Order isomorphisms for convex functions on windows

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Joint work with Shiri Artstein-Avidan and Vitali Milman

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Definition

Let S_1, S_2 be two partially ordered sets. A bijection $T : S_1 \longrightarrow S_2$ is called an **order-isomorphism** *if either:*

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Let $T : Cvx(\mathbb{R}^n) \to Cvx(\mathbb{R}^n)$ be an order isomorphism. Then:

- $(Tf)(x) = C_1(\mathcal{L}f)(Bx + v_0) + \langle v_1, x \rangle + C_0 \qquad$ *(reversing).*

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- $Cvx(K) = \{f : K \to \mathbb{R} \cup \{\infty\} : f \text{ is convex, l-s-c}\}$

F.L. - Functional form vs. characterizing properties

A fractional linear (f.l.) map is defined by:

$$F(x) = \frac{1}{\langle c, x\rangle + d}(Ax + b),$$

where $A \in M_{n\times n}$, $b, c \in \mathbb{R}^n$ and $d \in \mathbb{R}$ satisfy

$$\begin{pmatrix} A & b \\ c^T & d \end{pmatrix} \in GL_{n+1}.$$

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Dom(F) is an open half space U with $H = \partial U = \{\langle c, x\rangle = -d\}$
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Theorem (Shiffman '95)

Let $n \geq 2$. Let $K \subset \mathbb{R}^n$ be a convex set of full dimension. Assume that $F : K \to \mathbb{R}^n$ is an injective interval preserving map. Then F is fractional linear.

Order preserving isomorphisms on windows, *Cvx*

Theorem (Artstein S., Florentin, Milman V.)

If $T : Cvx(K_1) \to Cvx(K_2)$ *is an order preserving isomorphism, then there exists a fractional linear map* $F : K_1 \times \mathbb{R} \to K_2 \times \mathbb{R}$*, (in particular,* K_2 *is a fractional linear image of* K_1*), such that* $epi(Tf) = F(epi(f))$.

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This implies that there exist affine linear $L_0, L_1 : \mathbb{R}^n \to \mathbb{R}$ and a bijective fractional linear $G : K_2 \to K_1$ s.t. $Tf = \left(\frac{f}{L_0} \right) \circ G + L_1$.

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Now we consider geometric convex functions in a window:

- If $0 \in K$, let: $Cvx_0(K) = \{f \in Cvx(K) : f \geq 0, f(0) = 0\}$

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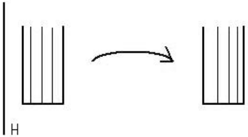
On this class appears another instance of F:

$F_{\mathcal{J}}(x, y) = (\frac{x}{y}, \frac{1}{y})$. It induces the transform $\mathcal{J}$ defined on $Cvx_0(K)$:

$$(\mathcal{J}f)(x) = \inf\{r > 0 : rf(\frac{x}{r}) \leq 1\},$$

and essentially there are no other order preserving isomorphisms on this class, but $\mathcal{J}$ and $\mathcal{I}$ - the identity.

- Affine hyperplanes; parallel mapped to parallel? Rarely...



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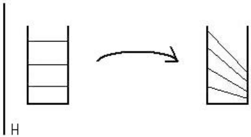

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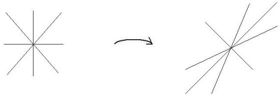

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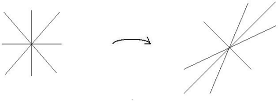
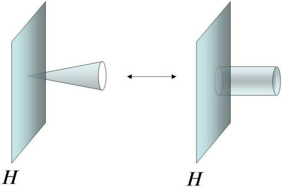


- Cones and half cylinders interchanged.



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Fact

Let F be an injective non-affine fractional linear map with $F(x_0) = y_0$. Then there exist $B, C \in GL_n$ such that $B(F(Cx + x_0) - y_0) = F_0(x)$.

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Fact (connection with classical polarity)

Let $K \subseteq H^- \subset \mathbb{R}^n$ be a closed convex set with $0 \in K$. Then:

$$F_0(K) = (e_1 - K^\circ)^\circ.$$

F.L. - Transitivity for $\triangle$, $\bigcirc$

Properties of fractional linear maps:

Fact

Let $\Delta_1, \Delta_2 \subset \mathbb{R}^n$ be two non degenerate open simplices. Let $p_1 \in \Delta_1$, and $p_2 \in \Delta_2$. There exists a bijective fractional linear map $F : \Delta_1 \to \Delta_2$ s.t. $F(p_1) = p_2$.

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Fact

Let B_n denote the open unit ball in $\mathbb{R}^n$, and $\mathcal{E}$ be some open ellipsoid, with $p \in \mathcal{E}$. Then there exists a bijective fractional linear map $F : B_n \to \mathcal{E}$ with $F(0) = p$.

F.L. - Rigidity for $\Diamond$, $\Box$

Fact

Let $K \subset \mathbb{R}^n$ be a symmetric, closed, convex set, and $F : K \longrightarrow K$ a bijective fractional linear map.

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Let $K \subset \mathbb{R}^n$ be a symmetric, closed, convex set, and $F : K \longrightarrow K$ a bijective fractional linear map. If $F(0) = 0$, then F is linear.

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Fact

Any bijective fractional linear map $F : B_\infty^n \to B_\infty^n$ is linear.

Fact

Any bijective fractional linear map $F : B_1^n \to B_1^n$ is linear.

Theorem (Artstein S., Florentin, Milman V.)

Let $K_1, K_2 \subseteq \mathbb{R}^n$ be convex sets with non empty interior, such that either $K_1 \neq \mathbb{R}^n$ or $K_2 \neq \mathbb{R}^n$. Then there does not exist any order reversing isomorphism $T : Cvx(K_1) \to Cvx(K_2)$.

Duality statements 1

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- Same for $T : Cvx_0(K_1) \to Cvx_0(K_2)$.
- Generalizing geometric convex functions ($T \subset K \subseteq \mathbb{R}^n$):

$$Cvx_T(K) = \{f \in Cvx(\mathbb{R}^n) : 1_K^\infty \leq f \leq 1_T^\infty\}$$

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- $\mathcal{A} : Cvx_T(K) \to Cvx_{K^\circ}(T^\circ)$ is an order reversing isomorphism.

Duality statements 2

Theorem (Artstein S., Florentin, Milman V.)

Let $n \geq 2$ and $T : Cvx_0(K) \to Cvx_{K^\circ}(\mathbb{R}^n)$ an order reversing isomorphism. Then $\exists F : K \times \mathbb{R}^+ \to K \times \mathbb{R}^+$, fractional linear, such that

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where $F(f)$ satisfies $epi(F(f)) = F(epi(f))$.
We say that T is essentially *the geometric duality $\mathcal{A}$.*

- Moreover, the same holds for $Cvx_T(K)$:

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Theorem (Artstein S., Florentin, Milman V.)

Let $n \geq 2$, *and* $A \subset int(B), C \subset int(D)$ *be compact convex sets in* $\mathbb{R}^n$. *If* $T : Cvx_A(B) \to Cvx_C(D)$ *is an order reversing isomorphism, then* T *is essentially the geometric duality* $\mathcal{A}$. *In particular,* C *is a fractional linear image of* B° *and* D *is a fractional linear image of* A°.

The End