The even Orlicz Minkowski problem

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joint work with E. Lutwak, D. Yang, and G. Zhang

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The Minkowski problem

The Minkowski problem concerns

- Existence
- Uniqueness
- Stability

of convex hypersurfaces whose Gauss curvature (possibly in a generalized sense) is prescribed as a function of the outer unit normals.

Definitions

Convex bodies

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Surface area measure of a convex body

$$S_K(\omega) = \mathcal{H}^{n-1}\{x \in \partial K : \ x \text{ has an outer unit normal in } \omega\}$$

The Minkowski problem

Which measures on the sphere are surface area measures?

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Theorem (Minkowski 1897, Fenchel & Jessen 1938)

If $\mu \in \mathcal{M}(S^{n-1})$ satisfies

$$\int_{S^{n-1}} u \, d\mu(u) = o$$

and $\mu(s) < \mu(S^{n-1})$ for each great subsphere s of S^{n-1}, then

$$\mu = S_K.$$

The L_p-Minkowski problem

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What are necessary and sufficient conditions on $\mu \in \mathcal{M}(S^{n-1})$ such that there exists $K \in \mathcal{K}^n$ with

$$h_K^{1-p}\, dS_K = d\mu?$$

(Chen, Chou, Guan, Hu, Hug, Jiang, Lin, Lutwak, Ma, Oliker, Shen, Stancu, Umanskiy, Wang, Yang, Zhang,...)

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- Even case ($p \neq n$): Lutwak '93
- $p > n$: Chou & Wang '06, Guan & Lin
- Polytopal case for $p > 1$: Chou & Wang '06
- different approach by Hug, Lutwak, Yang & Zhang

An application of the even L_p-Minkowski problem

The sharp L_p Sobolev inequality (Aubin '76, Talenti '76)

For $1 < p < n$

$$\|f\|_{\frac{np}{n-p}} \leq c_{n,p}\|\nabla f\|_p.$$

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The affine L_p Sobolev inequality (Lutwak, Yang, Zhang '02)

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$$\|f\|_{\frac{np}{n-p}} \leq c_{n,p}\,\mathcal{E}_p(f).$$

$$\mathcal{E}_p(f)^{-n} = d_{n,p}\int_{S^{n-1}} \|u \cdot \nabla f\|_p^{-n}\, du$$

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The affine inequality is stronger than the classical one

$$\mathcal{E}_p(f) \leq \|\nabla f\|_p.$$

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- H., Schuster '09: Further strengthening of the affine L_p Sobolev inequality
- Affine inequalities for $p \geq n$ (Cianchi, Lutwak, Yang, Zhang, H., Schuster, Xiao, Bastero, Romance, Alonso,...)

The geometry behind $\mathcal{E}_p(f)$:

An application of the even L_p-Minkowski problem

THE GEOMETRY BEHIND $\mathcal{E}_p(f)$: Let μ^t be the even measure

$$\int_{S^{n-1}} g(v)\, d\mu^t(v) = \int_{\{|f|=t\}} g\left(\frac{\nabla f(x)}{|\nabla f(x)|}\right) |\nabla f(x)|^{p-1}\, d\mathcal{H}^{n-1}(x)$$

for every $g \in C_e(S^{n-1})$.

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$$\begin{aligned} \|u \cdot \nabla f\|_p^p &= \int_{\mathbb{R}^n} |u \cdot \nabla f(x)|^p\, dx \\ &= \int_0^\infty \int_{\{|f|=t\}} \frac{|u \cdot \nabla f(x)|^p}{|\nabla f(x)|}\, d\mathcal{H}^{n-1}(x)\, dt \end{aligned}$$

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&= \int_0^\infty \int_{S^{n-1}} |u \cdot v|^p h_{K_t}^{1-p}(v) dS_K(v)\, dt \\
&= \int_0^\infty h(\Pi_p K_t, v)^p\, dt
\end{aligned}$$

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Towards an Orlicz Brunn-Minkowski theory

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Orlicz Brunn-Minkowski theory

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Uncovered elements of an Orlicz Brunn-Minkowski theory

- Lutwak, Yang, Zhang '10: Orlicz projection and centroid bodies
- Ludwig, Reitzner '10: Orlicz affine surface areas

Suppose $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and $\mu \in \mathcal{M}_e(S^{n-1})$ is not concentrated on a great subsphere of S^{n-1}.

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$$c\varphi(h_K)\, dS_K = d\mu$$

for some positive number c?

Chou & Wang '06: Smooth setting

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Theorem (H., Lutwak, Yang, Zhang '10)

If φ is decreasing, then there exists an origin symmetric convex body K such that

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$\varphi \equiv 1$: Solution to classical even Minkowski problem.

Sketch of the proof

$$\phi(t) := \int_0^t \frac{1}{\varphi(s)}\, ds.$$

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Crucial functional

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Goal

Find a function where Φ attains maximum.

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ϕ increasing, $V(h) = V(h_K) \Longrightarrow \Phi(h) \leq \Phi(h_K)$.

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$$\begin{aligned}
\int_{S^{n-1}} \phi(h_K) \, d\mu &\geq \int_{S^{n-1}} \phi(r_K h_{[-\bar{v}_K, \bar{v}_K]}) \, d\mu \\
&\geq |\mu| \phi \left(\frac{1}{|\mu|} \int_{S^{n-1}} r_K h_{[-\bar{v}_K, \bar{v}_K]} \, d\mu \right) \\
&\geq |\mu| \phi(c r_K).
\end{aligned}$$

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$$\begin{aligned} \Phi(h_K) &= 2nV(K)^{\frac{1}{2n}} - \int_{S^{n-1}} \phi(h_K)\, d\mu \\ &\leq 2nr_K^{1/2} V(B)^{\frac{1}{2n}} - |\mu|\phi(cr_K) \end{aligned}$$

$$r_K > r \Longrightarrow \Phi(h_K) < 0.$$

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$$2nV(K_0)^{\frac{1}{2n}} = \lim_{i\to\infty} 2nV(h_{K_i})^{\frac{1}{2n}} \geq \lim_{i\to\infty} \Phi(h_{K_i}) > 0.$$

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Hence K_0 has non-empty interior and

$$\Phi(f) \leq \Phi(h_{K_0}).$$

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$$V(K_0)^{\frac{1}{2n}-1} \varphi(h_{K_0}) \, dS_{K_0} = d\mu.$$

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Theorem (H., Lutwak, Yang, Zhang '10)

If $\phi(t) = \int_0^t 1/\varphi(s)\,ds$ exists and is unbounded for $t \to \infty$, then there exists an origin symmetric convex body K and $c > 0$ such that

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$\varphi(t) = t^{1-p}$: Solution of even L_p-Minkowski problem for $0 < p \neq n$