Roots of Steiner Polynomials

Martin Henk

joint work with
María A. Hernández Cifre and Eugenia Saorín

Otto-von-Guericke Universität Magdeburg

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  - Steiner polynomials, H.& Hernández Cifre&Saorín, Jetter, Katsnelson, Teissier, ...

- Steiner, 1840: Let $K, E \subset \mathbb{R}^n$ be two convex bodies, where we always assume $\dim(K + E) = n$.
For $\lambda \in \mathbb{R}_{\geq 0}$, the volume of $K + \lambda E$ is a polynomial of degree at most n in λ, the so called **Steiner polynomial** of K w.r.t. the gauge body E,

$$\mathrm{vol}(K + \lambda E) = \sum_{i=0}^{n} \binom{n}{i} \mathrm{W}_i(K; E)\lambda^i.$$

The coefficients $\mathrm{W}_i(K; E) = \mathrm{V}\big(K, \overset{(n-i)}{\ldots}, K, E, \overset{(i)}{\ldots}, E\big)$ are the so called **quermassintegrals** of K w.r.t. E, $0 \leq i \leq n$.

- For complex $z \in \mathbb{C}$ we denote the Steiner polynomial by

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- $\mathrm{W}_0(K;E) = \mathrm{vol}(K)$, $\mathrm{W}_n(K;E) = \mathrm{vol}(E)$.

- $\mathrm{W}_i(\mu_1\, K; \mu_2\, E) = \mu_1^{n-i} \mu_2^{i}\, \mathrm{W}_i(K;E),\ \ \mu_1, \mu_2 \geq 0.$

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- $\mathrm{W}_i(\mu_1 K; \mu_2 E) = \mu_1^{n-i} \mu_2^i \, \mathrm{W}_i(K;E), \ \ \mu_1, \mu_2 \geq 0$.

- $\mathrm{W}_i(K;E) \geq 0$ with equality if and only if $\dim K < n - i$ or $\dim E < i$. Hence

$$f_{K;E}(z) = \sum_{i=n-\dim K}^{\dim E} \binom{n}{i} \mathrm{W}_i(K;E) z^i$$

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$$f_{K;E}(z) = \mathrm{vol}(E) \sum_{i=0}^{n} \binom{n}{i} \mu^{n-i}\, z^i = \mathrm{vol}(E)\, (z+\mu)^n \, .$$

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- H.&H.C., 2008. $-\mu$ is an n-fold root of $f_{K;E}(z)$ if and only if $K = t + \mu\, E$.

- Favard, 1933. Let $E \subseteq K \subset \mathbb{R}^n$, $\dim E = n$, be convex bodies. K is a so called p-tangential body of E, $p \in \{0, \ldots, n-1\}$, if and only if

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- Let K be a p-tangential body of E. Then

$$f_{K;E}(z) = \mathrm{vol}(K)\left(\sum_{i=0}^{n-p}\binom{n}{i}z^i\right) + \sum_{i=n-p+1}^{n}\binom{n}{i}\mathrm{W}_i(K;E)\,z^i.$$

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- In particular, let K be a 1-tangential body of E. Then

$$\gamma_k^{-1} = \sqrt[n]{1 - \frac{\mathrm{vol}(E)}{\mathrm{vol}(K)}}\, \mathrm{e}^{\frac{2\pi k}{n}\mathrm{i}} - 1$$

are the roots of $f_{K;E}(z)$.

- Let B_n be the n-unit ball of volume κ_n. For the cube $C_n = [-1,1]^n$ we have

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$$\mathrm{W}_i(T_n; B_n) = \kappa_i \frac{n+1}{(n-i)!} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-(n-i+1)x^2} \left(\frac{1}{\sqrt{\pi}} \int_{-\infty}^{x} \mathrm{e}^{-y^2} \mathrm{d}y \right)^i \mathrm{d}x.$$

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  - Here

$$\mathrm{r}(K;E) = \max\{r \geq 0 : \text{some translate of } rE \subseteq K\},$$
$$\mathrm{R}(K;E) = \min\{R > 0 : \text{some translate of } K \subseteq RE\},$$

  are the inradius and circumradius of K w.r.t. E, respectively.

- Teissier, 1982. Let $\gamma_1, \ldots, \gamma_n$ be the roots of $f_{K;E}(z)$ with $\mathrm{Re}(\gamma_1) \leq \cdots \leq \mathrm{Re}(\gamma_n)$. For which convex bodies $K, E \subset \mathbb{R}^n$ does it hold

$$\mathrm{Re}(\gamma_1) \leq -\mathrm{R}(K;E) \leq -\mathrm{r}(K;E) \leq \mathrm{Re}(\gamma_n) \leq 0\,?$$

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- Teissier, 1982, also pointed out that for $n \leq 5$, Steiner-polynomials are stable, i.e., the real parts of all roots are non-positive ($\mathrm{Re}(\gamma_n) \leq 0$).

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- Katsnelson, 2009. Another family of high dimensional convex bodies contradicting the stability.

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- Jetter, 2011. Let $n \leq 9$ and let K be a C^2-convex body. Let ρ_{min} and ρ_{max} be the minimum and maximum values of the principal radii of curvature of K, and let γ_i, $1 \leq i \leq n$, be the roots of $f_{K;B_n}(z)$. Then

$$-\rho_{\mathsf{max}} \leq \mathrm{Re}(\gamma_i) \leq -\rho_{\mathsf{min}} \quad \text{for all} \quad i = 1, \ldots, n.$$

Where are the roots?

- Let

$$\mathcal{R}(n) := \left\{ z \in \mathbb{C}^{+} : f_{K;E}(z) = 0,\ K, E \subset \mathbb{R}^n,\ \dim(K+E) = n \right\}$$

be the set of all roots of Steiner polynomials of convex bodies $K, E \subset \mathbb{R}^n$, where we are just interested in the ones lying in the upper complex half plane $\mathbb{C}^{+}$.

- H.&H.C., 2011, H.&H.C.&S. 2011+.

  Let γ be a root of $f_{K;E}(z)$ and $\lambda > 0$. Then

  - $\lambda\gamma$ is a root of $f_{\lambda K;E}(z)$.

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  - $\gamma - \lambda$ is a root of $f_{K+\lambda E;E}(z)$.

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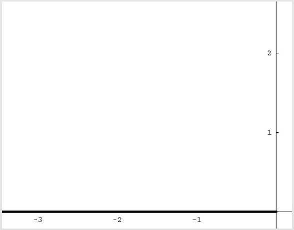

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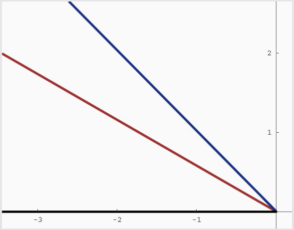

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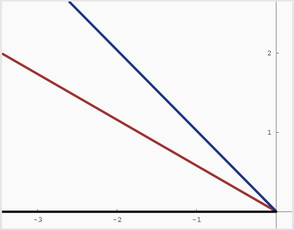

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  - For $n = 3, 4$, the "complex" boundary of $\mathcal{R}(n)$ is generated by a root of a truncated binomial polynomial:

$$\sum_{i=0}^{2} \binom{3}{i} z^i \quad \text{and} \quad \sum_{i=0}^{3} \binom{4}{i} z^i,$$

which can be realized as 1-tangential bodies K of lower dimensional bodies E.

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- Is $\operatorname{int} \mathcal{R}(n)$ independent of the gauge body E ($\dim E = n$)?
- Which bodies K, E form the "complex" boundary?

- Let

$$C(n) = \left\{ a = (a_0, \dots, a_n)^\intercal \in \mathbb{R}^{n+1}_{\geq 0} : \sum_{i=0}^{n} \binom{n}{i} a_i\, z^i = f_{K;E}(z) \right.$$
$$\left. \text{for convex bodies } K, E \text{ with } \dim(K+E) = n \right\}$$

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- In view of the Alexsandrov-Fenchel inequality we have

$$C(n) \subset \left\{ a \in \mathbb{R}^{n+1}_{\geq 0} : a_i^2 \geq a_{i-1} \, a_{i+1} \right\},$$

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but for $n \geq 3$ the inclusion is strict.
For instance, $(1, 0, 0, 1)$ cannot be the coefficients of a Steiner polynomial.

- Shephard, 1960.

$$C(n)\cap\mathbb{R}^{n+1}_{>0}=\Big\{a\in\mathbb{R}^{n+1}_{>0} : a_i^2\geq a_{i-1}\,a_{i+1}\Big\},$$

$$C(n)=\Big\{a\in\mathbb{R}^{n+1}_{\geq 0} : a_i\,a_j\geq a_{i-1}\,a_{j+1}\Big\}.$$

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- For more than two bodies a complete description of the "set of mixed volumes" is not known. Shephard also showed that "only" the Alexsandrov-Fenchel inequalities do not form a complete system for $n+2$ (n-dimensional) bodies.

- Shephard, 1960.

$$C(n) \cap \mathbb{R}^{n+1}_{>0} = \left\{ a \in \mathbb{R}^{n+1}_{>0} : a_i^2 \geq a_{i-1}\, a_{i+1} \right\},$$

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- For more than two bodies a complete description of the "set of mixed volumes" is not known. Shephard also showed that "only" the Alexsandrov-Fenchel inequalities do not form a complete system for $n+2$ (n-dimensional) bodies.

- Question: Find a complete description for more than 2 bodies?

- Idea of Shephard's proof: Let $\lambda_1 \geq \cdots \geq \lambda_n > 0$ and let

$$K = \operatorname{conv}\{0, e_1, \ldots, e_n\} \text{ and } E = \operatorname{conv}\{0, \lambda_1 e_1, \ldots, \lambda_n e_n\}$$

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Based on a dissection of $K + E$ into $n + 1$ simplices he finds

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Then $a_0 = W_0(K; E)$ and with

$$\lambda_1 = \frac{a_1}{a_0} \quad \lambda_i = \frac{a_{i-2}\, a_i}{a_{i-1}^2} \lambda_{i-1}$$

we have $\lambda_i \geq \lambda_{i-1}$ and $\mathrm{W}_i(K; E) = a_i$.

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  - Let $\gamma \in \mathcal{R}(n)$ and $K, E \subset \mathbb{R}^n$ such that $f_{K;E}(\gamma) = 0$. With $E' = E \times \mathrm{conv}\,\{0, e_{n+1}\} \subset \mathbb{R}^{n+1}$ we have

$$\mathrm{vol}(K + \lambda\, E') = \mathrm{vol}\big((K + \lambda E) \times \lambda \mathrm{conv}\,\{0, e_{n+1}\}\big) = \lambda\, \mathrm{vol}_n(K + \lambda E),$$

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- From the complete characterization of $C(n)$ one can deduce

  A real polynomial $\sum_{i=0}^{n} a_i z^i$, $a_i \geq 0$, is a Steiner polynomial $f_{K;E}(z)$ for $K, E \subset \mathbb{R}^n$, with

  $$\dim E = r, \quad \dim K = s, \quad \dim(K + E) = n,$$

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ii) the sequence $a_0, \ldots, a_n$ is ultra-logconcave, i.e.,

$$c_{i,n}\, a_i^2 \geq a_{i-1}\, a_{i+1} \ \text{ for } 1 \leq i \leq n-1.$$

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- Gurvits, 2009. "Steiner polynomial proof" of Liggett's theorem on the convolution of ultra-logconcave sequences.

- The proof of the strict monotonicity implies

H.&H.C.&S. 2011+ For $n \geq 3$, let $\gamma \in \mathrm{bd}\,\mathcal{R}(n) \backslash \mathbb{R}_{\leq 0}$ and let $K, E \subset \mathbb{R}^n$ with $f_{K;E}(\gamma) = 0$. Then there exists $i \in \{1, \dots, n-1\}$ such that K, E satisfy

$$\mathrm{W}_i(K;E)^2 = \mathrm{W}_{i-1}(K;E)\mathrm{W}_{i+1}(K;E),$$

with $\mathrm{W}_i(K;E) > 0$, i.e., they are extremal sets for at least one Aleksandrov-Fenchel inequality.

Possible candidates for the boundary?

- Any sequence

$$a = (\underbrace{0, \dots, 0}_{n-s}, \underbrace{1, \dots, 1}_{r+s+1-n}, \underbrace{0 \dots, 0}_{n-r})$$

of length $n+1$ corresponds to a Steiner polynomial

$$f_{K;E}(z) = \sum_{i=n-s}^{r} \binom{n}{i} z^i =: P^n_{n-s,r}(z)$$

with $\dim(K+E) = n$, $\dim K = s$, $\dim E = r$, which is a truncated binomial polynomial.

- $P^n_{0,r}(z) = \sum_{i=0}^{r} \binom{n}{i} z^i$

  can be realized (for instance) as an $(n-r)$-fold pyramid over an r-dimensional polytope E, which is an $(n-r)$-tangential body K of E.

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- In dimensions 3 and 4

$$P^3_{0,2}(z) \text{ and } P^4_{0,3}(z)$$

  have roots on the "complex boundary".

- But they are not "extremal" in dimension ≥ 5.

Optimal $P^n_{j,k}(z)$ in dimension ≤ 15:

$n = 3$:	$j = 0,\ k = 2$	$\gamma = -1.5000 + 0.8660i$	$\alpha = 2.6179$
$n = 4$:	$j = 0,\ k = 3$	$\gamma = -1.0000 + 1.0000i$	$\alpha = 2.3561$
$n = 5$:	$j = 1,\ k = 4$	$\gamma = -0.5000 + 0.8660i$	$\alpha = 2.0943$
$n = 6$:	$j = 1,\ k = 5$	$\gamma = -0.3856 + 0.9226i$	$\alpha = 1.9667$
$n = 7$:	$j = 1,\ k = 5$	$\gamma = -0.3249 + 1.2279i$	$\alpha = 1.8294$
$n = 8$:	$j = 2,\ k = 6$	$\gamma = -0.1464 + 0.9892i$	$\alpha = 1.7177$
$n = 9$:	$j = 2,\ k = 7$	$\gamma = -0.0698 + 0.9975i$	$\alpha = 1.6406$
$n = 10$:	$j = 2,\ k = 7$	$\gamma = 0.0158 + 1.1903i$	$\alpha = 1.5574$
$n = 11$:	$j = 3,\ k = 8$	$\gamma = 0.0854 + 0.9963i$	$\alpha = 1.4852$
$n = 12$:	$j = 3,\ k = 8$	$\gamma = 0.1533 + 1.1549i$	$\alpha = 1.4388$
$n = 13$:	$j = 3,\ k = 9$	$\gamma = 0.2127 + 1.1256i$	$\alpha = 1.3840$
$n = 14$:	$j = 4,\ k = 10$	$\gamma = 0.2400 + 0.9707i$	$\alpha = 1.3284$
$n = 15$:	$j = 4,\ k = 10$	$\gamma = 0.3139 + 1.0864i$	$\alpha = 1.2895$

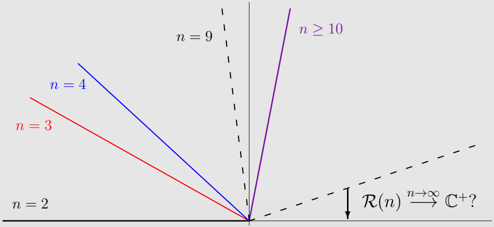
...

- Question: Which convex bodies $K, E \subset \mathbb{R}^n$ satisfy

$$f_{K;E}(z) = P^n_{j,k}(z)$$

for $j, k \in \{1, \dots, n-1\}$?

Where do they go?



- H.&H.C.&S. 2011+. Let $\gamma \in \mathbb{C}^+ \setminus \mathbb{R}_{>0}$. Then there exists an $n_\gamma \in \mathbb{N}$ with $\gamma \in \mathcal{R}(n)$ for all $n \geq n_\gamma$.

- It was shown by Ostrovskii, 2000 and Janson&Norfolk, 2009 that for any sequence $(k_n) \in \mathbb{N}$ such that $\alpha = \lim_{n\to\infty} k_n/n \in (0,1)$ the set of accumulation points of $\bigcup_{n=1}^{\infty}\left\{z \in \mathbb{C} : P^n_{0,k_n}(z) = 0\right\}$ coincide with the set

$$\left\{z \in \mathbb{C} : |z| = \alpha\,(1-\alpha)^{1/\alpha-1}\,|1+z|^{1/\alpha} \text{ and } \left|z - \frac{\alpha}{1-\alpha^2}\right| \leq \frac{\alpha^2}{1-\alpha^2}\right\}.$$

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- Hence, for $k_n = \lfloor n/2 \rfloor$, i.e., $\alpha = 1/2$, the number 1 is an accumulation point, and so there exists a sequence $\gamma_n \in \mathbb{C}^+ \setminus \mathbb{R}_{>0}$ such that $\lim_{n\to\infty} \gamma_n = 1$ and

$$P^n_{0,\lfloor n/2 \rfloor}(\gamma_n) = 0 \quad \text{for all } \; n \in \mathbb{N}.$$

- Since $P^n_{0,\lfloor n/2 \rfloor}$ are Steiner polynomials we are done. □

Questions:

- $\mathcal{R}(n)$ closed? Yes
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- Do the roots carry information about the geometric structure of the set?

- For instance:
  - H.&H.C., 2008. $-\mu$ is an n-fold root of $f_{K;E}(z)$ if and only if $K = t + \mu\, E$.
  - 0 is a k-fold root of $f_{K;E}(z)$ if and only if $\dim K \leq n - k$.

- H.&H.C., 2008. Let $K, E \subset \mathbb{R}^n$, $E \subset K$, $\dim E = n$, and let $\gamma_1, \ldots, \gamma_n$ be the roots of $f_{K;E}(z)$. Then K is a 1-tangential body of E if and only if there exists $\alpha \in (0,1)$ such that

$$\gamma_k^{-1} = \alpha^{1/n}\, \mathrm{e}^{\frac{2\pi(k-1)}{n}\mathrm{i}} - 1, \qquad \text{for} \;\; k = 1, \ldots, n$$

- H.&H.C., 2010.
  - Let $\mathrm{b} > 0$. Convex bodies $K, E \subset \mathbb{R}^n$ verifies the relations

$$\mathrm{W}_{n-k}(K;E) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} \mathrm{b}^{k-i} \mathrm{W}_{n-i}(K;E) \qquad (\star)$$

for $k = 0, 1, \ldots, n$, if and only if all the roots of its Steiner polynomial are symmetric with respect to $-\mathrm{b}/2$, i.e., γ root if and only if $-b-\gamma$ is a root.

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    for $k = 0, 1, \ldots, n$, if and only if all the roots of its Steiner polynomial are symmetric with respect to $-\mathrm{b}/2$, i.e., γ root if and only if $-b - \gamma$ is a root.

  - When $E = B_n$, bodies K of constant width verify $(\star)$. Thus, a constant width set K with breath b verifies that all the roots of $f_{K;B_n}(z)$ are symmetric with respect to $-\mathrm{b}/2$.

- Recall that for K with $\dim K = n$, $-\mathrm{r}(K; B_n)$ is an n-fold root of $f_{K;B_n}(z)$ if and only if $K = x + \mathrm{r}(K; B_n) B_n$ for some $x \in \mathbb{R}^n$.

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- Conjecture. Let $K \subset \mathbb{R}^n$, and let $m \in \{0, \ldots, n-1\}$. Then $-\mathrm{r}(K; B_n)$ is an $(n-m)$-fold root of $f_{K;B_n}(z)$ if and only if

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- H.&H.C., 2010. Let $n \geq 2$. Then K is a sausage with inradius $\mathrm{r}(K; B_n)$ if and only if $-\mathrm{r}(K; B_n)$ is an $(n-1)$-fold root of $f_{K;B_n}(z)$ and all its 2-dimensional projections have inradius $\mathrm{r}(K; B_n)$.

- Let $-\mathrm{r} = -\mathrm{r}(K; B_n)$ be a root of $f_{K;B_n}^{(n-2)}(z)$, i.e.,

$$\mathrm{W}_{n-2}(K) - 2\mathrm{r}\mathrm{W}_{n-1}(K) + \mathrm{r}^2\mathrm{W}_n(K) = 0.$$

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- Kubota's integral recursion formula implies

$$\int_{\mathcal{L}_2^n} f_{K|L;B_2}(-\mathrm{r})\, d\sigma(L) = \int_{\mathcal{L}_2^n} \mathrm{W}_0^{(2)}(K|L)\, d\sigma(L) - 2\mathrm{r} \int_{\mathcal{L}_2^n} \mathrm{W}_1^{(2)}(K|L)\, d\sigma(L)$$
$$+ \mathrm{r}^2 \int_{\mathcal{L}_2^n} \mathrm{W}_2^{(2)}(K|L)\, d\sigma(L)$$
$$= \frac{\mathrm{vol}_2(B_2)}{\mathrm{vol}(B_n)} \left[\mathrm{W}_{n-2}(K) - 2\mathrm{r}\mathrm{W}_{n-1}(K) + \mathrm{r}^2\mathrm{W}_n(K)\right] = 0.$$

- Since $\mathrm{r} = \mathrm{r}(K|L; B_n \cap L)$ for all $L \in \mathcal{L}_2^n$, Bonnesen's inequality states that

$$f_{K|L;B_2}(-\mathrm{r}) = \mathrm{W}_0^{(2)}(K|L) - 2\mathrm{W}_1^{(2)}(K|L)\mathrm{r} + \mathrm{W}_2^{(2)}(K|L)\mathrm{r}^2 \leq 0,$$

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- Hence $K|L$ is a 2-dimensional sausage with inradius r for any $L \in \mathcal{L}_2^n$.

- Then $\mathrm{r}\, B_n$ is a summand of K, i.e., there exists a convex body M with $K = M + \mathrm{r}\, B_n$. Since any 2-dimensional projection is a sausage, $\dim M = 1$.

The End

Thank you for your attention!