Steiner symmetrization and convergence

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Steiner symmetrization

Let $K \subseteq \mathbb{R}^n$ be a compact convex set, and let $u \in \mathbb{R}^n$ be a unit vector. Think of K as a family of line segments parallel to u.

Translate each of these line segments along the direction u until they are all balanced symmetrically around the plane $u^{\perp}$. The result is a new convex set $\mathrm{s}_u K$, called the *Steiner symmetrization* of K with respect to the direction u.

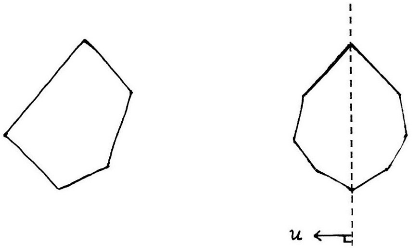


Figure: A convex set and its Steiner symmetral

- $\mathrm{s}_u K$ is symmetric under reflection across $u^\perp$.
- $V_n(\mathrm{s}_u K) = V_n(K)$
  – by Cavalieri's principle – here V_n denotes volume in $\mathbb{R}^n$.
- If B is a Euclidean ball then $\mathrm{s}_u B$ is a translate of B.
- $\mathrm{s}_u K$ is a translate of K iff K has reflectional symmetry in the direction of u.
- The only compact set stable under *every* Steiner symmetrization is a Euclidean ball centered at the origin.

Steiner symmetrization - Elementary Properties

- *Monotonicity* with respect to inclusion:

$$\text{If } K \subseteq L \text{ then } \mathrm{s}_u K \subseteq \mathrm{s}_u L.$$

- *Preserves* convexity, volume, shadow (projection) onto $u^\perp$.
- *Decreases* surface area (or perimeter), diameter, circumradius.
- *Increases* inradius.
- *Super-additivity* with respect to Minkowski sum:

$$\mathrm{s}_u(K + L) \supseteq \mathrm{s}_u K + \mathrm{s}_u L.$$

Steiner symmetrization may increase or decrease minimum width.

Steiner symmetrization - Continuity (sometimes)

Steiner symmetrization is continuous in the following limited sense:

If K has *non-empty interior*, and $K_i \longrightarrow K$ in the Hausdorff topology, then

$$\mathrm{s}_u K_i \longrightarrow \mathrm{s}_u K.$$

However, if K has empty interior (measure zero), there may be discontinuous behavior.

Idempotence and non-commutativity

Note that Steiner symmetrization is idempotent: $\mathrm{s}_u \mathrm{s}_u K = \mathrm{s}_u K$

But it is typically *non-commutative* when more than one direction is used; that is, usually:

$$\mathrm{s}_u \mathrm{s}_v K \ncong \mathrm{s}_v \mathrm{s}_u K.$$

Moreover, while $\mathrm{s}_v K$ is symmetric under reflection about $v^\perp$, the symmetral $\mathrm{s}_u \mathrm{s}_v K$ may no longer have this kind of symmetry.

Non-commutativity

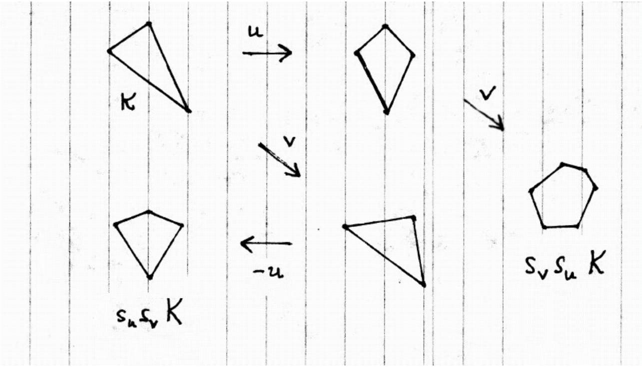
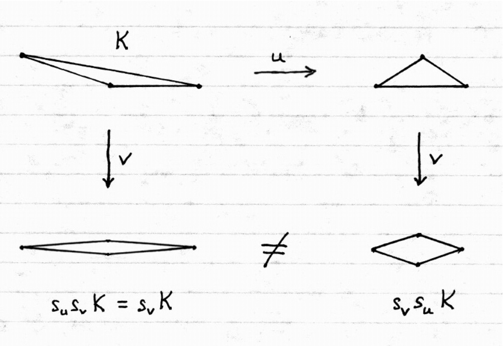


Figure: $\mathrm{s}_u \mathrm{s}_v K \not\cong \mathrm{s}_v \mathrm{s}_u K$ in general.

Accumulating symmetry

Exception: when $u \perp v$ the accumulated symmetry is retained.

(Although we may still have $\mathrm{s}_u \mathrm{s}_v K \not\cong \mathrm{s}_v \mathrm{s}_u K$.)



A fundamental convergence theorem

Theorem (Steiner, 1838 (?); Gross, 1917 (?))

Given a convex body K, there exists a sequence of directions u_i such that the sequence of Steiner symmetrals

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Recently, Bianchi and Gronchi (2003), Klartag and Milman (2003), and Klartag (2004) have given estimates on rates of convergence.

An analogous theorem replaces Steiner symmetrization with *shaking* and the Euclidean ball with a specified *simplex*. Campi, Colesanti, and Gronchi (2001) extended this Shaking Theorem to all *compact* sets.

Random symmetrization

Mani (1986)

Given a sequence of unit directions u_i chosen uniformly at random, the corresponding sequence of Steiner symmetrals of K converges to a ball almost surely; that is, with unit probability.

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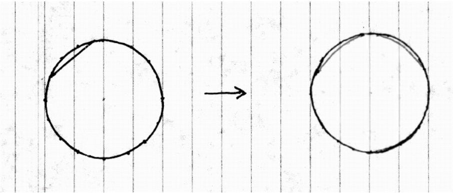
Van Schaftingen (2006) and Volčič (2009) have both given extensions of Mani's theorem to *compact* sets.

Proof of the convergence theorem (Sketch)

A classical proof of the convergence theorem runs roughly as follows: If K is not a ball already, the B denote the smallest ball that contains K.

Since K is compact, some relatively open set on the boundary of B avoids K.



Steiner symmetrization will enlarge the portion avoided on the boundary of B.

Proof of the convergence theorem (Sketch continued)

By judicious choice of directions, a finite sequence of Steiner symmetrizations will yield a symmetral $\widetilde{K}$ that avoids the boundary of B altogether, so that the circumradius of $\widetilde{K}$ is *strictly* less than that of K.

The argument then proceeds by minimizing circumradius over all successive Steiner symmetrals of K and applying standard compactness arguments (Blaschke selection) along with the continuity and/or monotonicity of Steiner symmetrization and circumradius.

A dense set of directions is sufficient

In a recent paper by Lutwak, Yang, and Zhang (2010), the authors required a sequence of Steiner symmetrizations that rounded out a given convex body, using only directions drawn from a *restricted dense set* of directions in the unit sphere.

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This leads to the question, however, of whether the *order* matters.

Can we apply a sequence Steiner symmetrizations in a dense set of directions willy-nilly to round out any convex body to a ball?

The answer this question is **No.**

A dense set of directions may not even work at all!

It also turns out the the *order* in which we apply Steiner symmetrizations in a countable dense set of directions $\{u_1, u_2, \ldots, \}$ will affect the *existence of a limit* for the sequence

$$K_i = \mathrm{s}_{u_i} \cdots \mathrm{s}_{u_2} \mathrm{s}_{u_1} K.$$

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It can be shown that **IF** the limit of the sequence $\{K_i\}$ exists, then it must be a ball. This follows from a very general theorem of Van Schaftingen (2005, 2006) as well as from more elementary arguments.

A dense set of directions may not even work at all!

However, it has recently been shown (Bianchi, K., Lutwak, Yang, Zhang 2010) that the limit of the sequence $\{K_i\}$ does *not* always exist.

This was also discovered independently by Burchard and Fortier (2011) and by Gronchi (2010).

An explicit counterexample

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For $m \geq 1$, let u_m denote the unit vector in $\mathbb{R}^2$ having counter-clockwise angle

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with the horizontal axis, measured in radians.

Since $\theta_m \to \infty$, while each successive incremental angle $\frac{\sqrt{2}}{p_m} \to 0$, the unit vectors u_m form a countable dense subset of the unit circle.

A counterexample (cont'd)

Applying Taylor's theorem and the Euler product formula, we obtain

$$\left(\prod_{i=1}^{\infty} \cos\left(\frac{\sqrt{2}}{p_i}\right)\right)^{-1} \leq \prod_{i=1}^{\infty}\left(\frac{1}{1-\frac{1}{p_i^2}}\right)$$

$$= \prod_{i=1}^{\infty}\left(1+\frac{1}{p_i^2}+\frac{1}{p_i^4}+\cdots\right) = \sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6},$$

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so that

$$\prod_{i=1}^{\infty} \cos\left(\frac{\sqrt{2}}{p_i}\right) \geq \frac{6}{\pi^2}.$$

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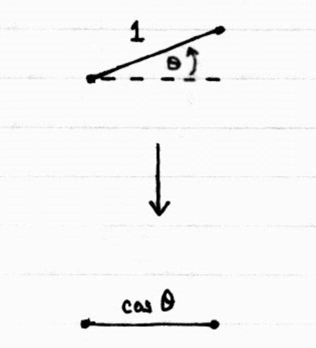
Each symmetrization has the effect of projecting the previous line segment onto the line perpendicular to u_m, thereby multiplying the previous length by the next incremental cosine, $\cos\left(\frac{\sqrt{2}}{p_m}\right)$.



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Since the limiting value of the cosine product is strictly positive (greater than 1/2, in fact), while the angles θ_m cycle around the circle forever, the iterated Steiner symmetrals of ℓ also spin in circles forever, while approaching a limiting positive length.

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In particular, the sequence of line segments

$$\ell_m = \mathrm{s}_{u_m} \cdots \mathrm{s}_{u_1} \ell$$

has no limit.

A counterexample (cont'd)

For an example with interior, let K be a cigar-shaped convex body of area ε containing that line segment ℓ as an axis of symmetry.

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By the monotonicity of Steiner symmetrization, each element in the sequence of Steiner symmetrals

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must contain the corresponding symmetral ℓ_m, so that the diameter of each K_m exceeds $\frac{6}{\pi^2}$.

Since each K_m has the same area ε as the original body K, which could be made arbitrarily small beforehand, it follows that the sequence K_m cannot approximate a ball.

A counterexample (cont'd)

Indeed, for $\varepsilon < \frac{9}{\pi^3}$ the sequence K_m has no limit, since the diameter line revolves forever, but does not shrink enough to accomodate the tiny given area ϵ.

A counterexample (cont'd)

Gronchi (2010) has shown independently that a more general family of examples can be constructed starting with any decreasing sequence of incremental angles θ_i provided that $\sum_{i=1}^{\infty} \theta_i^2$ converges and $\sum_{i=1}^{\infty} \theta_i$ diverges.

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Counterexamples to convergence are also described in a recent paper by Burchard and Fortier (2011).

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Alternatively, one might ask: do the directions of symmetrization used to attain a ball *need* to be dense?
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Again the answer is No.

Eggleston (1958) has shown that, given a basis of directions $u_1, \ldots, u_n$ for $\mathbb{R}^n$ having mutually irrational angle differences, the sequence

$$\mathrm{S}_{u_n} \cdots \mathrm{S}_{u_2} \mathrm{S}_{u_1}$$

iterated infinitely many times to any compact convex set K will result in a sequence of bodies converging to a ball of the same volume as K.

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This result is extended to *compact* sets in recent work of Burchard and Fortier (2011).

A related convergence theorem

Theorem (K. (2010))

Let $\mathcal{F} = \{v_1, \ldots, v_m\}$ be a finite set of unit vectors in $\mathbb{R}^n$. If each symmetral direction u_i is taken from the finite set $\mathcal{F}$, then the limit

$$L = \lim_{j \to \infty} \mathrm{s}_{u_j} \cdots \mathrm{s}_{u_1} K$$

exists for all compact convex $K \subseteq \mathbb{R}^n$. Moreover, the limit L is symmetric under reflection through each of the $v_i \in \mathcal{F}$ that is used infinitely often in the sequence.

In particular, if the sequence of symmetral directions $\{u_i\}$ uses each of the v_i infinitely often, then the resulting operator on compact convex sets is idempotent.

Open questions

Earlier we described a convex body K and a sequence of directions u_i for which the sequence of Steiner symmetrals

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However, the bodies K_i appear to *converge in shape*: there appears to be a corresponding sequence of isometries ψ_i such that the sequence $\{\psi_i K_i\}$ converges.

Is this always the case?

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Is this always the case?

Very recent news: Bianchi, Burchard, Campi, Gronchi, and Volčič have found a proof that this is true.

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A related open question: What happens if K is permitted to be an arbitrary (possibly non-convex) compact set?

More generally, under what conditions is the limit

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For a particular set K?
For all compact convex sets K?
For all compact sets K?

The End

Thank you