ is continuous.

# Further results

**Theorem (Gritzmann & Klee, 1992)**

If $K$ is 0-symmetric,

$$\mathrm{R}_i(K)\mathrm{r}_i(K^*) = 1.$$

P. Gritzmann, V. Klee: Inner and outer $j$-radii of convex bodies in finite-dimensional normed spaces, *Discrete Comput. Geom.* **7** (3) (1992).

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**Theorem (Pukhov, 1979; Perel'man, 1987)**

$$\frac{\mathrm{R}_{n-i+1}(K)}{\mathrm{r}_i(K)} < i + 1.$$

G. Ya. Perel'man: On the $k$-radii of a convex body, *Sibirsk. Mat. Zh.* **28** (1987); Pukhov, S. V.: Inequalities for the Kolmogorov and Bernšteĭn widths in Hilbert space, *Mat. Zametki* **25** (1979).

# Further results

## An improvement (Pukhov, 1979)

For $K$ centrally symmetric,

$$\frac{\mathrm{R}_{n-i+1}(K)}{\mathrm{r}_i(K)} < \sqrt{e}\,\min\left\{\sqrt{i}, \sqrt{n-i+1}\right\}.$$

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U. Betke, M. Henk: Estimating sizes of a convex body by successive diameters and widths, *Mathematatika* **39** (1992), 247-257

**Theorem**

Let $K \in \mathcal{K}^n$. Then:

$$\frac{2^n}{n!}\mathrm{r}_1(K)\cdots\mathrm{r}_n(K) \leq \mathrm{vol}(K) \leq 2^n\mathrm{R}_1(K)\cdots\mathrm{R}_n(K),$$

$$\kappa_n\bar{\mathrm{r}}_1(K)\cdots\bar{\mathrm{r}}_n(K) \leq \mathrm{vol}(K) \leq \kappa_n\overline{\mathrm{R}}_1(K)\cdots\overline{\mathrm{R}}_n(K).$$

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—, M. Henk: Intrinsic volumes and successive radii *J. Math. Anal. Appl.* **343** (2008)

We generalized these inequalities to the **intrinsic volumes**.

# Successive radii and Minkowski addition

For $K, K' \in \mathcal{K}^n$, it is well known that

$$\mathrm{r}(K+K') \geq \mathrm{r}(K)+\mathrm{r}(K'), \qquad \mathrm{R}(K+K') \leq \mathrm{R}(K)+\mathrm{R}(K'),$$

$$\omega(K+K') \geq \omega(K)+\omega(K'), \quad \mathrm{D}(K+K') \leq \mathrm{D}(K)+\mathrm{D}(K'),$$

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which can be rewritten in terms of the successive radii:

$$\mathrm{r}_1(K+K') \leq \mathrm{r}_1(K) + \mathrm{r}_1(K')$$

$$\mathrm{r}_n(K+K') \geq \mathrm{r}_n(K) + \mathrm{r}_n(K')$$

$$\mathrm{R}_1(K+K') \geq \mathrm{R}_1(K) + \mathrm{R}_1(K')$$

$$\mathrm{R}_n(K+K') \leq \mathrm{R}_n(K) + \mathrm{R}_n(K')$$

$$\overline{\mathrm{r}}_1(K+K') \geq \overline{\mathrm{r}}_1(K) + \overline{\mathrm{r}}_1(K')$$

$$\overline{\mathrm{r}}_n(K+K') \geq \overline{\mathrm{r}}_n(K) + \overline{\mathrm{r}}_n(K')$$

$$\overline{\mathrm{R}}_1(K+K') \leq \overline{\mathrm{R}}_1(K) + \overline{\mathrm{R}}_1(K')$$

$$\overline{\mathrm{R}}_n(K+K') \leq \overline{\mathrm{R}}_n(K) + \overline{\mathrm{R}}_n(K')$$

# Successive radii and Minkowski addition

For $K, K' \in \mathcal{K}^n$, it is well known that

$$\mathrm{r}(K+K') \geq \mathrm{r}(K) + \mathrm{r}(K'), \qquad \mathrm{R}(K+K') \leq \mathrm{R}(K) + \mathrm{R}(K'),$$

$$\omega(K+K') \geq \omega(K) + \omega(K'), \quad \mathrm{D}(K+K') \leq \mathrm{D}(K) + \mathrm{D}(K'),$$

which can be rewritten in terms of the successive radii:

$$\mathrm{r}_1(K+K') \leq \mathrm{r}_1(K) + \mathrm{r}_1(K')$$
$$\vdots \; ?$$
$$\mathrm{r}_n(K+K') \geq \mathrm{r}_n(K) + \mathrm{r}_n(K')$$

$$\mathrm{R}_1(K+K') \geq \mathrm{R}_1(K) + \mathrm{R}_1(K')$$
$$\vdots \; ?$$
$$\mathrm{R}_n(K+K') \leq \mathrm{R}_n(K) + \mathrm{R}_n(K')$$

$$\bar{\mathrm{r}}_1(K+K') \geq \bar{\mathrm{r}}_1(K) + \bar{\mathrm{r}}_1(K')$$
$$\vdots \; ?$$
$$\bar{\mathrm{r}}_n(K+K') \geq \bar{\mathrm{r}}_n(K) + \bar{\mathrm{r}}_n(K')$$

$$\overline{\mathrm{R}}_1(K+K') \leq \overline{\mathrm{R}}_1(K) + \overline{\mathrm{R}}_1(K')$$
$$\vdots \; ?$$
$$\overline{\mathrm{R}}_n(K+K') \leq \overline{\mathrm{R}}_n(K) + \overline{\mathrm{R}}_n(K')$$

How do the other successive radii relate with the Minkowski sum?

# Successive radii and Minkowski addition

B. González, —: Successive radii and Minkowski addition, *to appear in Monatsh. Math.* (2011).

**Theorem:** Let $K, K' \in \mathcal{K}^n$.

- If $1 \leq i \leq n-1$, then $\sqrt{2}\,\mathrm{r}_i(K+K') \geq \mathrm{r}_i(K) + \mathrm{r}_i(K')$.

# Successive radii and Minkowski addition

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**Theorem:** Let $K, K' \in \mathcal{K}^n$.

- If $1 \leq i \leq n-1$, then $\sqrt{2}\,\mathrm{r}_i(K+K') \geq \mathrm{r}_i(K) + \mathrm{r}_i(K')$.
- If $2 \leq i \leq n$, then $\sqrt{2}\,\mathrm{R}_i(K+K') \geq \mathrm{R}_i(K) + \mathrm{R}_i(K')$.

# Successive radii and Minkowski addition

B. González, —: Successive radii and Minkowski addition, *to appear in Monatsh. Math.* (2011).

**Theorem:** Let $K, K' \in \mathcal{K}^n$.

- If $1 \leq i \leq n-1$, then $\sqrt{2}\,\mathrm{r}_i(K+K') \geq \mathrm{r}_i(K) + \mathrm{r}_i(K')$.
- If $2 \leq i \leq n$, then $\sqrt{2}\,\mathrm{R}_i(K+K') \geq \mathrm{R}_i(K) + \mathrm{R}_i(K')$.

All inequalities are best possible: equality e.g. when $K$, $K'$ are the

$$\text{unit} \left\{ \begin{array}{l} i\text{-balls} \\ (n-i+1)\text{-balls} \end{array} \right\} \text{ of particular } \left\{ \begin{array}{l} i\text{-planes} \\ (n-i+1)\text{-planes} \end{array} \right\}.$$

# Successive radii and Minkowski addition

**For $1 \leq i \leq n-1$**

$$\sqrt{2}\,\mathrm{r}_i(K+K') \geq \mathrm{r}_i(K) + \mathrm{r}_i(K')$$

$$\mathrm{r}_n(K+K') \geq \mathrm{r}_n(K) + \mathrm{r}_n(K')$$

**For $2 \leq i \leq n$**

$$\mathrm{R}_1(K+K') \geq \mathrm{R}_1(K) + \mathrm{R}_1(K')$$

$$\sqrt{2}\,\mathrm{R}_i(K+K') \geq \mathrm{R}_i(K) + \mathrm{R}_i(K')$$

# Successive radii and Minkowski addition

**For $1 \leq i \leq n-1$**

$$\sqrt{2}\,\mathrm{r}_i(K+K') \geq \mathrm{r}_i(K) + \mathrm{r}_i(K')$$

$$\mathrm{r}_n(K+K') \geq \mathrm{r}_n(K) + \mathrm{r}_n(K')$$

$$\mathrm{r}_1(K+K') \leq \mathrm{r}_1(K) + \mathrm{r}_1(K')$$

**For $2 \leq i \leq n$**

$$\mathrm{R}_1(K+K') \geq \mathrm{R}_1(K) + \mathrm{R}_1(K')$$

$$\sqrt{2}\,\mathrm{R}_i(K+K') \geq \mathrm{R}_i(K) + \mathrm{R}_i(K')$$

$$\mathrm{R}_n(K+K') \leq \mathrm{R}_n(K) + \mathrm{R}_n(K')$$

# Successive radii and Minkowski addition

**For $1 \leq i \leq n-1$**

$$\sqrt{2}\,\mathrm{r}_i(K+K') \geq \mathrm{r}_i(K) + \mathrm{r}_i(K')$$

$$\mathrm{r}_n(K+K') \geq \mathrm{r}_n(K) + \mathrm{r}_n(K')$$

$$\mathrm{r}_1(K+K') \leq \mathrm{r}_1(K) + \mathrm{r}_1(K')$$

Opposite inequality $i = 2, \dots, n$?

**For $2 \leq i \leq n$**

$$\mathrm{R}_1(K+K') \geq \mathrm{R}_1(K) + \mathrm{R}_1(K')$$

$$\sqrt{2}\,\mathrm{R}_i(K+K') \geq \mathrm{R}_i(K) + \mathrm{R}_i(K')$$

$$\mathrm{R}_n(K+K') \leq \mathrm{R}_n(K) + \mathrm{R}_n(K')$$

Opposite inequality $i = 1, \dots, n-1$?

# Successive radii and Minkowski addition

**For $1 \le i \le n-1$**

$$\sqrt{2}\,\mathrm{r}_i(K+K') \ge \mathrm{r}_i(K) + \mathrm{r}_i(K')$$

$$\mathrm{r}_n(K+K') \ge \mathrm{r}_n(K) + \mathrm{r}_n(K')$$

$$\mathrm{r}_1(K+K') \le \mathrm{r}_1(K) + \mathrm{r}_1(K')$$

Opposite inequality $i = 2, \dots, n$? ✗

**For $2 \le i \le n$**

$$\mathrm{R}_1(K+K') \ge \mathrm{R}_1(K) + \mathrm{R}_1(K')$$

$$\sqrt{2}\,\mathrm{R}_i(K+K') \ge \mathrm{R}_i(K) + \mathrm{R}_i(K')$$

$$\mathrm{R}_n(K+K') \le \mathrm{R}_n(K) + \mathrm{R}_n(K')$$

Opposite inequality $i = 1, \dots, n-1$? ✗

**Proposition**

**There exists no constant** $C > 0$ such that for $2 \le i \le n$,

$$C\,\mathrm{r}_i(K+K') \le \mathrm{r}_i(K) + \mathrm{r}_i(K').$$

**There exists no constant** $C > 0$ such that for $1 \le i \le n-1$,

$$C\,\mathrm{R}_i(K+K') \le \mathrm{R}_i(K) + \mathrm{R}_i(K').$$

# Successive radii and Minkowski addition

Finally we consider the radii $\bar{\mathrm{r}}_i$ and $\overline{\mathrm{R}}_i$.

$$\bar{\mathrm{r}}_1(K+K') \geq \bar{\mathrm{r}}_1(K) + \bar{\mathrm{r}}_1(K')$$

$$\vdots \;?$$

$$\bar{\mathrm{r}}_n(K+K') \geq \bar{\mathrm{r}}_n(K) + \bar{\mathrm{r}}_n(K')$$

$$\overline{\mathrm{R}}_1(K+K') \leq \overline{\mathrm{R}}_1(K) + \overline{\mathrm{R}}_1(K')$$

$$\vdots \;?$$

$$\overline{\mathrm{R}}_n(K+K') \leq \overline{\mathrm{R}}_n(K) + \overline{\mathrm{R}}_n(K')$$

# Successive radii and Minkowski addition

Finally we consider the radii $\bar{\mathrm{r}}_i$ and $\overline{\mathrm{R}}_i$.

**Theorem:** Let $K, K' \in \mathcal{K}^n$.

For all $1 \leq i \leq n$,

$$\bar{\mathrm{r}}_i(K+K') \geq \bar{\mathrm{r}}_i(K) + \bar{\mathrm{r}}_i(K'),$$

$$\overline{\mathrm{R}}_i(K+K') \leq \overline{\mathrm{R}}_i(K) + \overline{\mathrm{R}}_i(K').$$

Equality e.g. when $K = K' = B_n$.

$$\bar{\mathrm{r}}_1(K+K') \geq \bar{\mathrm{r}}_1(K) + \bar{\mathrm{r}}_1(K')$$

$$\bar{\mathrm{r}}_i(K+K') \geq \bar{\mathrm{r}}_i(K) + \bar{\mathrm{r}}_i(K')$$

$$\bar{\mathrm{r}}_n(K+K') \geq \bar{\mathrm{r}}_n(K) + \bar{\mathrm{r}}_n(K')$$

$$\overline{\mathrm{R}}_1(K+K') \leq \overline{\mathrm{R}}_1(K) + \overline{\mathrm{R}}_1(K')$$

$$\overline{\mathrm{R}}_i(K+K') \leq \overline{\mathrm{R}}_i(K) + \overline{\mathrm{R}}_i(K')$$

$$\overline{\mathrm{R}}_n(K+K') \leq \overline{\mathrm{R}}_n(K) + \overline{\mathrm{R}}_n(K')$$

# Successive radii and Minkowski addition

Finally we consider the radii $\bar{\mathrm{r}}_i$ and $\overline{\mathrm{R}}_i$.

**Theorem:** Let $K, K' \in \mathcal{K}^n$.

For all $1 \leq i \leq n$,

$$\bar{\mathrm{r}}_i(K + K') \geq \bar{\mathrm{r}}_i(K) + \bar{\mathrm{r}}_i(K'),$$

$$\overline{\mathrm{R}}_i(K + K') \leq \overline{\mathrm{R}}_i(K) + \overline{\mathrm{R}}_i(K').$$

Equality e.g. when $K = K' = B_n$.

$$\bar{\mathrm{r}}_1(K + K') \geq \bar{\mathrm{r}}_1(K) + \bar{\mathrm{r}}_1(K')$$

$$\bar{\mathrm{r}}_i(K + K') \geq \bar{\mathrm{r}}_i(K) + \bar{\mathrm{r}}_i(K')$$

$$\bar{\mathrm{r}}_n(K + K') \geq \bar{\mathrm{r}}_n(K) + \bar{\mathrm{r}}_n(K')$$

$$\overline{\mathrm{R}}_1(K + K') \leq \overline{\mathrm{R}}_1(K) + \overline{\mathrm{R}}_1(K')$$

$$\overline{\mathrm{R}}_i(K + K') \leq \overline{\mathrm{R}}_i(K) + \overline{\mathrm{R}}_i(K')$$

$$\overline{\mathrm{R}}_n(K + K') \leq \overline{\mathrm{R}}_n(K) + \overline{\mathrm{R}}_n(K')$$

Opposite inequality for $i = 1, \ldots, n$?

# Successive radii and Minkowski addition

**Proposition**

**There exists no constant** $C > 0$ such that for $1 \leq i \leq n$

$$C\,\overline{\mathrm{r}}_i(K+K') \leq \overline{\mathrm{r}}_i(K) + \overline{\mathrm{r}}_i(K').$$

$$\overline{\mathrm{r}}_1(K+K') \geq \overline{\mathrm{r}}_1(K) + \overline{\mathrm{r}}_1(K')$$

$$\overline{\mathrm{r}}_i(K+K') \geq \overline{\mathrm{r}}_i(K) + \overline{\mathrm{r}}_i(K')$$

$$\overline{\mathrm{r}}_n(K+K') \geq \overline{\mathrm{r}}_n(K) + \overline{\mathrm{r}}_n(K')$$

$$\overline{\mathrm{R}}_1(K+K') \leq \overline{\mathrm{R}}_1(K) + \overline{\mathrm{R}}_1(K')$$

$$\overline{\mathrm{R}}_i(K+K') \leq \overline{\mathrm{R}}_i(K) + \overline{\mathrm{R}}_i(K')$$

$$\overline{\mathrm{R}}_n(K+K') \leq \overline{\mathrm{R}}_n(K) + \overline{\mathrm{R}}_n(K')$$

Opposite inequality for $i = 1, \ldots, n$?

# Successive radii and Minkowski addition

**Proposition**

**There exists no constant** $C > 0$ such that for $1 \leq i \leq n$

$$C\,\bar{\mathrm{r}}_i(K + K') \leq \bar{\mathrm{r}}_i(K) + \bar{\mathrm{r}}_i(K').$$

**However, there exists a constant** $C_{n,i} > 0$ such that for $1 \leq i \leq n$

$$C_{n,i}\,\overline{\mathrm{R}}_i(K + K') \leq \overline{\mathrm{R}}_i(K) + \overline{\mathrm{R}}_i(K').$$

$$\bar{\mathrm{r}}_1(K + K') \geq \bar{\mathrm{r}}_1(K) + \bar{\mathrm{r}}_1(K')$$

$$\bar{\mathrm{r}}_i(K + K') \geq \bar{\mathrm{r}}_i(K) + \bar{\mathrm{r}}_i(K')$$

$$\bar{\mathrm{r}}_n(K + K') \geq \bar{\mathrm{r}}_n(K) + \bar{\mathrm{r}}_n(K')$$

$$\overline{\mathrm{R}}_1(K + K') \leq \overline{\mathrm{R}}_1(K) + \overline{\mathrm{R}}_1(K')$$

$$\overline{\mathrm{R}}_i(K + K') \leq \overline{\mathrm{R}}_i(K) + \overline{\mathrm{R}}_i(K')$$

$$\overline{\mathrm{R}}_n(K + K') \leq \overline{\mathrm{R}}_n(K) + \overline{\mathrm{R}}_n(K')$$

Opposite inequality for $i = 1, \ldots, n$?

# Successive radii and Minkowski addition

**Proposition**

**There exists no constant** $C > 0$ such that for $1 \leq i \leq n$

$$C\,\overline{\mathrm{r}}_i(K+K') \leq \overline{\mathrm{r}}_i(K) + \overline{\mathrm{r}}_i(K').$$

**However, there exists a constant** $C_{n,i} > 0$ such that for $1 \leq i \leq n$

$$C_{n,i}\,\overline{\mathrm{R}}_i(K+K') \leq \overline{\mathrm{R}}_i(K) + \overline{\mathrm{R}}_i(K').$$

(We know $C_{n,1} = \sqrt{2}$ and $\sqrt{2} < C_{n,i} < 2$, $i > 1$).

$$\overline{\mathrm{r}}_1(K+K') \geq \overline{\mathrm{r}}_1(K) + \overline{\mathrm{r}}_1(K')$$

$$\overline{\mathrm{r}}_i(K+K') \geq \overline{\mathrm{r}}_i(K) + \overline{\mathrm{r}}_i(K')$$

$$\overline{\mathrm{r}}_n(K+K') \geq \overline{\mathrm{r}}_n(K) + \overline{\mathrm{r}}_n(K')$$

$$\overline{\mathrm{R}}_1(K+K') \leq \overline{\mathrm{R}}_1(K) + \overline{\mathrm{R}}_1(K')$$

$$\overline{\mathrm{R}}_i(K+K') \leq \overline{\mathrm{R}}_i(K) + \overline{\mathrm{R}}_i(K')$$

$$\overline{\mathrm{R}}_n(K+K') \leq \overline{\mathrm{R}}_n(K) + \overline{\mathrm{R}}_n(K')$$

Opposite inequality for $i = 1, \ldots, n$?

## New outer and inner successive radii

**Classical successive outer and inner radii**

For $K \in \mathcal{K}^n$ and $i = 1, \dots, n$ let

$$\mathrm{R}_i(K) = \min_{L \in \mathcal{L}_i^n} \mathrm{R}(K|L), \qquad \mathrm{r}_i(K) = \max_{L \in \mathcal{L}_i^n} \max_{x \in L^\perp} \mathrm{r}\big(K \cap (x + L); x + L\big),$$

$$\overline{\mathrm{R}}_i(K) = \max_{L \in \mathcal{L}_i^n} \mathrm{R}(K|L), \qquad \bar{\mathrm{r}}_i(K) = \min_{L \in \mathcal{L}_i^n} \max_{x \in L^\perp} \mathrm{r}\big(K \cap (x + L); x + L\big).$$

# New outer and inner successive radii

**Classical successive outer and inner radii**

For $K \in \mathcal{K}^n$ and $i = 1, \ldots, n$ let

$$\mathrm{R}_i(K) = \min_{L \in \mathcal{L}_i^n} \mathrm{R}(K|L), \qquad \mathrm{r}_i(K) = \max_{L \in \mathcal{L}_i^n} \mathrm{r}(K|L; L),$$

$$\overline{\mathrm{R}}_i(K) = \max_{L \in \mathcal{L}_i^n} \mathrm{R}(K|L), \qquad \bar{\mathrm{r}}_i(K) = \min_{L \in \mathcal{L}_i^n} \mathrm{r}(K|L; L).$$

# New outer and inner successive radii

**We define:**

$$\widetilde{\mathrm{R}}_i(K) = \int_{\mathcal{L}_i^n} \mathrm{R}(K|L)\, d\sigma(L), \qquad \widetilde{\mathrm{r}}_i(K) = \int_{\mathcal{L}_i^n} \mathrm{r}(K|L)\, d\sigma(L).$$

$\sigma(L)$ is the Haar measure on $\mathcal{L}_i^n$ such that $\sigma(\mathcal{L}_i^n) = 1$.

**Classical successive outer and inner radii**

For $K \in \mathcal{K}^n$ and $i = 1, \dots, n$ let

$$\mathrm{R}_i(K) = \min_{L \in \mathcal{L}_i^n} \mathrm{R}(K|L), \qquad \mathrm{r}_i(K) = \max_{L \in \mathcal{L}_i^n} \mathrm{r}(K|L; L),$$

$$\overline{\mathrm{R}}_i(K) = \max_{L \in \mathcal{L}_i^n} \mathrm{R}(K|L), \qquad \bar{\mathrm{r}}_i(K) = \min_{L \in \mathcal{L}_i^n} \mathrm{r}(K|L; L).$$

# New outer and inner successive radii

**We define:**

$$\widetilde{\mathrm{R}}_i(K) = \int_{\mathcal{L}_i^n} \mathrm{R}(K|L)\, d\sigma(L), \qquad \widetilde{\mathrm{r}}_i(K) = \int_{\mathcal{L}_i^n} \mathrm{r}(K|L)\, d\sigma(L).$$

**Some properties:**

- $\widetilde{\mathrm{R}}_i, \widetilde{\mathrm{r}}_i : \mathcal{K}^n \longrightarrow \mathbb{R}_{\geq 0}$ are continuous.

# New outer and inner successive radii

**We define:**

$$\widetilde{\mathrm{R}}_i(K) = \int_{\mathcal{L}_i^n} \mathrm{R}(K|L)\, d\sigma(L), \qquad \widetilde{\mathrm{r}}_i(K) = \int_{\mathcal{L}_i^n} \mathrm{r}(K|L)\, d\sigma(L).$$

**Some properties:**

- $\widetilde{\mathrm{R}}_i, \widetilde{\mathrm{r}}_i : \mathcal{K}^n \longrightarrow \mathbb{R}_{\geq 0}$ are continuous.
- $\widetilde{\mathrm{r}}_n(K) \leq \cdots \leq \widetilde{\mathrm{r}}_1(K) \qquad \widetilde{\mathrm{R}}_1(K) \leq \cdots \leq \widetilde{\mathrm{R}}_n(K)$

# New outer and inner successive radii

**We define:**

$$\widetilde{\mathrm{R}}_i(K) = \int_{\mathcal{L}_i^n} \mathrm{R}(K|L)\, d\sigma(L), \qquad \widetilde{\mathrm{r}}_i(K) = \int_{\mathcal{L}_i^n} \mathrm{r}(K|L)\, d\sigma(L).$$

**Some properties:**

- $\widetilde{\mathrm{R}}_i, \widetilde{\mathrm{r}}_i : \mathcal{K}^n \longrightarrow \mathbb{R}_{\geq 0}$ are continuous.
- $\mathrm{r}(K) = \widetilde{\mathrm{r}}_n(K) \leq \cdots \leq \widetilde{\mathrm{r}}_1(K) = \dfrac{\mathrm{b}(K)}{2} = \widetilde{\mathrm{R}}_1(K) \leq \cdots \leq \widetilde{\mathrm{R}}_n(K) = \mathrm{R}(K)$

# New outer and inner successive radii

**We define:**

$$\widetilde{\mathrm{R}}_i(K) = \int_{\mathcal{L}_i^n} \mathrm{R}(K|L)\, d\sigma(L), \qquad \widetilde{\mathrm{r}}_i(K) = \int_{\mathcal{L}_i^n} \mathrm{r}(K|L)\, d\sigma(L).$$

**Some properties:**

- $\widetilde{\mathrm{R}}_i, \widetilde{\mathrm{r}}_i : \mathcal{K}^n \longrightarrow \mathbb{R}_{\geq 0}$ are continuous.
- $\mathrm{r}(K) = \widetilde{\mathrm{r}}_n(K) \leq \cdots \leq \widetilde{\mathrm{r}}_1(K) = \dfrac{\mathrm{b}(K)}{2} = \widetilde{\mathrm{R}}_1(K) \leq \cdots \leq \widetilde{\mathrm{R}}_n(K) = \mathrm{R}(K)$

**Regarding Minkowski addition:**

- $\widetilde{\mathrm{R}}_1(K+K') = \widetilde{\mathrm{R}}_1(K) + \widetilde{\mathrm{R}}_1(K')$, $\quad \widetilde{\mathrm{r}}_1(K+K') = \widetilde{\mathrm{r}}_1(K) + \widetilde{\mathrm{r}}_1(K')$

# New outer and inner successive radii

**We define:**

$$\widetilde{\mathrm{R}}_i(K) = \int_{\mathcal{L}_i^n} \mathrm{R}(K|L)\,d\sigma(L), \qquad \widetilde{\mathrm{r}}_i(K) = \int_{\mathcal{L}_i^n} \mathrm{r}(K|L)\,d\sigma(L).$$

**Some properties:**

- $\widetilde{\mathrm{R}}_i, \widetilde{\mathrm{r}}_i : \mathcal{K}^n \longrightarrow \mathbb{R}_{\geq 0}$ are continuous.
- $\mathrm{r}(K) = \widetilde{\mathrm{r}}_n(K) \leq \cdots \leq \widetilde{\mathrm{r}}_1(K) = \dfrac{\mathrm{b}(K)}{2} = \widetilde{\mathrm{R}}_1(K) \leq \cdots \leq \widetilde{\mathrm{R}}_n(K) = \mathrm{R}(K)$

**Regarding Minkowski addition:**

- $\widetilde{\mathrm{R}}_1(K+K') = \widetilde{\mathrm{R}}_1(K) + \widetilde{\mathrm{R}}_1(K')$, $\widetilde{\mathrm{r}}_1(K+K') = \widetilde{\mathrm{r}}_1(K) + \widetilde{\mathrm{r}}_1(K')$
- $\widetilde{\mathrm{R}}_i(K+K') \leq \widetilde{\mathrm{R}}_i(K) + \widetilde{\mathrm{R}}_i(K')$, $\widetilde{\mathrm{r}}_i(K+K') \geq \widetilde{\mathrm{r}}_i(K) + \widetilde{\mathrm{r}}_i(K')$, $2 \leq i \leq n$

# New outer and inner successive radii

**We define:**

$$\widetilde{\mathrm{R}}_i(K) = \int_{\mathcal{L}_i^n} \mathrm{R}(K|L)\, d\sigma(L), \qquad \widetilde{\mathrm{r}}_i(K) = \int_{\mathcal{L}_i^n} \mathrm{r}(K|L)\, d\sigma(L).$$

**Some properties:**

- $\widetilde{\mathrm{R}}_i, \widetilde{\mathrm{r}}_i : \mathcal{K}^n \longrightarrow \mathbb{R}_{\geq 0}$ are continuous.
- $\mathrm{r}(K) = \widetilde{\mathrm{r}}_n(K) \leq \cdots \leq \widetilde{\mathrm{r}}_1(K) = \dfrac{\mathrm{b}(K)}{2} = \widetilde{\mathrm{R}}_1(K) \leq \cdots \leq \widetilde{\mathrm{R}}_n(K) = \mathrm{R}(K)$

**Regarding Minkowski addition:**

- $\widetilde{\mathrm{R}}_1(K+K') = \widetilde{\mathrm{R}}_1(K) + \widetilde{\mathrm{R}}_1(K')$, $\quad \widetilde{\mathrm{r}}_1(K+K') = \widetilde{\mathrm{r}}_1(K) + \widetilde{\mathrm{r}}_1(K')$
- $\widetilde{\mathrm{R}}_i(K+K') \leq \widetilde{\mathrm{R}}_i(K) + \widetilde{\mathrm{R}}_i(K')$, $\quad \widetilde{\mathrm{r}}_i(K+K') \geq \widetilde{\mathrm{r}}_i(K) + \widetilde{\mathrm{r}}_i(K')$, $2 \leq i \leq n$
- **There exists no** $C > 0$ with $C\,\widetilde{\mathrm{r}}_i(K+K') \leq \widetilde{\mathrm{r}}_i(K) + \widetilde{\mathrm{r}}_i(K')$.
- **There exists** $C_{n,i} > 0$ with $C_{n,i}\,\widetilde{\mathrm{R}}_i(K+K') \leq \widetilde{\mathrm{R}}_i(K) + \widetilde{\mathrm{R}}_i(K')$.

# New outer and inner successive radii

**We define:**

$$\widetilde{\mathrm{R}}_i(K) = \int_{\mathcal{L}_i^n} \mathrm{R}(K|L)\, d\sigma(L), \qquad \widetilde{\mathrm{r}}_i(K) = \int_{\mathcal{L}_i^n} \mathrm{r}(K|L)\, d\sigma(L).$$

**Some properties:**

- $\widetilde{\mathrm{R}}_i, \widetilde{\mathrm{r}}_i : \mathcal{K}^n \longrightarrow \mathbb{R}_{\geq 0}$ are continuous.
- $\mathrm{r}(K) = \widetilde{\mathrm{r}}_n(K) \leq \cdots \leq \widetilde{\mathrm{r}}_1(K) = \dfrac{\mathrm{b}(K)}{2} = \widetilde{\mathrm{R}}_1(K) \leq \cdots \leq \widetilde{\mathrm{R}}_n(K) = \mathrm{R}(K)$

**Regarding the volume:**

- $\mathrm{vol}(K) \leq \kappa_n \widetilde{\mathrm{R}}_1(K) \cdots \widetilde{\mathrm{R}}_n(K).$

# New outer and inner successive radii

**We define:**

$$\widetilde{\mathrm{R}}_i(K) = \int_{\mathcal{L}_i^n} \mathrm{R}(K|L)\, d\sigma(L), \qquad \widetilde{\mathrm{r}}_i(K) = \int_{\mathcal{L}_i^n} \mathrm{r}(K|L)\, d\sigma(L).$$

**Some properties:**

- $\widetilde{\mathrm{R}}_i, \widetilde{\mathrm{r}}_i : \mathcal{K}^n \longrightarrow \mathbb{R}_{\geq 0}$ are continuous.
- $\mathrm{r}(K) = \widetilde{\mathrm{r}}_n(K) \leq \cdots \leq \widetilde{\mathrm{r}}_1(K) = \dfrac{\mathrm{b}(K)}{2} = \widetilde{\mathrm{R}}_1(K) \leq \cdots \leq \widetilde{\mathrm{R}}_n(K) = \mathrm{R}(K)$

**Regarding the volume:**

- $\mathrm{vol}(K) \leq \kappa_n \widetilde{\mathrm{R}}_1(K) \cdots \widetilde{\mathrm{R}}_n(K)$.
- We believe that $\mathrm{vol}(K) \geq \kappa_n \widetilde{\mathrm{r}}_1(K) \cdots \widetilde{\mathrm{r}}_n(K)$.

# New outer and inner successive radii

**We define:**

$$\widetilde{\mathrm{R}}_i(K) = \int_{\mathcal{L}_i^n} \mathrm{R}(K|L)\, d\sigma(L), \qquad \widetilde{\mathrm{r}}_i(K) = \int_{\mathcal{L}_i^n} \mathrm{r}(K|L)\, d\sigma(L).$$

**Some properties:**

- $\widetilde{\mathrm{R}}_i, \widetilde{\mathrm{r}}_i : \mathcal{K}^n \longrightarrow \mathbb{R}_{\geq 0}$ are continuous.
- $\mathrm{r}(K) = \widetilde{\mathrm{r}}_n(K) \leq \cdots \leq \widetilde{\mathrm{r}}_1(K) = \dfrac{\mathrm{b}(K)}{2} = \widetilde{\mathrm{R}}_1(K) \leq \cdots \leq \widetilde{\mathrm{R}}_n(K) = \mathrm{R}(K)$

**Regarding the volume:**

- $\mathrm{vol}(K) \leq \kappa_n \widetilde{\mathrm{R}}_1(K) \cdots \widetilde{\mathrm{R}}_n(K)$.
- We believe that $\mathrm{vol}(K) \geq \kappa_n \widetilde{\mathrm{r}}_1(K) \cdots \widetilde{\mathrm{r}}_n(K)$.
- We have shown the weaker inequality

$$\mathrm{vol}(K) \geq \kappa_n \widetilde{\mathrm{r}}_{n-1}(K)^{n-1} \widetilde{\mathrm{r}}_n(K).$$

# Successive radii of convex bodies

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